Variation of Bergman metrics in different views

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We shall show two views on the variation of Bergman metrics. The one view comes from a quasiconformal deformation of a Riemann surface holomorphically depended and the other view comes from a holomorphic family of Riemann surfaces shaping a pseudoconvex domain.

1 Quasiconformally holomorphic movement of a Riemann surface

Let R_t be a Riemann surface which moves quasiconformally depended on a complex parameter t about 0 in the complex plane. We say R_t has quasiconformally holomorphic movement if the Beltrami differential

$$\mu_t = \mu(z,t)\frac{d\bar{z}}{dz} = \frac{(h_t)_{\bar{z}}d\bar{z}}{(h_t)_z dz}$$

of quasiconfomal mapping

$$h_t : R_0 \longrightarrow R_t \quad (w = h_t(z))$$

satisfies the following conditions:

(i) $\mu(z,0) = 0$, $\mu(z,t)$ is measurable, $esssup_R|\mu(z,t)| < 1$,

(ii) For every t there exist constants ϵ_t and M_t such that

$$|\epsilon| < \epsilon_t \implies esssup_R|\mu(z,t+\epsilon) - \mu(z,t)| < |\epsilon|M_t,$$

(iii) For almost all $z \in R_0$ $\mu(z,t)$ is holomorphic
    with respect to t.

2   holomorphic family

Let

$$\pi : S \longrightarrow B$$

be a holomorphic family of a Riemann surfaces

$$R_t = \pi^{-1}(t), t \in B,$$

where S is a 2-dimensional analytic space, B is a disk in $\mathbb{C}$ and R_t is irreducible.

Suppose that S is unramified domain over $B \times \mathbb{C}$ with smooth boundary.

Let a defining function

$$\Phi(t,z)$$

be a real valued C^2 function in a neighborhood of ∂S such that it is positive inside of S, negative outside of S and $\dfrac{\partial \Phi(t,z)}{\partial z}$ doesn't vanish on the boundary of S.

For $(t,z) \in \partial S$ set

$$k_1(t,z) = \frac{\partial \Phi(t,z)}{\partial t} / |\frac{\partial \Phi(t,z)}{\partial z}|,$$

$$k_2(t,z) = \frac{\partial^2\Phi}{\partial t\partial\bar{t}}|\frac{\partial\Phi}{\partial z}|^2 - 2Re\{\frac{\partial^2\Phi}{\partial\bar{t}\partial z}\frac{\partial\Phi}{\partial t}\frac{\partial\Phi}{\partial\bar{z}}\} + |\frac{\partial\Phi}{\partial t}|^2\frac{\partial^2\Phi}{\partial z\partial\bar{z}}/|\frac{\partial\Phi}{\partial z}|^3.$$

These do not depend on the choice of defining function $\Phi(t,z)$. If S is pseudoconvex, then $k_2(t,z) \geq 0$.

3 Behavior space

Let

$$\Lambda = \Lambda(R)$$

be a real Hilbert space which consists of square integrable complex differentials on a Riemann surface R. Its inner product is given as follows:

$$< \omega, \sigma > = Real\ part\ of\ \iint_R \omega \wedge \ ^*\bar{\sigma} = \Re(\omega, \sigma),$$

where $\bar{\sigma}$ is the complex conjugate differential of σ, $^*\bar{\sigma}$ is the conjugate differential of $\bar{\sigma}$, (ω, σ) denotes the integral itself of above second expression which means the complex inner product and $\Re(\omega, \sigma)$ means its real part. Typical subspaces of Λ are the following:

$\Lambda_h = \{\lambda \in \Lambda : \lambda \text{ is harmonic}\}$,

$\Lambda_{eo} = \{\lambda \in \Lambda : \lambda \text{ is orthogonal to } \Lambda_h \text{ and closed differential}\}$,

$\Lambda_{hse} = \{\lambda \in \Lambda_h : \int_\gamma \lambda = 0 \text{ for any dividing cycle } \gamma\}$,

$\Lambda_{he} = \{\lambda \in \Lambda_h : \lambda \text{ is exact}\}$, $\Lambda_{ho} = \ ^*\Lambda_{he}^\perp$, $\Lambda_{hm} = \ ^*\Lambda_{hse}^\perp$,

where $\Lambda_\chi^\perp$ is the orthogonal complement of Λ_χ in Λ_h and $^*\Lambda_\chi = \{\ ^*\omega : \omega \in \Lambda_\chi\}$. Note the following orthogonal decompositions:

$$\Lambda_h = \Lambda_{he} \oplus\ ^*\Lambda_{ho} = \Lambda_{hse} \oplus\ ^*\Lambda_{hm} = \ ^*\Lambda_{he} \oplus \Lambda_{ho} = \ ^*\Lambda_{hse} \oplus \Lambda_{hm}.$$

Let

$$\Gamma_h = \{\lambda \in \Lambda_h : \lambda \text{ is real}\}, \ \Gamma_{hse} = \Gamma_h \cap \Lambda_{hse},$$

$$\Gamma_{he} = \Gamma_h \cap \Lambda_{he}, \ \Gamma_{ho} = {}^*\Gamma_{he}^{\perp}, \ \Gamma_{hm} = {}^*\Gamma_{hse}^{\perp},$$

where $\Gamma_\chi^\perp$ is the orthogonal complement of Γ_χ in Γ_h and ${}^*\Gamma_\chi = \{{}^*\omega; \omega \in \Gamma_\chi\}$.

We say a subspace

$$\Lambda_x = \Gamma_x + i\,{}^*\Gamma_x^\perp$$

behavior space, where Γ_x is a subspace of Γ_h and ${}^*\Gamma_x^\perp$ consists of conjugate differentials of orthogonal complement of Γ_x in Γ_h. We assume that

$$\Lambda_x(R_t) \circ h_t \subseteq \Lambda_x(R_0) + \Lambda_{eo}(R_0)$$

For example, $\Gamma_{he} + i\Gamma_{ho}$, $\Gamma_{hm} + i\Gamma_{hse}$, $i\Gamma_h$ and Γ_h are behavior spaces.

4 Elementary differentials

Suppose that there is a point p excluded the support of μ_t, take a local disk $V = \{z : |z - \zeta| < 1\}$ around $\zeta = p$ which does not meet the support of μ_t.

There exist meromorphic differentials φ_n^t, ψ_n^t on R_t with pole only at $h_t(p)$ such that

(i) $\varphi_n^t - \dfrac{dz}{(z-\zeta)^{n+1}}, \ \psi_n^t - \dfrac{dz}{(z-\zeta)^{n+1}}$
      is holomorphic on $V_t = h_t(V)$,
      where z is a local variable on V_t.

(ii) φ_n^t (ψ_n^t) coincides with a differential

in $i\Gamma_h + \Lambda_{eo}$ ($\Gamma_h + \Lambda_{eo}$) on $R_t - V_t$.

$\varphi_n^t + \overline{\varphi_n^t}$ and $\psi_n^t - \overline{\psi_n^t}$ coincide with differentials in Λ_{eo}.

Set
$$K_n^t = n!(\varphi_n^t - \psi_n^t)/4\pi = \hat{K}_n^t(z,\zeta)dz,$$
$$L_n^t = n!(\varphi_n^t + \psi_n^t)/4\pi = \hat{L}_n^t(z,\zeta)dz.$$
Let write φ^t for φ_1^t and ψ^t for ψ_1^t.

Theorem 1

$$(\omega, K_{n+1}^t) = \frac{d^n\hat{\omega}}{dz^n}(\zeta)$$

for a square integrable holomorphic differential

$$\omega = \hat{\omega}(z)dz \text{ on } R_t.$$

$K^t = K_1^t$ is called a Bergman kernel and $\hat{K}^t(\zeta,\zeta)d\zeta \wedge d\bar{\zeta}$ is the Bergman metric.

Since $\varphi_n^t + \overline{\varphi_n^t} - \psi_n^t + \overline{\psi_n^t} \in \Lambda_{eo}$ has a singularity $\dfrac{2d\bar{z}}{(z-\zeta)^{n+1}}$,

there exists a potential $P_n^t(z,\zeta)$ with a singularity $\dfrac{-1}{n(z-\zeta)^n}$

such that

$$dP_n^t = \frac{1}{2}(\varphi_n^t - \psi_n^t + \overline{\varphi_n^t} + \overline{\psi_n^t}) = \frac{2\pi}{n!}(K_n^t + \overline{L_n^t}).$$

Let write P^t for P_1^t.

Let $G^t = G^t(z,\zeta)$ be the Green function with pole at $\zeta = h_t(p)$ on R_t. Set $\varphi_0^t = dG^t + i * dG^t$. We say

$$\gamma^t(\zeta) = \frac{1}{2\pi i}\int_{|z-\zeta|=\epsilon} G^t(z,\zeta)\frac{dz}{z-\zeta}$$

a Robin's constant at $h_t(p)$. The following theorem shows that the Bergman metric closely relate to the Robin constant.

Theorem 2 (Suita's theorem)

$$\hat{K}^t(\zeta, \zeta) = -\frac{1}{2\pi}\frac{\partial^2 \gamma^t(\zeta)}{\partial \zeta \partial \bar{\zeta}}.$$

5   Variation of a meromorphic differential

The elementary meromorphic differentials φ_n^t, ψ_n^t may behave smoothly in our circumstance.

Theorem 3

Suppose that R_t has quasiconformally holomorphic movement. Let a meromorphic differential ϕ^t on R_t satisfy that

$$\phi^t \circ h_t - \phi^0 \in \Lambda_x + \Lambda_{eo}$$

where the pole of ϕ^0 is excluded the support of μ_t.

There exist differentials $\phi_u^t, \phi_v^t \in \Lambda_x + \Lambda_{eo}$ on R_t such that

$$\lim_{u \to 0} \left\| \frac{\phi^{t+u} \circ h_{t+u} \circ h_t^{-1} - \phi^t}{u} - \phi_u^t \right\| = 0,$$

$$\lim_{v \to 0} \left\| \frac{\phi^{t+iv} \circ h_{t+iv} \circ h_t^{-1} - \phi^t}{v} - \phi_v^t \right\| = 0.$$

And $\phi_u^t + i\phi_v^t = i * (\phi_u^t + i\phi_v^t)$ is a holomorphic differential.

Set

$$\frac{\partial \phi^t}{\partial t} = \frac{1}{2}(\phi_u^t - i\phi_v^t), \quad \frac{\partial \phi^t}{\partial \bar{t}} = \frac{1}{2}(\phi_u^t + i\phi_v^t).$$

6 Variation of Robin's constant

In the one view we have the first and second variational formulas of Robin's constant.

Theorem 4

Suppose that R_t has quasiconformally holomorphic movement.

$$\frac{\partial \gamma^t(\zeta)}{\partial t} = \frac{1}{4\pi}(\frac{\partial \varphi_0^t}{\partial t}, \overline{\varphi_0^t})$$

$$= \frac{i}{\pi} \iint_{R_0} (\frac{\partial G^t}{\partial w}\frac{\partial w}{\partial z})^2 \frac{\partial \mu}{\partial t}(h_t)_z^2 dz d\bar{z},$$

$$\frac{\partial^2 \gamma^t(\zeta)}{\partial \bar{t} \partial t} = -\frac{1}{2\pi}(\frac{\partial \varphi_0^t}{\partial \bar{t}}, \frac{\partial \varphi_0^t}{\partial \bar{t}})$$

$$= \frac{i}{\pi} \iint_{R_0} (\frac{\partial \varphi_0^t}{\partial \bar{t}}\frac{\partial G^t}{\partial w}\frac{\partial w}{\partial z})^2 \frac{\partial \mu}{\partial t}(h_t)_z^2 dz d\bar{z} \le 0.$$

In the other view they are given by the following forms.

Theorem 5

Suppose that S is unramified domain over $B \times C$ with smooth boundary.

$$\frac{\partial \gamma^t(\zeta)}{\partial t} = \frac{-1}{\pi}\int_{\partial R_t} k_1(t,z)|\frac{\partial G^t(z,\zeta)}{\partial z}|^2 ds_z,$$

$$\frac{\partial^2 \gamma^t(\zeta)}{\partial \bar{t} \partial t} = \frac{-1}{\pi}\int_{\partial R_t} k_2(t,z)|\frac{\partial G^t(z,\zeta)}{\partial z}|^2 ds_z$$

$$- \frac{2i}{\pi} \iint_{R_t} |\frac{\partial^2 G^t(z,\zeta)}{\partial \bar{t} \partial z}|^2 dz d\bar{z}.$$

Hence $\frac{\partial^2 \gamma^t(\zeta)}{\partial \bar{t} \partial t} \le 0$ *if S is pseudoconvex.*

The first formula may be regarded as Hadamard's variational formula.

7 Variation of Bergman metrics

In the one view we have the first and second variational formulas of Bergman metrics.

Theorem 6

Suppose that R_t has quasiconformally holomorphic movement.

$$\frac{\partial \hat{K}^t(\zeta, \zeta)}{\partial t} = (K^t, \frac{\partial K^t}{\partial \bar{t}}),$$

$$\frac{\partial^2 \hat{K}^t(\zeta, \zeta)}{\partial \bar{t} \partial t} = (\frac{\partial K^t}{\partial \bar{t}}, \frac{\partial K^t}{\partial \bar{t}}) + (\frac{\partial L^t}{\partial \bar{t}}, \frac{\partial L^t}{\partial \bar{t}}) \geq 0,$$

$$\frac{\partial^2 \log \hat{K}^t(\zeta, \zeta)}{\partial \bar{t} \partial t} = \frac{1}{\hat{K}^t(\zeta, \zeta)} \{(\frac{\partial K^t}{\partial \bar{t}}, \frac{\partial K^t}{\partial \bar{t}}) + (\frac{\partial L^t}{\partial \bar{t}}, \frac{\partial L^t}{\partial \bar{t}})\}$$

$$- \frac{1}{\hat{K}^t(\zeta, \zeta)^2} |(\frac{\partial K^t}{\partial \bar{t}}, K^t)|^2 \geq 0.$$

In the other view they are given by the following forms.

Theorem 7

Suppose that S is unramified domain over $B \times \mathbf{C}$ with smooth boundary.

$$\frac{\partial \hat{K}^t(\zeta, \zeta)}{\partial t} = -i \iint_{\partial R_t} \hat{K}^t(z, \zeta) \overline{\frac{\partial \hat{K}^t(z, \zeta)}{\partial t}} dz d\bar{z},$$

$$\frac{\partial^2 \hat{K}^t(\zeta, \zeta)}{\partial \bar{t} \partial t} = \frac{1}{2} \int_{\partial R_t} k_2(t, z)(|\hat{K}^t(z, \zeta)|^2 + |\hat{L}^t(z, \zeta)|^2) ds_z$$

$$+ i \iint_{R_t} (|\frac{\partial^2 \hat{K}^t(z, \zeta)}{\partial \bar{t}}|^2 + |\frac{\partial^2 \hat{L}^t(z, \zeta)}{\partial \bar{t}}|^2) dz d\bar{z}.$$

Hence $\dfrac{\partial^2 \hat{K}^t(\zeta, \zeta)}{\partial \bar{t} \partial t} \geq 0$, *if S is pseudoconvex.*

8 Application

Using the above variational formulas we have several applications.

Theorem 8 (Lewittes)

Let R be a non-planar Riemann surface. If the Gaussian curvature of the Bergman metric has zero, then R is an (ultra) hyper-elliptic Riemann surface of parabolic type. Conversely, if R is an (ultra) hyper-elliptic Riemann surface of parabolic type, then the branch points coincide with the zeros of the Gaussian curvature of the Bergman metric.

Theorem 9

Let a compact bordered Riemann surface R_t of genus g with m (> 0) boundary component have a quasiconformally holomorphic movement. If $2g + m + 1$ Robin's constants $\gamma^t(p_i)$ are harmonic with respect to t, all R_t are conformally equivalent.

Theorem 10

Let a compact bordered Riemann surface R_t of finite type have quasiconformally holomorphic movement. If $\log \hat{K}^t(\zeta, \zeta)$ is harmonic with respect to t. All R_t are conformally equivalent.

Theorem 11

Let S be an unramified pseudoconvex domain over $B \times \mathbf{C}$ with smooth boundary. Then $\log \hat{K}^t(\zeta, \zeta)$ is plurisubharmonic on S. Further, if, for each $t \in B$, ∂R has at least one strictly pseudoconvex point $(t, \alpha(t))$, then $\log \hat{K}^t(\zeta, \zeta)$ is a strictly plurisubharmonic function on S.

Theorem 12 *(S.Hamano and H.Yamaguchi)*

Let S be an unramified pseudoconvex domain over $B \times \mathbf{C}$ with smooth boundary. If there exists a holomorphic section $\zeta = \zeta(t)$ on $\mathbf{B}$ of S such that $\log \hat{K}^t(\zeta, \zeta)$ is harmonic on $\mathbf{B}$, then S is biholomorphic to the product $\mathbf{B} \times R_0$ by the transformation of the form $t = t$, $w = f(t, z)$.

9 Variational formulas of Rauch type

Let R_t be compact and represented as a covering surface on $\hat{\mathbf{C}}$. Let $\{\zeta_j(t)\}_{j=1}^{\nu}$ be the branch points of order $\{k_j\}$ and the local parameters $\{w_j = (z - \zeta_j(t))^{1/k_j}\}$.

Theorem 13

$$\frac{\partial \hat{K}^t(\zeta, \zeta)}{\partial t} = \frac{1}{2\pi} \sum_{j=1}^{\nu} \frac{\zeta_j'(t)}{k_j(k_j - 2)!} [\frac{\partial^{k_j-2}}{\partial w_j^{k_j-2}} (\frac{\partial P^t(w_j, \zeta)}{\partial w_j} \frac{\partial \overline{P^t(w_j, \zeta)}}{\partial w_j})]_{w_j=0}$$

$$\frac{\partial^2 \hat{K}^t(\zeta, \zeta)}{\partial \bar{t} \partial t} = \frac{1}{4\pi^2}(\| \sum_{j=1}^{\nu} \frac{\overline{\zeta_j'(t)}}{k_j(k_j-2)!} \frac{\partial \Omega_j^t(z)}{\partial z} dz \|_{R_t}^2$$

$$+ \| \sum_{j=1}^{\nu} \frac{\overline{\zeta_j'(t)}}{k_j(k_j-2)!} \frac{\partial \Xi_j^t(z)}{\partial z} dz \|_{R_t}^2) \geq 0,$$

where

$$\Omega_j^t(z) = [\frac{\partial^{k_j-2}}{\partial \overline{w_j}^{k_j-2}} (\frac{\partial P^t(w_j, \zeta)}{\partial w_j} P^t(z, w_j))]_{w_j=0},$$

$$\Xi_j^t(z) = [\frac{\partial^{k_j-2}}{\partial \overline{w_j}^{k_j-2}} (\frac{\partial \overline{P^t(w_j, \zeta)}}{\partial \overline{w_j}} P^t(z, w_j))]_{w_j=0}.$$

References

[1] J. Lewittes : Differentials and metrics on Riemann surfaces, *Trans. Amer. Math. Soc.,139* 1969, 311-318

[2] N. Suita : On capacities and kernels of Riemann surfaces, *Arch. Rat. Mech. Anal.,46* 1972, 212-217

[3] H. Yamaguchi : Sur le mouvement des constantes de Robin, *J. Math. Kyoto Univ.,15* 1975, 53-71

[4] H. Yamaguchi : Parabolicité d'une fonction entière, *J. Math. Kyoto Univ.,16* 1976, 71-92

[5] H. Yamaguchi : Calcul des variations analytiques, *Jap. J. Math. New Seris.,* 1981, 319-377

[6] F. Maitani : Variation of meromorphic differentials under quasiconformal deformations, *J. Math. Kyoto Univ., 24* 1984, 49-66

[7] F. Maitani : Covering properties of extremal vertical slit mappings, *Kodai Math. J., 11* 1988, 361-371

[8] F. Maitani and H. Yamaguchi : Variation of Bergman metrics on Riemann Surfaces, *Math. Ann., 330* 2004, 477-489

[9] S.Hamano and H. Yamaguchi : A note on Variation of Bergman Metrics on Riemann Surfaces under Pseudoconvexity, *Complex Variables, 49* 2004, 673-679