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<th>Title</th>
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</thead>
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Projective embeddings and Lagrangian fibrations of Kummer varieties

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1 Introduction

Let $(X, \omega)$ be a compact Kähler manifold and $(L, h) \rightarrow X$ a Hermitian line bundle with $c_1(M, h) = \omega$. Then for sufficiently large integer $k$, $X$ can be embedded into a projective space by basis $s_0, \ldots, s_{N_k}$ of $H^0(X, L^k)$:

$$\iota_k : X \hookrightarrow \mathbb{C}P^{N_k} = \mathbb{P}H^0(X, L^k)^*.$$

Set $\omega_k = \frac{1}{k} \iota_k^* \omega_{\text{FS}}$, where $\omega_{\text{FS}}$ is the Fubini-Study metric on $\mathbb{C}P^{N_k}$. Then Tian [10] and Zelditch [12] proved that $\omega_k$ converge to $\omega$ under appropriate choices of basis of $H^0(X, L^k)$. More precisely,

**Theorem 1.1 (Zelditch [12]).** Suppose that the basis $s_0, \ldots, s_{N_k} \in H^0(X, L^k)$ are orthonormal with respect to the $L^2$-inner product for each $k \gg 1$. Then there exist constants $C_\eta > 0$ independent of $k$ such that

$$\|\omega - \omega_k\|_{C^\eta} \leq \frac{C_\eta}{k}.$$ 

In this article, we study asymptotic behavior of projective embeddings and the amoebas of abelian varieties and Kummer varieties. We can think of this as a Lagrangian fibration version of the above theorem.

We consider a natural torus action on $\mathbb{C}P^{N_k}$. Then we have a moment map

$$\mu_k : \mathbb{C}P^{N_k} \rightarrow \Delta_k \subset \text{Lie}(T^{N_k})^*$$

of the $T^{N_k}$-action. Note that $\mu_k$ is a Lagrangian fibration of $\mathbb{C}P^{N_k}$ with respect to the Fubini-Study metric $\omega_{\text{FS}}$. We denote $B_k = \mu_k(\iota_k(X))$. $B_k$ is called a compactified amoeba. We are interested in the asymptotic behavior of the restriction $\pi_k : X \rightarrow B_k$ of the moment map $\mu_k$. Amoebas heavily depend on the choice of projective embeddings. Thus the choice of basis of holomorphic sections is an important problem. Of course, there is not a natural choice of basis in general. However, we have some natural choice of basis in special cases.

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such as the case of toric varieties and abelian varieties. In these cases, the basis are related to Lagrangian fibrations \(\pi : (X, \omega) \rightarrow B\) of \(X\). We compare \(\pi\) and \(\pi_k\).

First we consider the simplest case, i.e. the case of toric varieties. Let \((X, L)\) be a polarized toric variety. In this case, \(H^0(X, L^k)\) is spanned by (Laurent) monomials \(z^I = z_1^{i_1} \cdots z_n^{i_n}\). Let \(\pi : X \rightarrow \Delta\) be a moment map of a natural torus action, where \(\Delta\) is the moment polytope of \(X\). Then each monomial corresponds to a lattice point in \(\Delta\):

\[
I = (i_1, \ldots, i_n) \in k\Delta \cap \mathbb{Z}^n \mapsto z^I \in H^0(X, L^k).
\]

We consider the projective embedding \(\iota_k : X \hookrightarrow \mathbb{C}P^{N_k}\) defined by the monomials. Then \(\pi_k : X \rightarrow \Delta_k\) is invariant under the \(T^n\)-action. Hence we have the following commutative diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{\iota_k} & \mathbb{C}P^{N_k} \\
\pi \downarrow & & \downarrow \pi_k \\
\Delta & \xrightarrow{\mu_k} & \Delta_k
\end{array}
\]

In particular, \(B_k\) is the image of the \(n\)-dimensional polytope \(\Delta\).

**Remark 1.2.** Note that \(\dim_{\mathbb{R}} B_k = 2n = \dim_{\mathbb{R}} X\) in general.

The case of abelian varieties is less trivial. Let \(A = \mathbb{C}^n / \mathbb{Z}^n + \mathbb{Z}^n\) be an abelian variety and \(L \rightarrow A\) a principally polarization. Then holomorphic sections of \(L^k\) are essentially given by the theta functions. There are some natural choices of basis of theta functions. For example,

\[
\vartheta \begin{bmatrix} 0 \\ -b \end{bmatrix}(k^{-1}\Omega, z), \quad b \in \frac{1}{k}\mathbb{Z}^n / \mathbb{Z}^n
\]

give a basis of \(H^0(A, L^k)\), where

\[
\vartheta \begin{bmatrix} a \\ b \end{bmatrix}(\Omega, z) = \sum_{i \in \mathbb{Z}^n} \exp \left( \pi \sqrt{-1}^i (l + a)\Omega(l + a) + 2\pi \sqrt{-1}^i (l + a)(z + b) \right).
\]

In particular, we have the following isomorphism

\[
H^0(A, L^k) \cong \bigoplus_{b \in \frac{1}{k}\mathbb{Z}^n / \mathbb{Z}^n} \mathbb{C} \cdot b.
\]

This isomorphism can be given by the Lagrangian fibration

\[
\pi : A \longrightarrow T^n, \quad z = \Omega x + y \longmapsto y,
\]

and this is interpreted in terms of geometric quantization ([11]) or mirror symmetry ([7], [4]). We consider the projective embeddings defined by the above basis:

\[
i_k : A \hookrightarrow \mathbb{C}P^{k^n - 1}, \quad z \mapsto \left( \vartheta \begin{bmatrix} 0 \\ -b_1 \\ \vdots \\ -b_k^n \end{bmatrix}(k^{-1}\Omega, z) : \cdots : \vartheta \begin{bmatrix} 0 \end{bmatrix}(k^{-1}\Omega, z) \right).
\]
In this case, the restriction

\[ \pi_k = \mu_k \circ \iota_k : A \rightarrow B_k \]

is not the same as the Lagrangian fibration \( \pi \). However, we can easily see that \( \pi_k \) is invariant under the translations

\[ \Omega x + y \mapsto \Omega(x + a) + y, \quad a \in \frac{1}{k} \mathbb{Z}^n / \mathbb{Z}^n. \]

Therefore, \( \pi_k \) looks "close" to \( \pi \) for large \( k \). In fact, this can be justified by using the notion of Gromov-Hausdorff distance.

We discuss this more precisely in the next section. The case of Kummer varieties is discussed in Section 3.

2 The case of abelian varieties

Let \( A = \mathbb{C}^n / \Omega \mathbb{Z}^n + \mathbb{Z}^n \) be an \( n \)-dimensional abelian variety as in the previous section. We take a principal polarization \( L \rightarrow A \) defined by

\[ L = (\mathbb{C}^n \times \mathbb{C}) / \sim, \]

where

\[ (z, \zeta) \sim (z + \lambda, e^{\pi i \lambda (\text{Im } \Omega)^{-1} z + \frac{\pi}{2} \lambda (\text{Im } \Omega)^{-1} \lambda} \zeta) \]

for \( \lambda \in \Omega \mathbb{Z}^n + \mathbb{Z}^n \). Then \( L \) is symmetric, i.e.

\[ (-1)^{A} L \cong L, \]

where

\[ (-1)^{A} : A \rightarrow A, \quad z \mapsto -z \]

is the inverse morphism.

Remark 2.1. The choice of \( L \) is not essential. In fact, any other principal polarization can be obtained as a pull-back of \( L \) by some translation. The symmetricity condition is important when we deal with the case of Kummer varieties.

Let \( \omega_0 \) be the flat Kähler metric in the class \( c_1(L) \) and fix a Hermitian metric \( h_0 \) of \( L \) such that \( c_1(L, h_0) = \omega_0 \).

Let \( T^f \) and \( T^b \) be \( n \)-dimensional tori \( \mathbb{R}^n / \mathbb{Z}^n \) and identify

\[ A \cong T^f \times T^b, \quad \Omega x + y \leftrightarrow (x, y). \]

Then the natural projection

\[ \pi : A \rightarrow T^b, \quad \Omega x + y \mapsto y \]

is a Lagrangian fibration with respect to \( \omega_0 \).
We denote the subgroups of $T^b$ of $k$-torsion points by
\[ T_k^b = \frac{1}{k} \mathbb{Z}^n / \mathbb{Z}^n = \{ b_i \}_{i=1,\ldots,k^n} \subset T^b. \]

Then
\[ s_{b_i} = C k^{-\frac{n}{4}} \exp\left( \frac{\pi}{2} k^t z(\text{Im} \Omega) z \right) \theta(k^{-1} \Omega, z), \quad i = 1, \ldots, k^n \]
give a basis of $H^0(A, L^k)$, where $C$ is a constant determined by $\Omega$ and $h_0$. It is known that $s_{b_i}$ has a peak along the fiber $\pi^{-1}(b_i)$. An important property for our purpose is the following:

**Proposition 2.2.** $s_1, \ldots, s_{k^n}$ are orthonormal basis of $H^0(X, L^k)$ with respect to the $L^2$-inner product.

We consider the projective embedding defined by these theta functions
\[ \iota_k : A \rightarrow \mathbb{C}P^{k^n-1}, \quad z \mapsto \left( \theta \left[ \begin{array}{c} 0 \\ -b_1 \end{array} \right] (k^{-1} \Omega, z) : \cdots : \theta \left[ \begin{array}{c} 0 \\ -b_{k^n} \end{array} \right] (k^{-1} \Omega, z) \right). \]

The moment map of $\mathbb{C}P^{k^n-1}$ is given by
\[ \mu_k : (Z^1 : \cdots : Z^{k^n}) \mapsto \frac{1}{\sum |Z^i|^2} \left( |Z^1|^2, \ldots, |Z^{k^n}|^2 \right), \]
where $(Z^1 : \cdots : Z^{k^n})$ is the homogeneous coordinate of $\mathbb{C}P^{k^n-1}$. We set
\[ B_k := \mu_k(\iota_k(X)), \quad \pi_k := \mu_k \circ \iota_k : A \rightarrow B_k \]
as before. We also denote the restriction of the Fubini-Study metric to $X$ by
\[ \omega_k := \frac{1}{k} \iota_k^* \omega_{FS}, \]
here we normalize the Fubini-Study metric in order to $\omega_k$ represents $c_1(L)$.

We compare $\pi : (A, \omega_0) \rightarrow T^b$ and $\pi_k : (A, \omega_k) \rightarrow B_k$ as maps between metric spaces. For that purpose, we need to define distances on $T^b$ and $B_k$. We define a metric on $T^b$ in such a way that $\pi : (A, \omega_0) \rightarrow T^b$ is a Riemannian submersion. The distance on $B_k$ is induced from a metric on the moment polytope $\Delta_k$. The metric on $\Delta_k$ is also defined in such a way that
\[ \mu_k : (\mathbb{C}P^{N_k}, \frac{1}{k} \omega_{FS}) \rightarrow \Delta_k \]
is a Riemannian submersion in the interior of $\Delta_k$.

**Theorem 2.3 ([5]).** $\pi_k : (A, \omega_k) \rightarrow B_k$ converge to $\pi : (A, \omega) \rightarrow T^b$ in the following sense.
\((1)\) \(\omega_k\) converge to \(\omega\) in \(C^\infty\) as \(k \to \infty\). In particular, the sequence \(\{(A, \omega_k)\}\) of Riemannian manifolds converges to \((A, \omega_0)\) with respect to the Gromov-Hausdorff distance.

\((2)\) \(B_k\) converge to \(T^b\) as \(k \to \infty\) with respect to the Gromov-Hausdorff distance.

\((3)\) \(\{\pi_k\}\) converges to \(\pi\) as maps between metric spaces.

Before the proof, we recall the notion of Gromov-Hausdorff convergence and convergence of maps.

First we recall the definition of Hausdorff distance. Let \(Z\) be a metric space and \(X, Y \subset Z\) be two subsets. We denote the \(\epsilon\)-neighborhood of \(X\) in \(Z\) by \(B(X, \epsilon)\). Then the Hausdorff distance between \(X\) and \(Y\) is given by

\[
d_{\text{H}}^Z(X, Y) = \inf \left\{ \epsilon > 0 \mid X \subset B(Y, \epsilon), Y \subset B(X, \epsilon) \right\}.
\]

For metric spaces \(X\) and \(Y\), the Gromov-Hausdorff distance is defined by

\[
d_{\text{GH}}(X, Y) = \inf \{d_{\text{H}}^Z(X, Y) \mid X, Y \leftrightarrow Z\ \text{are isometric embeddings.}\}.
\]

Next we recall the notion of convergence of maps (see also [6]). Let \(f_k : X_k \to Y_k, f : X \to Y\) be maps between metric spaces. Suppose that \(X_k\) and \(Y_k\) converge to \(X\) and \(Y\) respectively with respect to the Gromov-Hausdorff distance. Then by definition, there exist isometric embeddings \(X, X_k \hookrightarrow Z\) and \(Y, Y_k \hookrightarrow W\) into some metric spaces such that \(X_i\) (resp. \(Y_i\)) converge to \(X\) (resp. \(Y\)) with respect to the Hausdorff topology in \(Z\) (resp. \(W\)). We say that \(\{f_i\}\) converges to \(f\) if for every sequence \(x_k \in X_k\) converging to \(x \in X\), \(f_k(x_k)\) converges to \(f(x)\) in \(W\).

**Outline of the proof**

(1) is a direct consequence of Theorem 1.1 and Proposition 2.2.

(2) Decompose \(T\mathbb{C}\mathbb{P}^{n_k}\) into horizontal and vertical parts:

\[
T_p\mathbb{C}\mathbb{P}^{n_k} = T_{\mathbb{C}\mathbb{P}^{n_k}/\Delta_k, p} \oplus (T_{\mathbb{C}\mathbb{P}^{n_k}/\Delta_k, p})^\perp
\]

where \(T_{\mathbb{C}\mathbb{P}^{n_k}/\Delta_k, p} = \ker d\mu_k\) is the tangent space to the fiber of \(\mu_k\) and \((T_{\mathbb{C}\mathbb{P}^{n_k}/\Delta_k, p})^\perp\) is the orthogonal complement with respect to the Fubini-Study metric. Similarly we decompose the tangent space of \(A\):

\[
T_zA = T_{A/T^{b}, z} \oplus (T_{A/T^{b}, z})^\perp,
\]

where \((T_{A/T^{b}, z})^\perp\) is the orthogonal complement of \(T_{A/T^{b}, z} = \ker d\pi\) with respect to the flat metric \(\omega_0\). Then the metrics on \(\Delta_k\) and \(T^b\) are given by the restriction of \(\omega_k\) and \(\omega_0\) on the horizontal subspaces respectively. Therefore we need to compare two horizontal and vertical spaces. We can prove that these two decompositions are close in the following sense:
Lemma 2.4. (1) If $\xi \in T_{A/T^{b},z}$, then
\[ |d_{\iota_{k}}(\xi)^{H}| \leq \frac{C}{\sqrt{k}}|\xi|. \]

(2) If $\eta \in (T_{A/T^{b},z})^\perp$, then
\[ |d_{\iota_{k}}(\eta)^{V}| \leq \frac{C}{\sqrt{k}}|\eta|. \]

This lemma follows from the asymptotic behavior of the theta functions. By using the above estimates, we have
\[ d_{GH}(T^{b}, B_{k}) \leq \frac{C}{\sqrt{k}}. \]

In fact, we can show that the composition
\[ \varphi_{k} = \pi_{k} \circ \sigma_{0} : T^{b} \rightarrow B_{k} \]

of the zero section $\sigma_{0} : T^{b} \rightarrow A$ and $\pi_{k}$ is "almost isometric" (a $\frac{C}{\sqrt{k}}$-Hausdorff approximation (see [3] for the definition)).

3 The case of Kummer varieties

Let $(A, L)$ be a polarized abelian variety as in the previous section. The Kummer variety of $A$ is defined by
\[ X = A/(-1)_{A}. \]

We take a line bundle $M \rightarrow X$ satisfying
\[ p^{*}M \cong L^{2}, \]

where $p : A \rightarrow X$ is the natural projection. From the fact that $p^{*} : \text{Pic}(X) \rightarrow \text{Pic}(A)$ is injective, we have
\[ p^{*}M^{k} \cong L^{2k}. \]

It is easy to see that $p^{*} : H^{0}(X, M^{k}) \rightarrow H^{0}(A, L^{2k})$ is injective and the image is spanned by
\[ s_{b_{i}} + s_{-b_{i}}, \quad b_{i} \in T_{2k}^{b} \]

(see [1] and [8]). Note that
\[ N_{k} + 1 = \dim H^{0}(X, M^{k}) = 2^{n-1}(k^{n} + 1). \]

Let $\omega$ be the orbifold Kähler metric induced from the flat metric $2\omega_{0}$ on $A$. Then $[\omega] = c_{1}(M).$ We also have a Lagrangian fibration
\[ \pi : (X, \omega) \rightarrow B = T^{b}/(-1) \]
induced by $\pi: A \to T^b$. We set
\[
t_i = \begin{cases} 
\frac{1}{\sqrt{2^n}}(s_{b_i} + s_{-b_i}), & \text{if } b_i \in T_{2k}^b \setminus T_2^b, \\
\frac{1}{\sqrt{2^{n-1}}}s_{b_i}, & \text{if } b_i \in T_2^b.
\end{cases}
\]
Then $\{t_i\}$ is an orthonormal basis of $H^0(X, M^k)$.

We denote by $\iota_k: X \to \mathbb{C}\mathbb{P}^{N_k}$ the projective embedding defined by $\{t_i\}$, $\pi_k: X \to B_k$ the restriction of the moment map, and $\omega_k = \frac{1}{k}\iota_k^*\omega_{\text{FS}}$ as before. Then the same theorem holds for $X$ as well.

**Theorem 3.1.** (1) $\{(X, \omega_k)\}$ converges to $(X, \omega)$ with respect to the Gromov-Hausdorff distance.

(2) $B_k$ converge to $B$ with respect to the Gromov-Hausdorff distance.

(3) $\{\pi_k\}$ converges to $\pi$ as maps between metric spaces.

**Outline of the proof**

(1) follows from the fact that $\{t_i\}$ are orthonormal and an orbifold version of Theorem 1.1:

**Theorem 3.2 (Song [9], Dai-Liu-Ma [2]).** Let $(X, \omega)$ be a compact Kähler orbifold of dimension $n \geq 2$ with only finite isolated singularities $\text{Sing}(X) = \{e_j\}_{j=1}^m$ and $(M, h) \to X$ be an orbifold Hermitian line bundle with $c_1(M, h) = \omega$. For $k \gg 1$, we consider the projective embedding $\iota_k: X \to \mathbb{C}\mathbb{P}^{N_k}$ defined by an orthonormal basis. We put $\omega_k = \frac{1}{k}\iota_k^*\omega_{\text{FS}}$ as before. Then
\[
\|\omega - \omega_k\|_{C^q, z} \leq C_q \left( \frac{1}{k} + k^\frac{3}{2}e^{-k\delta r(z)^2} \right),
\]
where $\|\cdot\|_{C^q, z}$ is the $C^q$-norm at $z \in X$ and $r(z)$ is the distance between $z$ and the singular set $\{e_j\}$.

(2) What we must take care of is the existence of singular fibers. For each $b \in \text{Sing}(B) = T_2^b/(\{-1\})$, we denote the $\sqrt{\frac{\log k}{\delta k}}$-neighborhood of the singular fiber $\pi^{-1}(b)$ by
\[
N_{b,k} = \left\{ z \in X \mid d(z, \pi^{-1}(b)) \leq \sqrt{\frac{\log k}{\delta k}} \right\}
\]
and put
\[
X(k) = X \setminus \bigcup_{b \in \text{Sing}(B)} N_{b,k}.
\]
Then we can show that $\pi(N_{b,k})$ and $\pi_k(N_{b,k})$ are small for large $k$ (in fact, their diameters can be bounded by $O\left(\sqrt{\frac{\log k}{k}}\right)$). Therefore we may “ignore” these parts. On the other hand, we have the same estimates as in Lemma 2.4 on $X(k)$. Hence we can apply the same arguments to this situation.
References


