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ON THE GEOMETRY OF MODULI SPACE OF POLARIZED CALABI-YAU MANIFOLDS

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1. Introduction

Let X be a compact Kähler manifold with zero first Chern class, and let L be an ample line bundle over X. The pair (X, L) is called a polarized Calabi-Yau manifold. By Yau's proof of the Calabi conjecture, we know such a manifold carries a unique Ricci flat metric compatible with the polarization (cf. [36]). Thus, the moduli space of such Ricci flat Kähler metrics is the moduli space of complex structures of (X, L).

By a theorem of Mumford, a Calabi-Yau moduli space (or any coarse moduli space of polarized Kähler manifolds) is a complex variety. In particular, most points of $\mathcal{M}$ are smooth points so that we can do differential geometry on them. Now, in Riemannian geometry, there is a natural metric on any moduli space of metrics, the Weil-Petersson metric, obtained by restriction from the metric on the space of metrics. Quite a lot is known about the local structure of the WP metric on Calabi-Yau moduli space. But much less is known about its global properties.

In this short paper, we study the integrals of the curvature invariants of the Weil-Petersson metric on a Calabi-Yau moduli space. In Theorem 4, we prove that these quantities are all finite. In Theorem 5 and in work to appear [7], we prove that they are rational numbers. Now if the moduli space had been compact, then this would be expected by the theorem of Gauss-Bonnet-Chern. But Calabi-Yau moduli spaces are not compact, making this result nontrivial.

Besides its mathematical interest, the geometry of Calabi-Yau moduli space is very interesting in string theory, and there are various physics arguments [16, 9, 6, 32] suggesting the finiteness of the volume and integrability of the curvature invariants of the Weil-Petersson metric.

Mathematically, this paper is a continuation of the previous works in [19, 21, 20, 22, 23, 14, 15], on the local and global geometry of the moduli space and the BCOV torsion of Calabi-Yau moduli.

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Before finishing this section, we write out explicitly the Calabi-Yau moduli of the most famous Calabi-Yau threefold: the quintic hypersurface in CP^4. Let this be

$$X = \{Z \mid Z_0^5 + \cdots + Z_4^5 + 5\lambda Z_0 \cdots Z_4 = 0\} \subset CP^4.$$

It is a smooth hypersurface if λ is not any of the fifth unit roots. To construct the moduli space, we define

$$V = \{f \mid f \text{ is a homogeneous quintic polynomial of } Z_0, \cdots, Z_4\}.$$

one can verify that $\dim V = 126$. Thus for any $t \in P(V) = CP^{125}$, t is represented by a hypersurface. However, if two hypersurfaces differ by an element in $Aut(CP^4)$, then they are considered the same. Let D be the divisor in CP^{125} characterizing the singular hypersurfaces in CP^4. Then the moduli space of X is

$$\mathcal{M} = CP^{125} \backslash D / Aut(CP^4).$$

The dimension of the moduli space is 101. But other than the dimension, we still know very little about this variety.

The organization of the paper is as follows: in Section 2, we give some physics background of our problems; in Section 3, we define the Weil-Petersson metric; in Section 4, we present the main results of this paper; then we introduce the Hodge metrics in Section 5; in the last section, we prove the main results of this paper.

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2. Physics background

In the original compactifications of heterotic string theory [24], as well as in many later constructions, the universe is a direct product of a $4d$ space-time and a tiny, compact Ricci flat six manifold M. Arguments from supersymmetry, as well as the fact that we know no other examples, suggest that M is a Calabi-Yau manifold. While we do not know which M to choose, we do know how to go from geometric properties of M, together with certain auxiliary data, to statements about observable physics. Then, if a particular choice of M and the auxiliary data implies statements which are in conflict with observation, we know this choice is incorrect. At present it is an open problem to show that any specific choice or "vacuum" is consistent with current observations. Given that such choices exist, we would like to go on to show that the number of vacua is finite, and estimate their number.

Suppose we assume a particular Calabi-Yau M; then the fact that Ricci flat metrics on M come in moduli spaces leads to the existence of approximate solutions, in which the moduli of M are slowly varying in four-dimensional space-time. These lead almost inevitably to corrections to Newton's (and Einstein's) laws of gravity which contradict observation, and thus we must somehow modify the construction by postulating additional background fields to remove these moduli.

One way to do this is flux compactification, described in [8] and references there. This construction picks out special points in moduli space, the flux vacua, and thus part of counting vacua is to count these points.

In their studies of type IIb flux compactification, the first author and his collaborators (cf. [1, 10, 11, 12, 13]) derived an asymptotic formula for the number of flux vacua, [13, Theorem 1.8]. It is a product of a coefficient determined by topological data of M, with an integral of a curvature invariant derived from the Weil-Petersson metric over the moduli space. Thus, the finiteness of such integrals implies the finiteness of the number of flux vacua (up to certain caveats explained in [13]), which was a primary motivation for us to write this paper.

We are also very interested to the duality between of the special Kähler manifolds and the Calabi-Yau moduli. In the proof of the finiteness of the volume of Calabi-Yau moduli [22] and in the proof of the incompleteness of special Kähler manifold [18], we use the generalized maximal principal. It would be interesting to answer the following questions: Are the volume of projective special Kähler manifolds finite? Are the Calabi-Yau moduli always incomplete with respect to the Weil-Petersson metric? We hope that not only one can answer these questions but also we can find relations of these two problems.

3. Weil-Petersson geometry

Let $\mathcal{M}$ be the moduli space of a polarized Calabi-Yau manifold of dimension n. Let $0 \subset F^n \subset F^{n-1} \subset \cdots \subset F^1 \subset F^0 = H$ be the Hodge bundles over $\mathcal{M}$. Since each point of $\mathcal{M}$ is represented by a Calabi-Yau manifold, the rank of F^n is 1. A natural Hermitian metric on F^n is given by the second Hodge-Riemann relation:

$$C \int_{X_t} \Omega \wedge \bar{\Omega} > 0,$$

where C is suitable constant and Ω is a nonzero $(n,0)$ form of $X_t, t \in \mathcal{M}$. By a theorem of Tian [30], we know that the curvature of the above Hermitian metric is positive, and the Weil-Petersson metric is equal to the curvature of F^n. Thus we can *define* the Weil-Petersson metric whose Kähler form is the curvature form of the line bundle. Let the Kähler form of the Weil-Petersson metric be ω_{WP}, then we have

$$\omega_{WP} = -\sqrt{-1}\partial\bar{\partial} \log \int_{X_t} \Omega \wedge \bar{\Omega},$$

where Ω is a local holomorphic section of the bundle F^n.

The Weil-Petersson geometry is composed of the moduli space $\mathcal{M}$, the Hodge bundles $F^k, k = 0, \cdots, n$, and the Weil-Petersson metric ω_{WP}. In order to understand the geometry of the moduli space, we need to study the curvature and the asymptotic behavior of the Weil-Petersson metric. Let $(\,,\,)$ be the quadratic form on H defined by the cup product. The quadratic form is nondegenerate but not positive definite. Let

$$F^k = H^{n,0} \oplus \cdots \oplus H^{k,n-k}$$

be the orthogonal splitting with respect to the quadratic form (,). Let $(\frac{\partial}{\partial t_1}, \cdots, \frac{\partial}{\partial t_m})$ be a local holomorphic frame near a smooth point x of $\mathcal{M}$. Define $\nabla_i\Omega$ to be the $H^{n-1,1}$ part of $\partial_i\Omega = \frac{\partial\Omega}{\partial t_i}$ and $\nabla_j\nabla_i\Omega$ to be the $H^{n-2,2}$ part of $\partial_j\partial_i\Omega$ or $\partial_j\nabla_i\Omega$. Then we have the following result:

Theorem 1. *The curvature tensor $R_{\alpha\bar\beta\gamma\bar\delta}$ of the Weil-Petersson metric is*

$$R_{\alpha\bar\beta\gamma\bar\delta} = g_{\alpha\bar\beta}g_{\gamma\bar\delta} + g_{\alpha\bar\delta}g_{\gamma\bar\beta} - \frac{(\nabla_\alpha\nabla_\gamma\Omega, \overline{\nabla}_\beta\overline{\nabla}_\delta\Omega)}{(\Omega,\bar\Omega)}. \tag{1}$$

□

If $n = 3$, then we have

$$\frac{(\nabla_\alpha\nabla_\gamma\Omega, \overline{\nabla}_\beta\overline{\nabla}_\delta\Omega)}{(\Omega,\bar\Omega)} = F_{\alpha\gamma m}\overline{F_{\beta\delta n}}g^{m\bar n}/(\Omega,\bar\Omega)^2,$$

where $\{F_{\alpha\beta\gamma}\}$ is the Yukawa coupling, which is a holomorphic section of the bundle $\mathrm{Sym}^{\otimes 2} F^n \otimes \mathrm{Sym}^{\otimes 3} T^*\mathcal{M}$, locally defined as

$$F_{\alpha\beta\gamma} = (\Omega, \partial_\alpha\partial_\beta\partial_\gamma\Omega).$$

So (1) can be written as

$$R_{\alpha\bar\beta\gamma\bar\delta} = g_{\alpha\bar\beta}g_{\gamma\bar\delta} + g_{\alpha\bar\delta}g_{\gamma\bar\beta} - F_{\alpha\gamma m}\overline{F_{\beta\delta n}}g^{m\bar n}/(\Omega,\bar\Omega)^2. \tag{2}$$

Remark 1. *Formula* (2) *was first given by Strominger* [29]. *In the general case, a Hodge theoretic proof of Theorem 1 was given by Wang* [34]. *Schumacher's paper* [26] *will lead another proof using the method of Siu* [27].

We know very little of the global behavior of the moduli space $\mathcal{M}$, except the following result of Viehweg [33].

Theorem 2 (Viehweg). *$\mathcal{M}$ is quasi-projective.*

Remark 2. *By the above theorem, after normalization and desingularization, there is a compact manifold $\overline{\mathcal{M}}$ such that $\overline{\mathcal{M}}\backslash\mathcal{M}$ is a divisor of normal crossings. In fact, since in general $\mathcal{M}$ is a complex variety, we can redefine $\mathcal{M}$ to be the regular part of $\mathcal{M}$. On such a setting, up to a finite cover, both $\mathcal{M}$ and $\overline{\mathcal{M}}$ are manifolds.*

For the extension of the Hodge bundles across the divisor at infinity, we have the following theorem of Schmid [25] or Steenbrink [28]:

Theorem 3. *Let $\mathfrak{X} \to \Delta^r \times (\Delta^*)^s$ be a family of polarized Calabi-Yau manifolds, where Δ and Δ^* are the unit disk and the punctured unit disk, respectively. Suppose that all the monodromy operators are unipotent. Then there is a natural extension of the Hodge bundles to Δ^{r+s}.*

By the following result on the Weil-Petersson metric, the extension of the bundles F^n, F^{n-1} will give us information of the limiting behaviors of the Weil-Petersson metric at infinity.

We need the following result of Tian [30] in the rest of this paper:

Proposition 1. *Let* $(g_{\alpha\bar{\beta}})$ *be the metric matrix of the Weil-Petersson metric under the frame* $(\frac{\partial}{\partial t_1}, \cdots, \frac{\partial}{\partial t_m})$. *Then we have*

$$g_{\alpha\bar{\beta}} = -\frac{(\nabla_\alpha\Omega, \overline{\nabla_\beta\Omega})}{(\Omega, \overline{\Omega})}.$$

The proof is a straightforward computation and is omitted.

□

4. THE MAIN RESULTS

There are several previous results related to the main results of this paper. It was proved in [22, Theorem 5.2] that the volume with respect to the Weil-Petersson metric and the Hodge metric is finite. In [23, 31], the rationality of the volume with respect to the Weil-Peterssoin metric was proved. Furthermore, in [23], it was proved that the integration of the n-th power of the Ricci curvature of the Weil-Petersson metric is a rational number. The main results of this paper are Theorem 4 and Theorem 5. The most general forms of these results will appear in our upcoming paper [7].

Theorem 4. *Let* R_{WP} *be the curvature tensor of* ω_{WP}. *Let* $R = R_{WP}\otimes 1 + 1\otimes\omega_{WP}$. *Let* f *be any invariant polynomial of* R. *Then we have*

$$\int_{\mathcal{M}} f(R) < +\infty.$$

The above theorem is equivalent to the following: let $f_1, \cdots, f_s$ be invariant polynomials of R_{WP} of degree $k_1, \cdots, k_s$, respectively. Then

$$\int_{\mathcal{M}} \sum_i f_i(R_{WP}) \wedge \omega_{WP}^{m-k_i} < +\infty,$$

where m is the complex dimension of $\mathcal{M}$. The result is a generalization of Theorem 5.2 in [22].

Theorem 5. *We assume that* $\dim \mathcal{M} = 2$. *Let* R *be the curvature operator of the Hodge bundle, and let* f *be an invariant polynomial with rational coefficients. Then we have*

$$\int_{\mathcal{M}} f(R) \in \mathbb{Q}.$$

If $\dim \mathcal{M} = 1$, or the rank of F^k is one, then the corresponding result follows from [23], because the only Chern class will be the first Chern class. The case that $\mathcal{M}$ is of arbitrary dimension is treated in [7].

5. THE HODGE METRIC

The curvature properties of the Weil-Petersson metric are not good. For example, even in the case when the moduli space is of dimension 1, from [4, page 65], we know that the sign of the Gauss curvature is not fixed. In [19], the second author introduced another natural metric, called the Hodge metric, on $\mathcal{M}$. We shall see that the Hodge metric is the bridge between the curvature invariants and finiteness.

The following definition of the Hodge metric is from [22, section 6], which is slightly different from that in [19].

Let $\mathcal{M}$ be the moduli space of any polarized compact Kahler manifold (not necessarily Calabi-Yau). Let $x \in \mathcal{M}$ be a smooth point of $\mathcal{M}$. Assume that the period map $p : \mathcal{M} \to D$ is an immersion near x.

Let $0 \subset F^n \subset \cdots \subset F^1 \subset F^0 = H$ be the Hodge bundles and let

$$T_t\mathcal{M} \to H^1(X_t, \Theta_t)$$

be the Kodaira-Spencer isomorphism. Let $\xi \in T_t\mathcal{M}$. Then ξ defines a map

$$\xi : H^1(X_t, \Theta_t) \times H^{p,q} \to H^{p-1,q+1}.$$

Let $||\xi||_{p,q}$ be the operator norm with respect to the metric on Hodge bundles $H^{p,q}$ and $H^{p-1,q+1}$. Then we define

$$||\xi||^2 = \sum_{p+q=n} ||\xi||^2_{p,q}.$$

From the above definition, we get a Hermitian metric on the smooth part of the moduli space $\mathcal{M}$. Let ω_H be the Kähler form of the metric. Then the properties of the Hodge metric can be summarized as follows [19]:

Theorem 6. *Using the above notations, we have*

1. *The Hodge metric is a Kähler metric;*
2. *The bisectional curvartures of the Hodge metric are nonpositive;*
3. *The holomorphic sectional curvatures of the Hodge metric are bounded from above by a negative constant;*
4. *The Ricci curvature of the Hodge metric is bounded above by a negative constant.*

Remark 3. *If p is not an immersion, then using the same definition, we get a semi-positive pseudo metric and the form ω_H is also well defined. In fact, up to a constant, the Hodge metric is the pull-back of the invariant Hermitian metric of D, the classifying space. Such a metric is Kähler in the sense that $d\omega_H = 0$. The pseudo metric is called a generalized Hodge metric in* [14].

The Hodge metric and the Weil-Petesson metric on the moduli space of polarized Calabi-Yau manifolds are closely related. Let the dimension of the moduli space be m. Then we have the following

Theorem 7. *Using the above notations, we have*

1. *By Proposition 1, we have*

$$2\omega_{WP} \leq \omega_H.$$

(2) *If $\mathcal{M}$ is the moduli space of algebraic K3 surfaces, then*

$$2\omega_{WP} = \omega_H.$$

(3) *If $\mathcal{M}$ is the moduli space of a Calabi-Yau threefold, then we have* [21]

$$\omega_H = (m+3)\omega_{WP} + \mathrm{Ric}(\omega_{WP}).$$

(4) *If $\mathcal{M}$ is the moduli space of a Calabi-Yau fourfold, then we have* [22]

$$\omega_H = 2(m+2)\omega_{WP} + 2\mathrm{Ric}(\omega_{WP}).$$

(5) *In general, the generalized Hodge metrics and the Weil-Petersson metric are related by the so-called BCOV torsion (cf.* [2, 3]*) in* [14]*:*

$$\sum_{i=1}^{n}(-1)^i\omega_{H^i} - \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log T = \frac{\chi}{12}\omega_{WP},$$

where T is the BCOV torsion, ω_{H^i} is the generalized Hodge metric with respect to the variation of Hodge structures of weight i, and χ is the Euler characteristic number of a generic fiber.

In order to study the asymptotic behavior of the Hodge metrics, we quote the following Schwarz lemma of Yau [35]:

Theorem 8. *Let M, N be Kähler manifolds. Suppose that M is complete and the Ricci curvature of M is bounded from below. Suppose that the bisectional curvatures of N are nonpositive and the holomorphic sectional curvatures are bounded from above by a negative constant. Then there is a constant C, depending only on the dimensions of the two manifolds and the above curvature bounds, such that*

$$f^*(\omega_N) \le C\omega_M,$$

where ω_M, ω_N are the Kähler forms of manifold M and N, respectively.

By Remark 2, the regular part of the complex variety $\mathcal{M}$ is quasi-projective. We construct a Kähler metric ω_P on a quasi-projective manifold, following Jost-Yau [17]. Let $U = (\Delta^*)^r \times \Delta^s$. We define a Kähler metric on U by

$$\sqrt{-1}\left(\sum_{i=1}^{r}\frac{dz_i\wedge d\bar{z}_i}{|z_i|^2(\log\frac{1}{|z_i|})^2} + \sum_{i=r+1}^{r+s} dz_i\wedge d\bar{z}_i\right).$$

Since $\mathcal{M}$ is quasi-projective, it can be covered by finitely many open sets of the form $(\Delta^*)^r \times \Delta^s$ (r is allowed to be zero). By Jost-Yau, we can glue the Kähler metrics of the above form and get a global Kähler metric ω_P on $\mathcal{M}$. The metric satisfies the following properties:

(1) ω_P is complete;
(2) The Ricci curvature of ω_P is bounded from below;
(3) the volume of the metric ω_P is finite.

Unlike the Weil-Petersson metric or Hodge metric, ω_P is not intrinsically defined.

If we let f in Theorem 8 be the identity map from $\mathcal{M}$ to itself, then using the Schwarz lemma, we get the following

Lemma 1. *Let ω_H, ω_P be the two metrics on $\mathcal{M}$. Then there is a constant C such that*

$$\omega_H \leq C\omega_P.$$

□

We remark that by [14, Theorem A.1], even for the generalized Hodge metric, the inequality in Lemma 1 is still valid. In particular, this implies that the Hodge volumes and the Weil-Petersson volume are all finite on a Calabi-Yau moduli space.

6. Proof of the results.

Proof of Theorem 4. Let $c_r(\omega_{WP})$ be the r-th elementary polynomial of the curvature matrix of the Weil-Petersson metric. We claim that

$$|c_r(\omega_{WP})| \leq C\omega_H^r. \tag{3}$$

The above inequality means that for any $v_1, \cdots, v_r \in T\mathcal{M}$, we have

$$|c_r(\omega_{WP})(v_1, \cdots, v_r, \bar{v}_1, \cdots, \bar{v}_r)| \leq C \prod_{i=1}^{r} ||v_i||^2$$

for some constant $C > 0$, where the norm of the right hand side is with respect to the metric ω_H.

To prove the claim, first we choose a normal coordinate system at $x \in \mathcal{M}$ such that

$$g_{i\bar{j}}(x) = \delta_{ij}, \quad dg_{i\bar{j}}(x) = 0. \tag{4}$$

Let

$$R_i^j = \sum_{kl} R_{ik\bar{l}}^j dz^k \wedge d\bar{z}^l, \quad \text{where } R_{ik\bar{l}}^j = g^{j\bar{p}} R_{i\bar{p}k\bar{l}}.$$

Then the r-th Chern class is given by

$$c_r(\omega_{WP}) = \frac{(-1)^r}{r!} \sum_{\tau \in S_r} sgn(\tau) R_{i_1}^{i_{\tau(1)}} \wedge \cdots \wedge R_{i_r}^{i_{\tau(r)}}, \tag{5}$$

where S_r is the symmetric group of the set $\{1, 2, \ldots, r\}$.

We define

$$h'_{\alpha\bar{\beta}} = \delta_{\alpha\beta} + \sum_{\gamma} (\nabla_\alpha \nabla_\gamma \Omega, \overline{\nabla}_\beta \overline{\nabla}_\gamma \Omega).$$

Then $(h'_{\alpha\bar{\beta}})$ defines a Kähler metric ω'. By [14, Proposition 2.8] and Theorem 1, we have

$$\omega' \leq \omega_H.$$

Thus in order to prove the claim, we only need to prove that

$$|c_r(\omega_{WP})| \leq C(\omega')^r.$$

Let $A_{ij} = \sum_k (\nabla_i \nabla_k \Omega, \overline{\nabla}_j \overline{\nabla}_k \Omega)$. Since the matrix (A_{ij}) is Hermitian, after suitable unitary change of basis, we can assume

$$A_{ij}(p) = \begin{cases} \lambda_i & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Since $(A_{ij}(p))$ is positive-semidefinite, $\lambda_i \geq 0$, and we can write

$$h'_{i\bar{j}}(p) = \delta_{ij}(1 + \lambda_i). \tag{6}$$

Clearly, we have

$$(1 + \lambda_{i_1}) \cdots (1 + \lambda_{i_\alpha}) \leq \det h' \tag{7}$$

for any $1 \leq i_1 < i_2 < \ldots < i_\alpha \leq n$, where $\det h' = \det(h'_{\alpha\bar{\beta}})$. We assume that $v_i = \frac{\partial}{\partial t_{k_i}}$, then by (5), we have

$$|c_r(\omega_{WP})(v_1, \cdots, v_r, \bar{v}_1, \cdots, \bar{v}_r)| \leq C \operatorname{Max} |R^{i_1}_{j_1 k_1 \overline{\sigma(k_1)}} \cdots R^{i_r}_{j_r k_r \overline{\sigma(k_r)}}|,$$

For fixed i, j, k, l, by the Cauchy-Schwartz inequality, we have

$$\begin{aligned} |R^j_{ik\bar{l}}| &\leq |\delta_{ij}\delta_{kl} + \delta_{il}\delta_{kj} - (\nabla_i \nabla_k \Omega, \overline{\nabla}_j \overline{\nabla}_l \Omega)| \\ &\leq 2 + \sqrt{(\nabla_i \nabla_k \Omega, \overline{\nabla}_i \overline{\nabla}_k \Omega)(\nabla_j \nabla_l \Omega, \overline{\nabla}_j \overline{\nabla}_l \Omega)} \\ &\leq 2 + \sqrt{h_{i\bar{i}} h_{j\bar{j}}} \leq 2\sqrt{(1 + \lambda_i)(1 + \lambda_j)}. \end{aligned}$$

So we get

$$\left| R^{i_1}_{j_1 k_1 \bar{k}_{\sigma(1)}} \cdots R^{i_r}_{j_r k_r \bar{k}_{\sigma(r)}} \right| \leq 2^m \prod_{\alpha=1}^{m} \left(\sqrt{(1 + \lambda_{k_i})(1 + \lambda_{\sigma(k_i)})} \right).$$

The claim follows from the above inequality and (7).

Let $r_0, \cdots, r_t \geq 0$ such that $\sum r_i = m$. Then using (3),

$$|c_{r_1}(\omega_{WP}) \wedge \cdots \wedge c_{r_t}(\omega_{WP}) \wedge \omega_{WP}^{r_0}| \leq C(\det h').$$

By Lemma 1, the left hand side of the above equation is integrable. Theorem 4 follows from the above inequality.

□

To prove Theorem 5, we first observe the following:

Let $E \to X$ be a holomorohic vector bundle over a compact manifold X. Let h_0, h_1 be two Hermitian metrics on the bundle. Let R_0, R_1 be the curvature tensors and let θ_0, θ_1 be the connection matrices. Let f be an invariant polynomial. Then we have

$$\int_X f(R_0, \cdots, R_0) = \int_X f(R_1, \cdots, R_1).$$

In fact,

$$\int_X f(R_1, \cdots, R_1) - \int_X f(R_0, \cdots, R_0) = \int_X \sum_{i=1}^{k} f(R_1, \cdots, R_1, R_1 - R_0, R_0, \cdots, R_0).$$

Since $R_1 - R_0 = \bar\partial(\theta_1 - \theta_0)$, we get

$$\begin{aligned}&\int_X f(R_1, \cdots, R_1) - \int_X f(R_0, \cdots, R_0)\\ &= \sum_{i=1}^k \int_X \bar\partial f(R_1, \cdots, R_1, \theta_1 - \theta, R_0, \cdots, R_0) = 0.\end{aligned}$$

A similar method can be used in the non-compact cases. The only difference is that we need various estimates of the curvatures and the connections near the infinity.

Proof of Theorem 5. As discussed in Section 3, up to a finite cover, we can assume that both $\mathcal{M}$ and $\overline{\mathcal{M}}$ are manifolds and $D = \overline{\mathcal{M}}\backslash\mathcal{M}$ is a divisor of normal crossings. Furthermore, by [23, Lemma 4.1], we assume that the monodromy operators of the divisor D are all unipotent. Since $\dim\mathcal{M} = 2$, if D_0 denotes the smooth part of D, then the singular part $D\backslash D_0$ are the set of finite points. Let $F = F^k$ be a Hodge bundle and let $\bar F$ be the Schmid extension of the bundle across D.

Let

$$D\backslash D_0 = \{x_1, \cdots, x_s\}.$$

Let h be the Hermitian metric of F. Let U be a neighborhood of D such that $x_i \notin \overline{U}$ for any i. Let $e_1, \cdots, e_k$ be a local holomorphic frame. Let $\langle e_i, e_j\rangle$ be the inner product induced from the metric h. Let (z_1, z_2) be the local coordinates of U such that $D \cap U = \{z_1 = 0\}$. Then by the Nilpotent orbit theorem of Schmid, we can write

$$e_i = \exp(N \log 1/z_1) A_i(z_1, z_2), \quad 1 \le i \le \operatorname{rank} F.$$

It follows that the determinant of the metric matrix $\langle e_i, e_j\rangle$ can be expanded as

$$\det\langle e_i, e_j\rangle = a(z_1, z_2)(\log\frac{1}{r_1})^\alpha + \cdots,$$

where $\cdots$ are the lower order terms and $a(z_1, z_2)$ is a real analytic function of (z_1, z_2). The zero set of $a(z_1, z_2)$ is finite, and is independent to the choices of local frames. Let $x_{s+1}, \cdots, x_t$ be such kind of zeros on D. Let $U_1, \cdots, U_t$ be neighborhoods of $x_i (1 \le i \le t)$. Assume that $U_i \cap U_j = \emptyset$ for $i \ne j$. These open sets are called the neighborhoods of the first kind. Let $\{U_1, \cdots, U_{t+t_1}\}$ be a cover of D. The neighborhood $U_i (t < i \le t + t_1)$ are called neighborhoods of the second kind. Let U_i be a neighborhood of the second kind and let (z_1, z_2) be the holomorphic coordinates. Let θ, R be the connection and the curvature matrices of the metric h. Let

$$\theta = \theta_i dz_i, \qquad R = R_{i\bar j} dz_i \wedge d\bar z_j.$$

Then there is a constant $C > 0$ such that

$$\begin{aligned}&|\theta_1| \le \frac{C}{r_1 \log\frac{1}{r_1}}, \quad |\theta_2| \le C;\\ &|R_{1\bar 1}| \le \frac{C}{(r_1\log\frac{1}{r_1})^2}, \quad |R_{1\bar 2}|, |R_{2\bar 1}| \le \frac{C}{r_1\log\frac{1}{r_1}}, \quad |R_{2\bar 2}| \le C.\end{aligned} \tag{8}$$

Let h' be a Hermitian metric on $\overline{F}$ and let θ', R' be the corresponding connection and curvature matrices. We assume that on the neighborhoods of the first kind, the metric is flat. That is, θ' and R' are identically zero on $U_i (1 \le i \le t)$. Let $U_1, \cdots, U_{t+t_1+t_2}$ be a cover of $\overline{\mathcal{M}}$ such that

(1) $U_i (1 \le i \le t)$ are neighborhoods of the first kind;
(2) $U_i (t \le i \le t+t_1)$ are neighborhoods of the second kind;
(3) $D \cap (\bigcup_{i=t+t_1+1}^{t+t_1+t_2} \overline{U}_i) = \emptyset$.

By [23, Theorem 3.1], we know that for any $\varepsilon > 0$, there is a cut-off function $\rho = \rho_\varepsilon$ such that

(1) $0 \le \rho_\varepsilon \le 1$;
(2) For any open neighborhood V of D in $\overline{\mathcal{M}}$, there is $\varepsilon > 0$ such that $\operatorname{supp}(1 - \rho_\varepsilon) \subset V$;
(3) For each $\varepsilon > 0$, there is a neighborhood V_1 of D such that $\rho_\varepsilon|_{V_1} \equiv 0$;
(4) $\rho_{\varepsilon'} \ge \rho_\varepsilon$ for $\varepsilon' \le \varepsilon$;
(5) There is a constant C, independent of ε such that

$$-C\omega_P \le \sqrt{-1}\partial\bar{\partial}\rho_\varepsilon \le C\omega_P, \quad \left|\frac{\partial\rho}{\partial z_1}\right| \le \frac{C}{r_1 \log\frac{1}{r_1}}, \quad \left|\frac{\partial\rho}{\partial z_2}\right| \le C.$$

Since

$$R - R_0 = \bar{\partial}(\theta - \theta_0),$$

we have

$$(9) \quad \int_{\mathcal{M}} \rho(f(R,R) - f(R_0,R_0)) = -\int_{\mathcal{M}} \bar{\partial}\rho \wedge f(R, \theta - \theta_0) - \int_{\mathcal{M}} \bar{\partial}\rho \wedge f(R_0, \theta - \theta_0),$$

where f is an invariant quadratic polynomial. For $\varepsilon > 0$ small enough, On $U_i (t + t_1 < i \le t + t_1 + t_2)$, $\rho \equiv 1$. So we have

$$\begin{aligned} &\left|\int_{\mathcal{M}} \rho(f(R,R) - f(R_0,R_0))\right| \\ &\le \sum_{i=1}^{t+t_1} \left(\left|\int_{U_i} \bar{\partial}\rho \wedge f(R, \theta - \theta_0)\right| + \left|\int_{U_i} \bar{\partial}\rho \wedge f(R_0, \theta - \theta_0)\right|.\right) \end{aligned}$$

We shall prove that the right hand side of the above goes to zero as $\varepsilon \to 0$. If U_i is a neighborhood of the second kind, then since θ_0 is bounded, by (8) and the definition of ρ, we know that

$$|\bar{\partial}\rho \wedge f(R, \theta - \theta_0)| + |\bar{\partial}\rho \wedge f(R_0, \theta - \theta_0)| \le \frac{C}{r_1^2 (\log\frac{1}{r_1})^2} |dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2|,$$

which is integrable. Thus we have

$$\begin{aligned} &\lim_{\varepsilon \to 0} \left|\int_{U_i} \bar{\partial}\rho \wedge f(R, \theta - \theta_0)\right| + \left|\int_{U_i} \bar{\partial}\rho \wedge f(R_0, \theta - \theta_0)\right| \\ &\le C \int_{\operatorname{supp} \nabla\rho} (r_1^2 (\log\frac{1}{r_1})^2)^{-1} |dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2| = 0 \end{aligned}$$

for $t < i \le t + t_1$. If U_i is a neighborhood of the first kind, then we have

$$\int_{U_i} \bar{\partial}\rho \wedge f(R_0, \theta - \theta_0) = 0$$

because $R_0 \equiv 0$ on U_i. On the other hand, we have

$$\int_{U_i} \bar{\partial}\rho \wedge f(R, \theta - \theta_0) = \int_{U_i} \bar{\partial}\rho \wedge f(R, \theta).$$

From [5, Proposition 5.22], we know that there is a gauge transform e such that we have

$$|Ad(e)\theta_1| \leq \frac{C}{r_1 \log \frac{1}{r_1}}, \quad |Ad(e)\theta_2| \leq C;$$

$$|Ad(e)R_{1\bar{1}}| \leq \frac{C}{(r_1 \log \frac{1}{r_1})^2}, \quad |Ad(e)R_{1\bar{2}}|, |Ad(e)R_{2\bar{1}}| \leq \frac{C}{r_1 \log \frac{1}{r_1}}, \quad |Ad(e)R_{2\bar{2}}| \leq C.$$

Since f is an invariant polynomial, using the transform, we have

$$\left|\int_{U_i} \bar{\partial}\rho \wedge f(R, \theta)\right| = \left|\int_{U_i} \bar{\partial}\rho \wedge f(Ad(e)R, Ad(e)\theta)\right|$$
$$\leq C \int_{\operatorname{supp} \nabla\rho} (r_1^2 (\log \frac{1}{r_1})^2)^{-1} |dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2| \to 0.$$

Thus from (9), we proved that

$$\lim_{\varepsilon \to 0} \int_{\mathcal{M}} \rho(f(R, R) - f(R_0, R_0)) = 0.$$

By the Gauss-Bonnet-Chern theorem,

$$\int_{\mathcal{M}} f(R_0, R_0)$$

is an integer. Thus

$$\int_{\mathcal{M}} f(R, R)$$

is also an integer.

□

Remark 4. *Theorem 5 is a rationality result of the Hodge bundles. However, by Proposition 1, the Weil-Petersson metric is the quotient of the Hermitian metrics on the Hodge bundles $H^{n-1,1}$ and $H^{n,0}$, respectively. Thus the corresponding result for the Weil-Petersson metric is also valid.*

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