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SZEGÖ PROJECTIONS AND NEW INVARIANTS FOR CR AND CONTACT MANIFOLDS

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ABSTRACT. In this paper we present the construction in [Po4] of several new invariants for CR and contact manifolds by looking at the noncommutative residue traces of various geometric Ψ_HDO projections. In the CR setting these operators arise from the $\overline{\partial}_b$-complex and include the Szegö projections acting on (p,q)-forms. In the contact setting they stem from the generalized Szegö projections at arbitrary integer levels of Epstein-Melrose and from the contact complex of Rumin. In particular, we recover and extend recent results of Hirachi and Boutet de Monvel and answer a question of Fefferman. Furthermore, we give simple algebro-geometric arguments proving that Hirachi's invariant vanishes on strictly pseudoconvex CR manifolds of dimension $4m+1$.

1. INTRODUCTION

Let $D\subset\mathbb{C}^{n+1}$ be a strictly pseudoconvex domain with boundary ∂D. Let θ be a pseudohermitian contact form on ∂D, i.e., if near a point of ∂D we let $\rho(z,\overline{z})$ be a local defining function for D with $\partial\overline{\partial}\rho>0$ then θ agrees up to a conformal factor with $i(\partial-\overline{\partial})\rho$.

We endow ∂D with the Levi metric defined by the Levi form associated to θ and we let $S_\theta: L^2(\partial D)\rightarrow L^2(\partial D)$ be the Szegö projection associated to this metric and let $k_{S_\theta}(z,\overline{w})d\theta^n\wedge\theta$ be the Schwartz kernel of S_θ. As shown by Fefferman [Fe1] and Boutet de Monvel-Sjöstrand [BS] near the diagonal $w=z$ we can write

$$k_{S_\theta}(z,\overline{w})=\varphi_\theta(z,\overline{w})\rho(z,\overline{w})^{-(n+1)}+\psi_\theta(z,\overline{w})\log\rho(z,\overline{w}), \tag{1.1}$$

where $\varphi_\theta(z,\overline{w})$ and $\psi_\theta(z,\overline{w})$ are smooth functions. Then Hirachi defined

$$L(S_\theta):=\int_M\psi_\theta(z,\overline{z})d\theta^n\wedge\theta. \tag{1.2}$$

Theorem 1.1 (Hirachi [Hi]). *1) $L(S_\theta)$ is a CR invariant, i.e., it does not depend on the choice of θ. In particular, this is a biholomorphic invariant of D.*

2) $L(S_\theta)$ is invariant under smooth deformations of the domain D.

Subsequently, Boutet de Monvel [Bo2] generalized Hirachi's result to the contact setting in terms of the generalized Szegö projections for contact manifolds introduced in [BGu]. Such operators are FIO's with complex phase and their kernels admit near the diagonal a singularity similar to (1.1). In this setting the integral of the leading logarithmic singularity defines a contact invariant.

It has been asked by Fefferman whether there would exist other invariants like $L(S_\theta)$, i.e., invariants arising from the integrals of the log singularities of geometric operators. The aim of this paper is to explain that there are many such invariants. These invariants can be classified into three families:

(i) CR invariants coming for the $\overline{\partial}_b$-complex of Kohn-Rossi ([KR], [Ko]);

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(ii) Contact invariants arising from the generalized Szegö projections at arbitrary integer level of Epstein-Melrose [EM];

(iii) Contact invariants coming from the contact complex of Rumin [Ru].

The construction of these invariants is based on two main tools:

- The Heisenberg calculus of Beals-Greiner [BG] and Taylor [Tay];
- The noncommutative residue trace for the Heisenberg calculus constructed in [Po1] and [Po5].

To date there is no known example of CR or contact manifold for which one of those invariants is not zero. The only known results are vanishing results: Hirachi [Hi] and Boutet de Monvel [Bo2] proved that their invariant vanish in dimension 3, and Boutet de Monvel [Bo3] has announced a proof of the vanishing of the invariant in any dimension, but the details of the proof have not appeared yet. In this paper, we give simple algebro-geometric arguments proving the vanishing of this invariant on strictly pseudoconvex CR manifolds of dimension $4m+1$ (see Section 7).

The talk is organized as follows. In Section 2 we recall few facts about Heisenberg manifolds and the Heisenberg calculus. In Section 3 we recall the construction and the main properties of the noncommutative residue for the Heisenberg calculus. In Section 4 we present the construction of the CR invariants for the $\overline{\partial}_b$-complex. In Section 5 we obtain contact invariants from the generalized Szegö projections of Epstein-Melrose. In Section 6 we construct contact invariants from Rumin's contact complex. Finally, in Section 7 we establish the vanishing of Hirachi's invariant on strictly pseudoconvex CR manifolds of dimension $4m+1$.

2. Heisenberg calculus

2.1. Heisenberg manifolds. A Heisenberg manifold is a pair (M,H) consisting of a manifold M together with a distinguished hyperplane bundle $H\subset TM$. Moreover, given another Heisenberg manifold (M',H') we say that a diffeomorphism $\phi: M\rightarrow M'$ is a Heisenberg diffeomorphism when $\phi_*H=H'$.

The main examples of Heisenberg manifolds include the following.

a) Heisenberg group. The $(2n+1)$-dimensional Heisenberg group $\mathbb{H}^{2n+1}$ is $\mathbb{R}^{2n+1}=\mathbb{R}\times\mathbb{R}^{2n}$ equipped with the group law,

$$x.y=(x_0+y_0+\sum_{1\leq j\leq n}(x_{n+j}y_j-x_jy_{n+j}),x_1+y_1,\ldots,x_{2n}+y_{2n}). \tag{2.1}$$

A left-invariant basis for its Lie algebra $\mathfrak{h}^{2n+1}$ is provided by the vector-fields,

$$X_0=\frac{\partial}{\partial x_0},\quad X_j=\frac{\partial}{\partial x_j}+x_{n+j}\frac{\partial}{\partial x_0},\quad X_{n+j}=\frac{\partial}{\partial x_{n+j}}-x_j\frac{\partial}{\partial x_0}, \tag{2.2}$$

with $j=1,\ldots,n$. For $j,k=1,\ldots,n$ and $k\neq j$ we have the relations,

$$[X_j,X_{n+k}]=-2\delta_{jk}X_0,\qquad [X_0,X_j]=[X_j,X_k]=[X_{n+j},X_{n+k}]=0. \tag{2.3}$$

In particular, the subbundle spanned by the vector fields $X_1,\ldots,X_{2n}$ defines a left-invariant Heisenberg structure on $\mathbb{H}^{2n+1}$.

(b) Codimension 1 foliations. These are the Heisenberg manifolds (M,H) such that H is integrable in Fröbenius' sense, i.e., $C^\infty(M,H)$ is closed under the Lie bracket of vector fields.

(c) Contact manifolds. A contact manifold is a Heisenberg manifold (M^{2n+1},H) such that near any point of M there exists a contact form annihilating H, i.e., a 1-form θ such that $d\theta_{|H}$ is non-degenerate. When M is orientable it is equivalent to require the existence of a globally defined contact form on M annihilating H. More specific examples of contact manifolds include the Heisenberg group $\mathbb{H}^{2n+1}$, boundaries of strictly pseudoconvex domains $D\subset\mathbb{C}^{2n+1}$, like the sphere S^{2n+1}, or even the cosphere bundle S^*M of a Riemannian manifold M^{n+1}.

d) Confoliations. The confoliations of Elyashberg and Thurston in [ET] interpolate between contact manifolds and foliations. They can be seen as oriented Heisenberg manifolds (M^{2n+1}, H) together with a non-vanishing 1-form θ on M annihilating H and such that $(d\theta)^n \wedge \theta \geq 0$.

e) CR manifolds. If $D \subset \mathbb{C}^{n+1}$ a bounded domain with boundary ∂D then the maximal complex structure, or CR structure, of $T(\partial D)$ is given by $T_{1,0} = T(\partial D) \cap T_{1,0}\mathbb{C}^{n+1}$, where $T_{1,0}$ denotes the holomorphic tangent bundle of $\mathbb{C}^{n+1}$. More generally, a CR structure on an orientable manifold M^{2n+1} is given by a complex rank n integrable subbundle $T_{1,0} \subset T_{\mathbb{C}}M$ such that $T_{1,0} \cap \overline{T_{1,0}} = \{0\}$. Besides on boundaries of complex domains, such structures naturally appear on real hypersurfaces in $\mathbb{C}^{n+1}$, quotients of the Heisenberg group $\mathbb{H}^{2n+1}$ by cocompact lattices, boundaries of complex hyperbolic spaces, and circle bundles over complex manifolds.

A real hypersurface $M = \{r = 0\} \subset \mathbb{C}^{n+1}$ is strictly pseudoconvex when the Hessian $\partial\overline{\partial}r$ is positive definite. In general, to a CR manifold M we can associate a Levi form $L_\theta(Z, W) = -id\theta(Z, \overline{W})$ on the CR tangent bundle $T_{1,0}$ by picking a non-vanishing real 1-form θ annihilating $T_{1,0} \oplus T_{0,1}$. We then say that M is strictly pseudoconvex (resp. κ-strictly pseudoconvex) when we can choose θ so that L_θ is positive definite (resp. is nondegenerate with κ negative eigenvalues) at every point. In particular, when this happens θ is non-degenerate on $H = \Re(T_{1,0} \oplus T_{0,1})$ and so (M, H) is a contact manifold.

2.2. **Tangent Lie group bundle.** The terminology Heisenberg manifold stems from the fact that the relevant tangent structure in this setting is that of a bundle GM of graded nilpotent Lie groups (see [BG], [Be], [EM], [FS], [Gr], [Po2], [Ro]). This tangent Lie group bundle bundle can be described as follows.

First, we can define an intrinsic Levi form as the 2-form $\mathcal{L} : H \times H \to TM/H$ such that, for any point $a \in M$ and any sections X and Y of H near a, we have

$$\mathcal{L}_a(X(a), Y(a)) = [X, Y](a) \mod H_a. \tag{2.4}$$

In other words the class of $[X, Y](a)$ modulo H_a depends only on $X(a)$ and $Y(a)$, not on the germs of X and Y near a (see [Po2]).

We define the tangent Lie algebra bundle $\mathfrak{g}M$ as the graded Lie algebra bundle consisting of $(TM/H) \oplus H$ together with the fields of Lie bracket and dilations such that, for sections X_0, Y_0 of TM/H and X', Y' of H and for $t \in \mathbb{R}$, we have

$$[X_0 + X', Y_0 + Y'] = \mathcal{L}(X', Y'), \qquad t.(X_0 + X') = t^2 X_0 + tX'. \tag{2.5}$$

Each fiber $\mathfrak{g}_aM$, $a \in M$, is a two-step nilpotent Lie algebra, so by requiring the exponential map to be the identity the associated tangent Lie group bundle GM appears as $(TM/H) \oplus H$ together with the grading above and the product law such that, for sections X_0, Y_0 of TM/H and X', Y' of H, we have

$$(X_0 + X').(Y_0 + Y') = X_0 + Y_0 + \frac{1}{2}\mathcal{L}(X', Y') + X' + Y'. \tag{2.6}$$

Moreover, if ϕ is a Heisenberg diffeomorphism from (M, H) onto a Heisenberg manifold (M', H') then, as $\phi_* H = H'$ we get linear isomorphisms from TM/H onto TM'/H' and from H onto H' which together give rise to a linear isomorphism $\phi'_H : TM/H \oplus H \to TM'/H' \oplus H'$. In fact ϕ'_H is a graded Lie group isomorphism from GM onto GM' (see [Po2]).

On the other hand, we have:

Proposition 2.1 ([Po2]). *1) At a point $x \in M$ we have $\operatorname{rk}\mathcal{L}(x) = 2n$ iff G_xM is isomorphic to $\mathbb{H}^{2n+1} \times \mathbb{R}^{d-2n}$.*

2) If $\dim M = 2n + 1$ then (M^{2n+1}, H) is a contact manifold iff GM is a fiber bundle with typical fiber $\mathbb{H}^{2n+1}$.

2.3. **Heisenberg calculus.** The Heisenberg calculus is the relevant pseudodifferential calculus to study hypoelliptic operators on Heisenberg manifolds. It was independently introduced by Beals-Greiner [BG] and Taylor [Tay] (see also [Bol], [Dy1], [Dy2], [EM], [FS], [Po3], [RS]).

The initial idea in the Heisenberg calculus, which is due to Stein, is to construct a class of operators on a Heisenberg manifold (M^{d+1}, H), called Ψ_HDO's, which at any point $a \in M$ are modeled on homogeneous left-invariant convolution operators on the tangent group G_aM.

Locally the ΨDO's can be described as follows. Let $U \subset \mathbb{R}^{d+1}$ be a local chart together with a frame $X_0, \ldots, X_d$ of TU such that $X_1, \ldots, X_d$ span H. Such a chart is called a Heisenberg chart. Moreover, on $\mathbb{R}^{d+1}$ we consider the dilations,

$$t.\xi = (t^2\xi_0, t\xi_1, \ldots, t\xi), \qquad \xi \in \mathbb{R}^{d+1}, \quad t > 0. \tag{2.7}$$

Definition 2.2. *1) $S_m(U \times \mathbb{R}^{d+1})$, $m \in \mathbb{C}$, is the space of functions $p(x,\xi)$ in $C^\infty(U \times \mathbb{R}^{d+1}\backslash 0)$ such that $p(x, t.\xi) = t^m p(x,\xi)$ for any $t > 0$.*

2) $S^m(U \times \mathbb{R}^{d+1})$, $m \in \mathbb{C}$, consists of functions $p \in C^\infty(U \times \mathbb{R}^{d+1})$ with an asymptotic expansion $p \sim \sum_{j\geq 0} p_{m-j}$, $p_k \in S_k(U \times \mathbb{R}^{d+1})$, in the sense that, for any integer N and for any compact $K \subset U$, we have

$$|\partial_x^\alpha \partial_\xi^\beta (p - \sum_{j<N} p_{m-j})(x,\xi)| \leq C_{\alpha\beta NK} \|\xi\|^{\Re m - \langle\beta\rangle - N}, \qquad x \in K, \quad \|\xi\| \geq 1, \tag{2.8}$$

where we have let $\langle\beta\rangle = 2\beta_0 + \beta_1 + \ldots + \beta_d$ and $\|\xi\| = (\xi_0^2 + \xi_1^4 + \ldots + \xi_d^4)^{1/4}$.

Next, for $j = 0, \ldots, d$ let $\sigma_j(x,\xi)$ denote the symbol (in the classical sense) of the vector field $\frac{1}{i}X_j$ and set $\sigma = (\sigma_0, \ldots, \sigma_d)$. Then for $p \in S^m(U \times \mathbb{R}^{d+1})$ we let $p(x, -iX)$ be the continuous linear operator from $C_c^\infty(U)$ to $C^\infty(U)$ such that

$$p(x,-iX)f(x) = (2\pi)^{-(d+1)} \int e^{ix.\xi} p(x, \sigma(x,\xi)) \hat{f}(\xi) d\xi, \qquad f \in C_c^\infty(U). \tag{2.9}$$

Definition 2.3. *$\Psi_H^m(U)$, $m \in \mathbb{C}$, consists of operators $P : C_c^\infty(U) \to C^\infty(U)$ which are of the form $P = p(x,-iX) + R$ for some p in $S^m(U \times \mathbb{R}^{d+1})$, called the symbol of P, and some smoothing operator R.*

For any $a \in U$ there is exists a unique affine change of variable $\psi_a : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ such that $\psi_a(a) = 0$ and $(\psi_a)_* X_j = \frac{\partial}{\partial x_j}$ at $x = 0$ for $j = 0, 1, \ldots, d+1$. Then, a continuous operator $P : C_c^\infty(U) \to C^\infty(U)$ is a Ψ_HDO of order m if, and only if, its kernel $k_P(x,y)$ has a behavior near the diagonal of the form,

$$k_P(x,y) \sim \sum_{j\geq -(m+d+2)} (a_j(x, \psi_x(y)) - \sum_{\langle\alpha\rangle = j} c_\alpha(x)\psi_x(x)^\alpha \log \|\psi_x(y)\|), \tag{2.10}$$

with $c_\alpha \in C^\infty(U)$ and $a_j(x,y) \in C^\infty(U \times (\mathbb{R}^{d+1}\backslash 0))$ such that $a_j(x, \lambda.y) = \lambda^j a_j(x,y)$ for any $\lambda > 0$. Moreover, $a_j(x,y)$ and $c_\alpha(x)$, $\langle\alpha\rangle = j$, depend only on the symbol of P of degree $-(j+d+2)$.

The class of Ψ_HDO's is invariant under changes of Heisenberg chart (see [BG, Sect. 16], [Po3, Appendix A]), so we may extend the definition of Ψ_HDO's to an arbitrary Heisenberg manifold (M, H) and let them act on sections of a vector bundle $\mathcal{E}$ over M. We let $\Psi_H^m(M, \mathcal{E})$ denote the class of Ψ_HDO's of order m on M acting on sections of $\mathcal{E}$.

Let $\mathfrak{g}^*M$ denote the (linear) dual of the Lie algebra bundle $\mathfrak{g}M$ of GM with canonical projection $\mathrm{pr} : M \to \mathfrak{g}^*M$. As shown in [Po3] (see also [EM]) the principal symbol of an operator $P \in \Psi_H^m(M, \mathcal{E})$ can be intrinsically defined as a symbol $\sigma_m(P)$ of the class below.

Definition 2.4. *$S_m(\mathfrak{g}^*M)$, $m \in \mathbb{C}$, consists of sections $p \in C^\infty(\mathfrak{g}^*M \setminus 0, \mathrm{End}\,\mathrm{pr}^*\mathcal{E})$ which are homogeneous of degree m with respect to the dilations in (2.5), i.e., we have $p(x, \lambda.\xi) = \lambda^m p(x,\xi)$ for any $\lambda > 0$.*

Next, for any $a \in M$ the convolution on G_aM gives rise under the (linear) Fourier transform to a bilinear product for homogeneous symbols,

$$*^a : S_{m_1}(\mathfrak{g}_a^*M, \mathcal{E}_a) \times S_{m_2}(\mathfrak{g}_a^*M, \mathcal{E}_a) \longrightarrow S_{m_1+m_2}(\mathfrak{g}_a^*M, \mathcal{E}_a), \tag{2.11}$$

This product depends smoothly on a as much so to yield a product,

$$* : S_{m_1}(\mathfrak{g}^*M, \mathcal{E}) \times S_{m_2}(\mathfrak{g}^*M, \mathcal{E}) \longrightarrow S_{m_1+m_2}(\mathfrak{g}^*M, \mathcal{E}), \tag{2.12}$$

$$p_{m_1} * p_{m_2}(a, \xi) = [p_{m_1}(a, .) *^a p_{m_2}(a, .)](\xi). \tag{2.13}$$

This provides us with the right composition for principal symbols, since we have

$$\sigma_{m_1+m_2}(P_1P_2) = \sigma_{m_1}(P_1) * \sigma_{m_2}(P_2) \qquad \forall P_j \in \Psi_H^{m_j}(M, \mathcal{E}). \tag{2.14}$$

for $P_1 \in \Psi_H^{m_1}(M, \mathcal{E})$ and $P_2 \in \Psi_H^{m_2}(M, \mathcal{E})$ such that one of them is properly supported.

Notice that when G_aM is not commutative, i.e., $\mathcal{L}_a \neq 0$, the product $*^a$ is not anymore the pointwise product of symbols and, in particular, is not commutative. Consequently, unless when H is integrable, the product for Heisenberg symbols is not commutative and, while local, it is not microlocal.

When the principal symbol of $P \in \Psi_H^m(M, \mathcal{E})$ is invertible with respect to the product $*$, the symbolic calculus of [BG] allows us to construct a parametrix for P in $\Psi_H^{-m}(M, \mathcal{E})$. In particular, although not elliptic, P is hypoelliptic with a controlled loss/gain of derivatives (see [BG]).

In general, it may be difficult to determine whether the principal symbol of a given operator P in $\Psi_H^m(M, \mathcal{E})$ is invertible with respect to the product $*$, but this can be completely determined in terms of a representation theoretic criterion on each tangent group G_aM, the so-called Rockland condition (see [Po3, Thm. 3.3.19]). In particular, if $\sigma_m(P)(a, .)$ is *pointwise* invertible with respect to the product $*^a$ for any $a \in M$ then $\sigma_m(P)$ is *globally* invertible with respect to $*$.

3. Noncommutative residue

Let (M^{d+1}, H) be a Heisenberg manifold equipped with a smooth positive density and let $\mathcal{E}$ be a Hermitian vector bundle over M. We let $\Psi_H^{\mathbb{Z}}(M, \mathcal{E})$ denote the space of Ψ_HDO of integer order acting on sections of $\mathcal{E}$.

3.1. Logarithmic singularity.

Let $P : C^\infty(M, \mathcal{E}) \to C^\infty(M, \mathcal{E})$ be a Ψ_HDO of integer order m. Then it follows from (2.10) that in a trivializing Heisenberg chart the kernel $k_P(x, y)$ of P has a behavior near the diagonal of the form,

$$k_P(x, y) = \sum_{-(m+d+2) \leq j \leq 1} a_j(x, -\psi_x(y)) - c_P(x) \log \|\psi_x(y)\| + \mathrm{O}(1), \tag{3.1}$$

where $a_j(x, y)$ is homogeneous of degree j in y with respect to the dilations (2.7). Furthermore, we have

$$c_P(x) = |\psi_x'| \int_{\|\xi\|=1} p_{-(d+2)}(x, \xi) d\xi, \tag{3.2}$$

where $p_{-(d+2)}(x, \xi)$ is the homogeneous symbol of degree $-(d+2)$ of P.

Let $|\Lambda|(M)$ be the bundle of densities on M. Then we have:

Proposition 3.1 ([Po1], [Po5]). *The coefficient $c_P(x)$ makes sense intrinsically on M as a section of $|\Lambda|(M) \otimes \operatorname{End} \mathcal{E}$.*

3.2. Noncommutative residue.

From now on we assume M compact. Therefore, for any $P \in \Psi_H^{\mathbb{Z}}(M, \mathcal{E})$ we can let

$$\operatorname{Res} P = \int_M \operatorname{tr}_{\mathcal{E}} c_P(x). \tag{3.3}$$

If P is in $\Psi_H^m(M, \mathcal{E})$ with $\Re m < -(d+2)$ then P is trace-class. It can be shown that we have an analytic continuation of the trace to Ψ_HDO's of non-integer orders which is analogous to that for classical ΨDO's in [KV]. Moreover, on Ψ_HDO's of integer orders this analytic extension of the trace induces a residual functional agreeing with (3.3), so that we have:

Proposition 3.2. *Let $P \in \Psi_H^{\mathbb{Z}}(M,\mathcal{E})$. Then for any family $(P(z))_{z\in\mathbb{C}} \subset \Psi_H^*(M,\mathcal{E})$ which is holomorphic in the sense of* [Po3] *and such that $P(0)=P$ and $\operatorname{ord}P(z) = z + \operatorname{ord}P$ we have*

$$\operatorname{Res} P = -\operatorname{res}_{z=0} \operatorname{Trace} P(z). \tag{3.4}$$

Thus the functional (3.3) is the analogue for the Heisenberg calculus of the noncommutative residue of Wodzicki ([Wo1], [Wo2]) and Guillemin [Gu1]. Furthermore, we have:

Proposition 3.3 ([Po1], [Po5]). *1) Let ϕ be a Heisenberg diffeomorphism from (M,H) onto a Heisenberg manifold (M',H'). Then for any $P \in \Psi_H^{\mathbb{Z}}(M,\mathcal{E})$ we have* $\operatorname{Res}\phi_* P = \operatorname{Res} P$.

2) Res *is a trace on the algebra $\Psi_H^{\mathbb{Z}}(M,\mathcal{E})$ which vanishes on differential operators and on Ψ_HDO's of integer order $\leq -(d+3)$.*

3) If M is connected then Res *is the unique trace up to constant multiple.*

Let $D \subset \mathbb{C}^{n+1}$ be a strictly pseudoconvex domain with boundary ∂D and let θ be a pseudohermitian contact form on ∂D. We endow ∂D with the associated Levi metric and we let $S_\theta : L^2(\partial D) \rightarrow L^2(\partial D)$ be the corresponding Szegö projection. Then S_θ is a Ψ_HDO of order 0 and with the notation of (1.1) we have $c_{S_\theta}(z) = -\frac{1}{2}\psi_\theta(z,\overline{z})d\theta^n \wedge \theta$. Thus,

$$\operatorname{Res} S_\theta = -\frac{1}{2} L(S_\theta). \tag{3.5}$$

This shows that Hirachi's invariant can be interpreted as a noncommutative residue.

4. CR invariants from the $\overline{\partial}_b$-complex

Let M^{2n+1} be a compact orientable CR manifold with CR tangent bundle $T_{1,0} \subset T_{\mathbb{C}}M$, so that $H = \Re(T_{1,0} \oplus T_{0,1}) \subset TM$ is a hyperplane bundle of TM admitting an (integrable) complex structure. Let θ be a global non-zero real 1-form annihilating H and let L_θ be the associated Levi form,

$$L_\theta(Z,W) = -id\theta(Z,\overline{W}) = i\theta([Z,\overline{W}]), \qquad Z,W \in C^\infty(M,T_{1,0}). \tag{4.1}$$

Let $\mathcal{N}$ be a supplement of H in TM. This is an orientable line bundle which gives rise to the splitting,

$$T_{\mathbb{C}}M = T_{1,0} \oplus T_{0,1} \oplus (\mathcal{N} \otimes \mathbb{C}). \tag{4.2}$$

Let $\Lambda^{1,0}$ and $\Lambda^{0,1}$ denote the annihilators in $T_{\mathbb{C}}^*M$ of $T_{0,1} \oplus (\mathcal{N}\otimes\mathbb{C})$ and $T_{1,0} \oplus (\mathcal{N}\otimes\mathbb{C})$ respectively and for $p,q = 0,\ldots,n$ let $\Lambda^{p,q} = (\Lambda^{1,0})^p \wedge (\Lambda^{0,1})^q$ be the bundle of (p,q)-forms. Then we have the splitting,

$$\Lambda^* T_{\mathbb{C}}^* M = (\bigoplus_{p,q=0}^{n} \Lambda^{p,q}) \oplus \theta \wedge \Lambda^* T_{\mathbb{C}}^* M. \tag{4.3}$$

Notice that this decomposition does not depend on the choice of θ, but it does depend on that of $\mathcal{N}$.

The complex $\overline{\partial}_b : C^\infty(M,\Lambda^{p,*}) \rightarrow C^\infty(M,\Lambda^{p,*+1})$ of Kohn-Rossi ([KR], [Ko]) is defined as follows. For any $\eta \in C^\infty(M,\Lambda^{p,q})$ we can uniquely decompose $d\eta$ as

$$d\eta = \overline{\partial}_{b;p,q}\eta + \partial_{b;p,q}\eta + \theta \wedge \mathcal{L}_{X_0}\eta, \tag{4.4}$$

where $\overline{\partial}_{b;p,q}\eta$ and $\partial_{b;p,q}\eta$ are sections of $\Lambda^{p,q+1}$ and $\Lambda^{p+1,q}$ respectively and X_0 is the section of $\mathcal{N}$ such that $\theta(X_0) = 1$. Thanks to the integrability of $T_{1,0}$ we have $\overline{\partial}_{b;p,q+1} \circ \overline{\partial}_{b;p,q} = 0$, so that we get a chain complex. Notice that this complex depends on the CR structure of M and on the choice of $\mathcal{N}$.

Assume now that M is endowed with a Hermitian metric h on $T_{\mathbb{C}}M$ which commutes with complex conjugation and makes the splitting (4.2) become orthogonal. The associated Kohn Laplacian is

$$\Box_{b;p,q} = \overline{\partial}^*_{b;p,q+1}\overline{\partial}_{b;p,q} + \overline{\partial}_{b;p,q-1}\overline{\partial}^*_{b;p,q}. \tag{4.5}$$

For $x \in M$ let $\kappa_+(x)$ (resp. $\kappa_-(x)$) be the number of positive (resp. negative) eigenvalues of L_θ at x. We then say that the condition $Y(q)$ holds when at every point $x \in M$ we have

$$q \notin \{\kappa_-(x), \ldots, n - \kappa_+(x)\} \cup \{\kappa_+(x), \ldots, n - \kappa_-(x)\}. \tag{4.6}$$

For instance, when M is κ-strictly pseudoconvex we have $\kappa_-(x) = n - \kappa_+(x) = \kappa$, so the condition $Y(q)$ exactly means that we must have $q \neq \kappa$ and $q \neq n - \kappa$.

Proposition 4.1 (see [BG, Sect. 21], [Po3, Sect. 3.5]). *The Kohn Laplacian $\Box_{b;p,q}$ admits a parametrix in $\Psi_H^{-2}(M, \Lambda^{p,q})$ iff the condition $Y(q)$ is satisfied.*

Let $S_{b;p,q}$ be the Szegö projection on (p,q)-forms, i.e., the orthogonal projection onto $\ker \Box_{b;p,q}$. We also consider the orthogonal projections $\Pi_0(\overline{\partial}_{b;p,q})$ and $\Pi_0(\overline{\partial}^*_{b;p,q})$ onto $\ker \overline{\partial}_{b;p,q}$ and $\ker \overline{\partial}^*_{b;p,q} = (\operatorname{im} \overline{\partial}_{b;p,q-1})^\perp$. In fact, as $\ker \overline{\partial}_{b;p,q} = \ker \Box_{b;p,q} \oplus \operatorname{im} \overline{\partial}_{b;p,q-1}$ we have $\Pi_0(\overline{\partial}_{b;p,q}) = S_{b;p,q} + 1 - \Pi_0(\overline{\partial}^*_{b;p,q})$, that is,

$$S_{b;p,q} = \Pi_0(\overline{\partial}_{b;p,q}) + \Pi_0(\overline{\partial}^*_{b;p,q}) - 1. \tag{4.7}$$

Let $N_{b;p,q}$ be the partial inverse of $\Box_{b;p,q}$, so that $N_{b;p,q}\Box_{b;p,q} = \Box_{b;p,q}N_{b;p,q} = 1 - S_{b;p,q}$. Then it can be shown (see, e.g., [BG, pp. 170–172]) that we have

$$\Pi_0(\overline{\partial}_{b;p,q}) = 1 - \overline{\partial}^*_{b;p,q}N_{b;p,q+1}\overline{\partial}_{b;p,q}, \qquad \Pi_0(\overline{\partial}^*_{b;p,q}) = 1 - \overline{\partial}_{b;p,q-1}N_{b;p,q-1}\overline{\partial}^*_{b;p,q-1}. \tag{4.8}$$

By Proposition 4.1 when the condition $Y(q)$ holds at every point the operator $\Box_{b;p,q}$ admits a parametrix in $\Psi_H^{-2}(M, \Lambda^{p,q})$ and then $S_{b;p,q}$ is a smoothing operator and $N_{b;p,q}$ is a Ψ_HDO of order -2. Therefore, using (4.8) we see that if the condition $Y(q+1)$ (resp. $Y(q-1)$) holds everywhere then $\Pi_0(\overline{\partial}_{b;p,q})$ (resp. $\Pi_0(\overline{\partial}^*_{b;p,q})$) is a Ψ_HDO.

Furthermore, in view of (4.7) we also see that if at every point the condition $Y(q)$ fails, but the conditions $Y(q-1)$ and $Y(q+1)$ hold, then the Szegö projection $S_{b;p,q}$ is a zero'th order Ψ_HDO projection. Notice that this may happen if, and only if, M is κ-strictly pseudoconvex with $\kappa = q$ or $\kappa = n - q$.

Bearing all this in mind we have:

Theorem 4.2 ([Po4]). *1) The following noncommutative residues are CR diffeomorphism invariants of M:*

(i) $\operatorname{Res} \Pi_0(\overline{\partial}_{b;p,q})$ *when the condition $Y(q+1)$ holds everywhere;*

(ii) $\operatorname{Res} \Pi_0(\overline{\partial}^*_{b;p,q})$ *when the condition $Y(q-1)$ holds everywhere;*

(iii) $\operatorname{Res} S_{b;p,\kappa}$ *and* $\operatorname{Res} S_{b;p,n-\kappa}$ *when M is κ-strictly pseudoconvex.*

In particular, they depend neither on the choice of the line bundle $\mathcal{N}$, nor on that of the Hermitian metric h.

2) The noncommutative residues (i)–(iii) are invariant under deformations of the CR structure coming from deformations of the complex structure of H.

Specializing Theorem 4.2 to the strictly pseudoconvex case we get:

Theorem 4.3 ([Po4]). *Suppose that M is a compact strictly pseudoconvex CR manifold. Then:*

1) $\operatorname{Res} S_{b;p,j}$, $j = 0, n$, *and* $\operatorname{Res} \Pi_0(\overline{\partial}_{b;p,q})$, $q = 1, \ldots, n-1$, *are CR diffeomorphism invariants of M. In particular, when M is the boundary of a strictly pseudoconvex domain $D \subset \mathbb{C}^n$ they give rise to bilholomorphism invariants of D.*

2) The above residues are invariant under deformations of the CR structure.

5. Invariants of generalized Szegö projections

Let (M^{2n+1}, H) be an orientable contact manifold. Given a contact form θ on M annihilating H we let X_0 be the Reeb vector field of θ, i.e., the unique vector field X_0 such that $\iota_{X_0}\theta = 1$ and $\iota_{X_0}d\theta = 0$. In addition, we let J be an almost complex structure on H which is calibrated in the sense that $d\theta(X, JX) > 0$ for any nonzero section X of H. Extending J to TM by requiring to have $JX_0 = 0$, we can equip TM with the Riemannian metric $g_{\theta,J} = d\theta(.,J.) + \theta^2$.

In this context Szegö projections have been defined by Boutet de Monvel and Guillemin in [BGu] as an FIO with complex phase. This construction has been further generalized by Epstein-Melrose [EM] as follows.

Let $\mathbb{H}^{2n+1}$ be the Heisenberg group of dimension $2n+1$ consisting of $\mathbb{R}^{2n+1}$ together with the group law (2.1). Let $\theta^0 = dx_0 + \frac{1}{2}\sum_{j=1}^n (x_j dx_{n+j} - x_{n+j}dx_j)$ be the standard left-invariant contact form of $\mathbb{H}^{2n+1}$; its Reeb vector field is $X_0^0 = \frac{\partial}{\partial x_0}$.

For $j = 1, \ldots, n$ let $X_j^0 = \frac{\partial}{\partial x_j} + \frac{1}{2}x_{n+j}\frac{\partial}{\partial x_0}$ and $X_{n+j}^0 = \frac{\partial}{\partial x_{n+j}} - \frac{1}{2}x_j\frac{\partial}{\partial x_0}$ then $X_1^0, \ldots, X_{2n}^0$ form a left-invariant frame of $H^0 = \ker\theta^0$ and sastify the relations (2.3). The standard CR structure of $\mathbb{H}^{2n+1}$ is then given by the complex structure J^0 on H^0 such that $J^0X_j^0 = X_{n+j}^0$ and $J^0X_{n+j} = -X_j$. Moreover, it follows from (2.3) that J^0 is calibrated with respect to θ^0 and that $X_0^0, X_1^0, \ldots, X_{2n}^0$ form an orthonormal frame of $T\mathbb{H}^{2n+1}$ with respect to the metric g_{θ^0,J^0}.

The scalar Kohn Laplacian on $\mathbb{H}^{2n+1}$ is equal to

$$\Box_{b,0}^0 = -\frac{1}{2}((X_1^0)^2 + \ldots + (X_{2n}^0)^2) + i\frac{n}{2}X_0^0. \tag{5.1}$$

For $\lambda \in \mathbb{C}$ the operator $-\frac{1}{2}((X_1^0)^2 + \ldots + (X_{2n}^0)^2) + i\lambda X_0^0$ is invertible if, and only if, we have $\lambda \notin \pm(\frac{n}{2} + \mathbb{N})$ (see [FS], [BG]). For $k = 0, 1, \ldots$ the orthogonal projection $\Pi_0(\Box_b + ikX_0^0)$ onto the kernel of $\Box_b + ikX_0^0$ is a left-invariant homogeneous Ψ_HDO of order 0 (see [BG, Thm. 6.61]). We then let $s_k^0 \in S_0((\mathfrak{h}^{2n+1})^*)$ denote its symbol, so that we have $\Pi_0(\Box_b + ikX_0^0) = s_k^0(-iX^0)$.

Next, since (M, H) is a contact manifold by Proposition 2.1 the tangent Lie group bundle GM is a fiber bundle with typical fiber $\mathbb{H}^{2n+1}$. A local trivialization near a given point $a \in M$ is obtained as follows.

Let $X_1, \ldots, X_{2n}$ be a local orthonormal frame of H on an open neighborhood U of a and which is admissible in the sense that $X_{n+j} = JX_j$ for $j = 1, \ldots, n$. In addition, let $\underline{X_0(a)}$ denote the class of $X_0(a)$ in T_aM/H_a. Then as shown in [Po2] the map $\phi_{X,a} : (T_aM/H_a) \oplus H_a \rightarrow \mathbb{R}^{2n+1}$ such that

$$\phi_{X,a}(x_0\underline{X_0(a)} + x_1X_1(a) + \ldots + x_{2n}X_{2n}(a)) = (x_0, \ldots, x_{2n}), \qquad x_j \in \mathbb{R}, \tag{5.2}$$

gives rise to a Lie group isomorphism from G_aM onto $\mathbb{H}^{2n+1}$. In fact, as $\phi_{X,a}$ depends smoothly on a we get a fiber bundle trivialization of $GM|_U \simeq U \times \mathbb{H}^{2n+1}$.

For $j = 0, \ldots, 2n$ let X_j^a be the model vector field of X_j at a as defined in [Po2]. This is the unique left-invariant vector field on G_aM which, in the coordinates provided by $\phi_{X,a}$, agrees with $\frac{\partial}{\partial x_j}$ at $x = 0$. Therefore, we have $X_j^a = \phi_{X,a}^* X_j^0$ and so we get $\phi_{X,a}^*\Box_b^0 = -\frac{1}{2}((X_1^a)^2 + \ldots + (X_{2n}^a)^2) + i\frac{n}{2}X_0^a$.

If $\tilde{X}_1, \ldots, \tilde{X}_{2n}$ is another admissible orthonormal frame of H near a, then we pass from $(\tilde{X}_1^a, \ldots, \tilde{X}_{2n}^a)$ to $(X_1^a, \ldots, X_{2n}^a)$ by an orthogonal linear transformation, which leaves the expression $(X_1^a)^2 + \ldots + (X_{2n}^a)^2$ unchanged. Therefore, the differential operator $\Box_b^a := \phi_{X,a}^*\Box_b^0$ makes sense independently of the choice of the admissible frame $X_1, \ldots, X_{2n}$ near a.

On the other hand, as $\phi_{X,a}$ induces a unitary transformation from $L^2(G_aM)$ onto $L^2(\mathbb{H}^{2n+1})$ we have $\Pi_0(\Box_b^a + ikX_0^a) = \Pi_0(\phi_{X,a}^*(\Box_b^0 + ikX_0^0)) = \phi_{X,a}^*\Pi_0(\Box_b^0 + ikX_0^0))$. Hence $\Pi_0(\Box_b^a + ikX_0^a)$ is a zero'th order left-invariant homogeneous Ψ_HDO on G_aM with symbol $s_k^a(\xi) = \phi_{X,a}^* s_k^0(\xi) = s_k^0((\phi_{X,a}^{-1})^t\xi)$. In fact, since $\phi_{X,a}$ depends smoothly on a we obtain:

Proposition 5.1. *For $k = 0, 1, \ldots$ there is a uniquely defined symbol $s_k \in S_0(\mathfrak{g}^*M)$ such that, for any admissible orthonormal frame $X_1, \ldots, X_d$ of H near a point $a \in M$, we have $s_k(a,\xi) = \phi^*_{X,a} s^0_k(\xi)$ for any $(a,\xi) \in \mathfrak{g}^*M \setminus 0$.*

We call s_k the *Szegö symbol at level k.* This definition *a priori* depends on the contact form θ and the almost complex structure J, but we have:

Lemma 5.2 ([EM], [Po4]). *(i) The symbol s_k is invariant under conformal changes of contact form.*

(ii) The change $(\theta, J) \to (-\theta, -J)$ transforms s_k into $s_k(x, -\xi)$.

(iii) The symbol s_k depends on J only up to homotopy of idempotents in $S_0(\mathfrak{g}^*M)$.

From now on we let $\mathcal{E}$ be a Hermitian vector bundle over M.

Definition 5.3 ([EM, Chap. 6]). *For $k = 0, 1, \ldots$ a generalized Szegö projection at level k is a $\Psi_H DO$ projection $S_k \in \Psi^0_H(M, \mathcal{E})$ with principal symbol $s_k \otimes \mathrm{id}_{\mathcal{E}}$.*

Generalized Szegö projections at level k always exist (see [EM], [Po4]). Moreover, when $k = 0$ and $\mathcal{E}$ is the trivial line bundle the above definition allows us to recover the Szegö projections of [BGu] (see [Po4]). In particular, when M is strictly pseudoconvex the Szegö projection $S_{b,0}$ is a generalized Szegö projection at level 0.

Given a generalized Szegö projection at level k we define

$$L_k(\mathcal{E}) = \operatorname{Res} S_k. \tag{5.3}$$

In fact, we have:

Proposition 5.4 ([Po4]). *The value of $L_k(\mathcal{E})$ does not depend on the choice of S_k.*

Next, recall that the K-group $K^0(M)$ can be described as the group of formal differences of stable homotopy classes of (smooth) vector bundles over M, where a stable homotopy between vector bundles $\mathcal{E}_1$ and $\mathcal{E}_2$ is given by an auxiliary vector bundle $\mathcal{F}$ and a vector bundle isomorphism $\phi : \mathcal{E}_1 \oplus \mathcal{F} \simeq \mathcal{E}_2 \oplus \mathcal{F}$. Then we obtain:

Theorem 5.5 ([Po4]). *1) $L_k(\mathcal{E})$ depends only on the Heisenberg diffeomorphism class of M and on the K-theory class of $\mathcal{E}$. In particular, it depends neither on the contact form θ, nor on the almost complex structure J.*

2) $L_k(\mathcal{E})$ invariant is under deformations of the contact structure.

6. Invariants from the contact complex

Let (M^{2n+1}, H) be an orientable contact manifold. Let θ be a contact form on M and let X_0 be its Reeb vector field of θ. We also let J be a calibrated almost complex structure on H and we endow TM with the Riemannian metric $g_{\theta,J} = d\theta(., J.) + \theta^2$.

Observe that the splitting $TM = H \oplus \mathbb{R}X_0$ allows us to identify H^* with the annihilator of X_0 in T^*M. More generally, identifying $\Lambda^k_{\mathbb{C}} H^*$ with $\ker \iota_{X_0}$, where ι_{X_0} denotes the contraction operator by X_0, gives the splitting

$$\Lambda^*_{\mathbb{C}} TM = \Big(\bigoplus_{k=0}^{2n} \Lambda^k_{\mathbb{C}} H^*\Big) \oplus \Big(\bigoplus_{k=0}^{2n} \theta \wedge \Lambda^k_{\mathbb{C}} H^*\Big). \tag{6.1}$$

For any horizontal form $\eta \in C^\infty(M, \Lambda^k_{\mathbb{C}} H^*)$ we can write $d\eta = d_b\eta + \theta \wedge \mathcal{L}_{X_0}\eta$, where $d_b\eta$ is the component of $d\eta$ in $\Lambda^k_{\mathbb{C}} H^*$. This does not provide us with a complex, for we have $d_b^2 = -\mathcal{L}_{X_0}\varepsilon(d\theta) = -\varepsilon(d\theta)\mathcal{L}_{X_0}$, where $\varepsilon(d\theta)$ denotes the exterior multiplication by $d\theta$.

The contact complex of Rumin [Ru] is an attempt to get a complex of horizontal differential forms by forcing the equalities $d_b^2 = 0$ and $(d_b^*)^2 = 0$.

A natural way to modify d_b to get the equality $d_b^2 = 0$ is to restrict d_b to the subbundle $\Lambda^*_2 := \ker \varepsilon(d\theta) \cap \Lambda^*_{\mathbb{C}} H^*$, since the latter is closed under d_b and is annihilated by d_b^2.

Similarly, we get the equality $(d_b^*)^2 = 0$ by restricting d_b^* to the subbundle $\Lambda_1^* := \ker \iota(d\theta) \cap \Lambda_{\mathbb{C}}^* H^* = (\operatorname{im} \varepsilon(d\theta))^{\perp} \cap \Lambda_{\mathbb{C}}^* H^*$, where $\iota(d\theta)$ denotes the interior product with $d\theta$. This amounts to replace d_b by $\pi_1 \circ d_b$, where π_1 is the orthogonal projection onto Λ_1^*.

In fact, since $d\theta$ is nondegenerate on H the operator $\varepsilon(d\theta) : \Lambda_{\mathbb{C}}^k H^* \to \Lambda_{\mathbb{C}}^{k+2} H^*$ is injective for $k \leq n-1$ and surjective for $k \geq n+1$. This implies that $\Lambda_2^k = 0$ for $k \leq n$ and $\Lambda_1^k = 0$ for $k \geq n+1$. Therefore, we only have two halves of complexes.

As observed by Rumin [Ru] we get a full complex by connecting the two halves by means of the operator $D_{R,n} : C^\infty(M, \Lambda_{\mathbb{C}}^n H^*) \to C^\infty(M, \Lambda_{\mathbb{C}}^n H^*)$ such that

$$D_{R,n} = \mathcal{L}_{X_0} + d_{b,n-1} \varepsilon(d\theta)^{-1} d_{b,n}, \tag{6.2}$$

where $\varepsilon(d\theta)^{-1}$ is the inverse of $\varepsilon(d\theta) : \Lambda_{\mathbb{C}}^{n-1} H^* \to \Lambda_{\mathbb{C}}^{n+1} H^*$. Notice that $D_{R,n}$ is a second order differential operator. This allows us to get the contact complex,

$$C^\infty(M) \overset{d_{R,0}}{\to} \ldots C^\infty(M, \Lambda^n) \overset{D_{R,n}}{\to} C^\infty(M, \Lambda^n) \ldots \overset{d_{R,2n-1}}{\to} C^\infty(M, \Lambda^{2n}). \tag{6.3}$$

where $d_{R,k}$ agrees with $\pi_1 \circ d_b$ for $k = 0, \ldots, n-1$ and with $d_{R,k} = d_b$ otherwise.

The contact Laplacian is defined as follows. In degree $k \neq n$ this is the differential operator $\Delta_{R,k} : C^\infty(M, \Lambda^k) \to C^\infty(M, \Lambda^k)$ such that

$$\Delta_{R,k} = \begin{cases} (n-k) d_{R,k-1} d_{R,k}^* + (n-k+1) d_{R,k+1}^* d_{R,k}, & k = 0, \ldots, n-1, \\ (k-n-1) d_{R,k-1} d_{R,k}^* + (k-n) d_{R,k+1}^* d_{R,k}, & k = n+1, \ldots, 2n. \end{cases} \tag{6.4}$$

For $k = n$ we have the differential operators $\Delta_{R,nj} : C^\infty(M, \Lambda_j^n) \to C^\infty(M, \Lambda_j^n)$, $j = 1, 2$, given by the formulas,

$$\Delta_{R,n1} = (d_{R,n-1} d_{R,n}^*)^2 + D_{R,n}^* D_{R,n}, \quad \Delta_{R,n2} = D_{R,n} D_{R,n}^* + (d_{R,n+1}^* d_{R,n}). \tag{6.5}$$

Observe that $\Delta_{R,k}$, $k \neq n$, is a differential operator order 2, whereas Δ_{Rn1} and Δ_{Rn2} are differential operators of order 4. Moreover, Rumin [Ru] proved that in every degree the contact Laplacian is maximal hypoelliptic. In fact, in every degree the contact Laplacian has an invertible principal symbol, hence admits a parametrix in the Heisenberg calculus (see [JK], [Po3, Sect. 3.5]).

Let $\Pi_0(d_{R,k})$ and $\Pi_0(D_{R,n})$ be the orthogonal projections onto $\ker d_{R,k}$ and $\ker D_{R,n}$, and let $\Delta_{R,k}^{-1}$ and $\Delta_{R,nj}^{-1}$ be the partial inverses of $\Delta_{R,k}$ and $\Delta_{R,nj}$. Then as in (4.8) we have

$$\Pi_0(d_{R,k}) = \begin{cases} 1 - (n-k-1)^{-1} d_{R,k+1}^* \Delta_{R,k+1}^{-1} d_{R,k}, & k = 0, \ldots, n-2, \\ 1 - d_{R,n}^* d_{R,n-1} d_{R,n}^* \Delta_{R,n1}^{-1} d_{R,n-1}, & k = n-1, \\ 1 - (k-n)^{-1} d_{R,k+1}^* \Delta_{R,k+1}^{-1} d_{R,k}, & k = n, \ldots, 2n-1, \end{cases} \tag{6.6}$$

$$\Pi_0(D_{R,n}) = 1 - D_{R,*} \Delta_{R,n2}^{-1} D_{R,n}. \tag{6.7}$$

As in each degree the principal symbol of the contact Laplacian is invertible, the operators $\Delta_{R,k}^{-1}$, $k \neq n$, and $\Delta_{R,nj}^{-1}$, $j = 1, 2$ are Ψ_HDO's of order -2 and order -4 respectively. Therefore, the above formulas for $\Pi_0(d_{R,k})$ and $\Pi_0(D_{R,n})$ show that these projections are zero'th order Ψ_HDO's.

Theorem 6.1 ([Po4]). *1)* $\operatorname{Res} \Pi_0(d_{R,k})$, $k = 1, \ldots, 2n-1$, *and* $\operatorname{Res} \Pi_0(D_{R,n})$ *are Heisenberg diffeomorphism invariants of* M, *hence their values depend neither on the contact form* θ, *nor on the almost complex structure* J.

2) These noncommutative residues are invariant under deformations of the contact structure.

7. Vanishing of Hirachi's invariant

In this section we give simple algebro-geometric arguments proving that Hirachi's invariant $L(S) = -\frac{1}{2} \operatorname{Res} S_{b;0,0}$ always vanishes on strictly pseudoconvex CR manifolds of dimension $4m+1$. First, we have:

Lemma 7.1. *(i) We have* $\operatorname{Res}\Pi_0(\overline{\partial}_{b;p,q}) = -\operatorname{Res}\Pi_0(\overline{\partial}^*_{b;p,q+2})$ *when the condition* $Y(q+1)$ *holds everywhere.*

(ii) We have $\operatorname{Res}\Pi_0(\overline{\partial}_{b;p,q}) = \operatorname{Res}\Pi_0(\overline{\partial}_{b;p,q+2})$ *when the condition* $Y(q+1)$ *and* $Y(q+2)$ *both hold everywhere.*

(iii) If M *is strictly pseudoconvex and* n *even we have* $\operatorname{Res} S_{b;p,0} = -\operatorname{Res} S_{b;p,n}$.

Proof. Suppose that the condition $Y(q)$ holds everywhere. Then from (4.8) and the fact that Res is a trace vanishing on 1 we get

$$\operatorname{Res}\Pi_0(\overline{\partial}_{b;p,q}) = -\operatorname{Res}(\overline{\partial}^*_{b;p,q+1}N_{b;p,q+1}\overline{\partial}_{b;p,q}) = -\operatorname{Res}(\overline{\partial}_{b;p,q}\overline{\partial}^*_{b;p,q+1}N_{b;p,q+1}), \tag{7.1}$$

$$\operatorname{Res}\Pi_0(\overline{\partial}^*_{b;p,q+2}) = -\operatorname{Res}(\overline{\partial}_{b;p,q+1}N_{b;p,q+1}\overline{\partial}^*_{b;p,q+2}) = -\operatorname{Res}(\overline{\partial}^*_{b;p,q+2}\overline{\partial}_{b;p,q+1}N_{b;p,q+1}). \tag{7.2}$$

Therefore, we see that $\operatorname{Res}\Pi_0(\overline{\partial}_{b;p,q}) + \operatorname{Res}\Pi_0(\overline{\partial}^*_{b;p,q+2})$ is equal to

$$\begin{aligned} -\operatorname{Res}[(\overline{\partial}_{b;p,q}\overline{\partial}^*_{b;p,q+1} + \overline{\partial}^*_{b;p,q+2}\overline{\partial}_{b;p,q+1})N_{b;p,q+1}] &= -\operatorname{Res}(\Box_{b;p,q+1}N_{b;p,q+1}) \\ &= -\operatorname{Res}(1 - S_{b;p,q+1}). \end{aligned} \tag{7.3}$$

Since the condition $Y(q+1)$ holds everywhere the operator $S_{b;p,q+1}$ is smoothing and so we have $\operatorname{Res}(1 - S_{b;p,q+1}) = 0$. It then follows that $\operatorname{Res}\Pi_0(\overline{\partial}_{b;p,q}) = -\operatorname{Res}\Pi_0(\overline{\partial}^*_{b;p,q+2})$.

Assume now that both conditions $Y(q+1)$ and $Y(q+2)$ hold everywhere. Then $S_{b;p,q+2}$ is smoothing and so from (4.7) we get

$$\operatorname{Res}\Pi_0(\overline{\partial}_{b;p,q+2}) + \operatorname{Res}\Pi_0(\overline{\partial}^*_{b;p,q+2}) = \operatorname{Res}(1 + S_{b;p,q+2}) = 0. \tag{7.4}$$

Hence $\operatorname{Res}\Pi_0(\overline{\partial}_{b;p,q+2}) = -\operatorname{Res}\Pi_0(\overline{\partial}^*_{b;p,q+2}) = \operatorname{Res}\Pi_0(\overline{\partial}_{b;p,q})$ as desired.

Finally, suppose that M is strictly pseudoconvex and that n is even. Then the condition $Y(q)$ holds for $q = 1, \ldots, n$. In particular, the conditions $Y(q+1)$ and $Y(q+2)$ hold everywhere simultanously for $q = 0, 2, \ldots, n-4$. Therefore, by the part (ii) we have $\operatorname{Res}\Pi_0(\overline{\partial}_{b;p,0}) = \operatorname{Res}\Pi_0(\overline{\partial}_{b;p,2}) = \ldots = \operatorname{Res}\Pi_0(\overline{\partial}_{b;p,n-2})$. Moreover, as the condition $Y(n-1)$ holds, by the part (i) we have $\operatorname{Res}\Pi_0(\overline{\partial}_{b;p,n-2}) = -\operatorname{Res}\Pi_0(\overline{\partial}^*_{b;p,n})$. Since $S_{b;p,0} = \Pi_0(\overline{\partial}_{b;p,0})$ and $S_{b;p,n} = \Pi_0(\overline{\partial}^*_{b;p,n})$ it follows that $\operatorname{Res} S_{b;p,0} = -\operatorname{Res} S_{b;p,n}$. □

Next, assume M strictly pseudoconvex and let θ be a contact form anihilating $T_{1,0} \oplus T_{0,1}$. Let X_0 be the Reeb vector field of θ so that $\iota_{X_0}\theta = 1$ ad $\iota_{X_0}d\theta = 0$. We endow $T_{\mathbb{C}}M$ with the Levi metric associated to θ, i.e., the Hermitian metric h_θ such that:

- The splitting $T_{\mathbb{C}}M = T_{1,0} \oplus T_{0,1} \oplus \mathbb{C}X_0$ is orthogonal with respect to h_θ;
- h_θ commutes with complex conjugation;
- h_θ agrees with L_θ on $T_{1,0}$ and we have $h_\theta(X_0, X_0) = 1$.

By duality this defines a Hermitian metric on $\Lambda^* T^*_{\mathbb{C}}M$, still denoted h_θ, and there is a uniquely defined Hodge operator $* : \Lambda^{p,q} \to \Lambda^{n-q,n-p}$ such that

$$\alpha \wedge \overline{*\beta} = h_\theta(\alpha, \beta) d\theta^n \qquad \forall \alpha, \beta \in C^\infty(M, \Lambda^{p,q}). \tag{7.5}$$

The operator $*$ is unitary and satisfies $*^2 = (-1)^{p+q}$ on $\Lambda^{p,q}$. Moreover, we have

$$\overline{\partial}^*_{b;p,q} = - * \partial_{b;n-q,n-p} *, \qquad \partial^*_{b;p,q} = - * \overline{\partial}_{b;n-q,n-p} * . \tag{7.6}$$

Lemma 7.2. *Assume that* M *is strictly pseudoconvex. Then we have* $\operatorname{Res} S_{b;0,0} = \operatorname{Res} S_{b;0,n}$.

Proof. First, let $\Pi_0(\partial_{b;0,0})$ be the orthogonal projection onto the kernel of $\partial_{b;0,0}$. Notice that the operators $\overline{\partial}_b$ and ∂_b in (4.4) are complex conjugates of each other, i.e., we have

$$\overline{\partial}_{b;p,q}\alpha = \overline{\partial_{b;p,q}\overline{\alpha}} \qquad \forall \alpha, \beta \in C^\infty(M, \Lambda^{p,q}). \tag{7.7}$$

Therefore $\Pi_0(\partial_{b;0,0})$ is the complex conjugate of $\Pi_0(\overline{\partial}_{b;0,0}) = S_{b;0,0}$. As $\overline{S_{b;0,0}} = (S^*_{b;0,0})^t = S^t_{b;0,0}$ and by the results of [Po5] we have $c_{S^t_{b;0,0}}(x) = c_{S_{b;0,0}}(x)$, we see that the densities $c_{\Pi_0(\partial_{b;0,0})}(x)$ and $c_{S_{b;0,0}}(x)$ agree.

On the other hand, let $\Pi_0(\overline{\partial}^*_{b;n,n})$ denote the orthogonal projection onto $\ker \overline{\partial}^*_{b;n,n}$. Since by (7.6) we have $\overline{\partial}^*_{b;n,n} = - * \partial_{b;0,0} *$ we see that $\Pi_0(\overline{\partial}^*_{b;n,n}) = *\Pi_0(\partial_{b;0,0})*$. As $*^2 = 1$ on $\Lambda^{n,n}$ we get $\operatorname{tr}_{\Lambda^{n,n}} c_{\Pi_0(\overline{\partial}^*_{b;n,n})}(x) = \operatorname{tr}_{\Lambda^{n,n}}[*c_{\Pi_0(\partial_{b;0,0})}(x)*] = c_{\Pi_0(\partial_{b;0,0})}(x)$. Therefore, we have $\operatorname{tr}_{\Lambda^{n,n}} c_{\Pi_0(\overline{\partial}^*_{b;n,n})}(x) = c_{S_{b;0,0}}(x)$, so that $\operatorname{Res}\Pi_0(\overline{\partial}^*_{b;n,n}) = \operatorname{Res} S_{b;0,0}$.

Next, let $Z_1, \ldots, Z_n$ be a local orthonormal frame of $T_{1,0}$. Then $\{X_0, Z_j, Z_{\bar{j}}\}$ is an orthonormal frame of $T_{\mathbb{C}}M$. Let $\{\theta, \theta^j, \theta^{\bar{j}}\}$ be the dual coframe on $T^*_{\mathbb{C}}M$. Let $\zeta = \theta^1 \wedge \ldots \wedge \theta^n$ and let $\tau(\eta) = \zeta \wedge \eta$. Then τ is a a (locally defined) vector bundle isomorphism from $\Lambda^{0,n}$ onto $\Lambda^{n,n}$ and we have

$$\overline{\partial}_b(\zeta \wedge \eta) = (\overline{\partial}_b\zeta) \wedge \eta + (-1)^n \zeta \wedge \overline{\partial}_b \eta, \tag{7.8}$$

$$\overline{\partial}_b\zeta = \sum_{1\leq j\leq n} (-1)^{j-1}\theta^1 \wedge \ldots \wedge \theta^{j-1} \wedge (\overline{\partial}_b\theta^j) \wedge \theta^{j+1} \wedge \ldots . \wedge \theta^n. \tag{7.9}$$

Let $\omega \in C^\infty(M, T^*M \otimes \operatorname{End}(T^*_{\mathbb{C}}M))$ be the connection 1-form of the Tanaka-Webster connection (see [Ta], [We]). Thus, if we let $\omega^j_k = h_\theta(\omega(Z_j), Z_k)$ then $d\theta^j = \theta^k \wedge \omega^j_k \mod \theta \wedge T^*M$. Let $\omega^{0,1}$ and $\omega^{j,0,1}_k$ be the respective $(0,1)$-components of ω and ω^j_k. Then we have $\overline{\partial}_b\theta^j = \theta^k \wedge \omega^{j,0,1}_k$. Combining this with (7.9) then gives

$$\overline{\partial}_b\zeta = \sum_{1\leq j\leq n} (-1)^{j-1}\theta^1 \wedge \ldots \wedge \theta^{j-1} \wedge \theta^j \wedge \omega^{j,0,1}_j \wedge \theta^{j+1} \wedge \ldots \wedge \theta^n = -\operatorname{Tr}\omega^{0,1} \wedge \zeta. \tag{7.10}$$

Hence $\overline{\partial}_b(\zeta \wedge \eta) = (-1)^n \zeta \wedge \overline{\partial}_b\eta - \operatorname{Tr}\omega^{0,1} \wedge \zeta \wedge \eta$. Thus,

$$\tau^{-1}\overline{\partial}_{b;n,q}\tau = D_{b;0,q}, \qquad D_{b;0,q}\eta = (-1)^n\overline{\partial}_{b;0,q}\eta - \operatorname{Tr}\omega^{0,1} \wedge \eta. \tag{7.11}$$

As τ is a unitary isomorphism we also have $\tau^{-1}\overline{\partial}^*_{b;n,q}\tau = D^*_{b;0,q}$. Using (4.8) we then deduce that $\tau^{-1}\Pi_0(\overline{\partial}^*_{b;n,n})\tau$ agrees with the orthogonal projection $\Pi_0(D_{b;0,n})$ onto $\ker D_{b;0,n}$. Therefore, the density $\operatorname{tr}_{\Lambda^{0,n}} c_{\Pi_0(D_{b;0,n})}(x)$ is equal to

$$\operatorname{tr}_{\Lambda^{0,n}}[\tau(x) c_{\Pi_0(\overline{\partial}^*_{b;n,n})} \tau(x)^{-1}] = \operatorname{tr}_{\Lambda^{n,n}} c_{\Pi_0(\overline{\partial}^*_{b;n,n})}(x). \tag{7.12}$$

Hence $\operatorname{Res}\Pi_0(D_{b;0,n}) = \operatorname{Res}\Pi_0(\overline{\partial}^*_{b;n,n})$.

On the other hand, it also follows from (4.8) that $\Pi_0(D_{b;0,n})$ and $\Pi_0(\overline{\partial}^*_{b;0,n})$ have same principal symbol, so by [Po4, Prop. 3.7] their noncommutative residues agree. Hence $\operatorname{Res}\Pi_0(\overline{\partial}^*_{b;n,n}) = \operatorname{Res}\Pi_0(\overline{\partial}^*_{b;0,n})$. As $S_{b;0,n} = \Pi_0(\overline{\partial}^*_{b;0,n})$ and we have shown above that $\operatorname{Res}\Pi_0(\overline{\partial}^*_{b;n,n}) = \operatorname{Res} S_{b;0,0}$, we see that $\operatorname{Res} S_{b;0,n} = \operatorname{Res} S_{b;0,0}$. The lemma is thus proved. □

We are now ready to prove:

Proposition 7.3. *The Hirachi invariant vanishes on strictly pseudoconvex CR manifolds of dimension* $4m + 1$.

Proof. Let M be strictly pseudoconvex CR manifolds of dimension $4m + 1$. By Lemma 7.1 we have $\operatorname{Res} S_{b;0,0} = -\operatorname{Res} S_{b;2m,2m}$ and by Lemma 7.2 we have $\operatorname{Res} S_{b;0,0} = \operatorname{Res} S_{b;2m,2m}$, so $\operatorname{Res} S_{b;0,0} = 0$. Hence the result. □

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