数理解析研究所講究録
1487 巻 2006 年 40-54

WEIGHTED BERGMAN KERNELS AND BALANCED METRICS

MIROSLAV ENGLIŠ

ABSTRACT. Motivated by the recent results for compact manifolds, we study the existence and uniqueness of balanced metrics in the noncompact setting, in particular, on smoothly bounded strictly pseudoconvex domains in $\mathbf{C}^n$. By an analysis of the boundary behaviour of weighted Bergman kernels, we show that, as with the solution to the Monger-Ampère equation, balanced metrics on such domains cannot be determined solely from their boundary singularities: in fact, we exhibit a whole family of metrics on the disc which are balanced up to an error term which is smooth up to the boundary. Finally, some applications are indicated which would follow once the existence and uniqueness of balanced metrics were established.

1. INTRODUCTION

Let Ω be a bounded domain in $\mathbf{C}^n$, $n \geq 1$, which we assume for simplicity to be contractible to a point, and w a positive continuous weight function on Ω. It is then well known that the subspace $L^2_{\text{hol}}(\Omega, w)$ of all holomorphic functions in $L^2(\Omega, w)$ (the *weighted Bergman space*) is closed and admits a reproducing kernel $K_w(x, y)$ (*weighted Bergman kernel*): that is,

$$f(x) = \int_\Omega f(y)\, K_w(x,y)\, w(y)\, dy \qquad \forall f \in L^2_{\text{hol}}(\Omega, w) \quad \forall x \in \Omega.$$

For brevity, we will usually write just $K_w(x)$ instead of $K_w(x, x)$.

Let now Φ be a strictly plurisubharmonic (or *strictly-PSH* for short) function on Ω, so that

$$g_{i\bar{j}} = \frac{\partial^2 \Phi}{\partial z_i \partial \bar{z}_j} \tag{1}$$

defines a Kähler metric on Ω. Let $\det[\partial\bar{\partial}\Phi] =: \det[g_{i\bar{j}}]$ be the associated volume density, and consider the weight function $w = e^{-\Phi} \det[\partial\bar{\partial}\Phi]$. Weighted Bergman kernels for such weights arise in certain approaches to quantization on Kähler manifolds, where one considers the family of these weights obtained by replacing Φ by Φ/h where h (Planck's constant) is a positive parameter, and the problem is to describe the asymptotic behavior of the corresponding kernels as h tends to zero. In this paper, however, we will be interested in another aspect of these weighted Bergman kernels.

1991 *Mathematics Subject Classification.* Primary 32A25; Secondary 32A36, 58E11.

Key words and phrases. Bergman kernel, balanced metric.

Research supported by GA AV ČR grant no. A1019304 and Ministry of Education research plan no. MSM4781305904.

Definition 1. The function Φ is called balanced if

$$e^{-\Phi(z)} K_w(z) = \text{const.} \tag{2}$$

Note that (2) is not really a property of Φ, but rather of the metric (1) defined by it. Indeed, if Φ' is another potential for $g_{i\bar{j}}$, then it follows from $\partial\bar{\partial}\Phi = \partial\bar{\partial}\Phi'$ and the contractibility of Ω that $\Phi' = \Phi - 2\operatorname{Re} F$ for some holomorphic function F on Ω. Thus $e^{-\Phi'} = e^{-\Phi}|e^F|^2$ and $w' := e^{-\Phi'} \det[\partial\bar{\partial}\Phi'] = w|e^F|^2$. Owing to the holomorphy of F, the mapping $f \mapsto e^F f$ is a unitary isomorphism of $L^2_{\text{hol}}(\Omega, w')$ onto $L^2_{\text{hol}}(\Omega, w)$, which implies that the corresponding reproducing kernels are related by $K_{w'} = |e^{-F}|^2 K_w$. Thus $e^{-\Phi'} K_{w'} = e^{-\Phi} K_w$. Consequently, the left-hand side of (2) depends only on the metric (1). The following definition is therefore consistent.

Definition 2. If (2) holds, then (1) is called a balanced metric.

Example. Let $\Omega = \mathbf{D}$, the unit disc in $\mathbf{C}$, and

$$\Phi = \alpha \log \frac{1}{1-|z|^2}, \qquad \alpha > 1. \tag{3}$$

Then $\det[\partial\bar{\partial}\Phi] = \alpha/(1-|z|^2)^2$, so $w = \alpha(1-|z|^2)^{\alpha-2}$. It is well known that the corresponding weighted Bergman kernel is

$$K_w(z) = \frac{\alpha-1}{\pi\alpha}(1-|z|^2)^{-\alpha}.$$

Thus

$$e^{-\Phi(z)} K_w(z) = \frac{\alpha-1}{\pi\alpha} \qquad \forall z \in \mathbf{D},$$

so Φ is balanced. $\square$

The two definitions extend in an obvious way also to domains which are not contractible, and more generally to polarized Kähler manifolds. Namely, let Ω be a complex manifold of dimension n, and ω a Kähler metric on Ω such that the second cohomology class $[\omega]$ is integral. Then there exists a holomorphic Hermitian line bundle $\mathcal{L}$ over Ω with compatible connection ∇ such that $\operatorname{curv}\nabla = \omega$. Let $\mathcal{L}^*$ be the dual bundle, and $L^2_{\text{hol}}(\mathcal{L}^*, \bigwedge^n \omega)$ the Bergman space of all square-integrable holomorphic sections of $\mathcal{L}^*$. Let $\{s_j\}$ be any orthonormal basis of this space, and

$$\epsilon(x) := \sum_j \|s_j(x)\|_x^2 \tag{4}$$

(where $\|\cdot\|_x$ denotes the fiber norm in $\mathcal{L}_x$). It is easy to see that this function does not depend on the choice of the orthonormal basis $\{s_j\}$, and a similar argument as above also shows that it does not really depend on the line bundle $\mathcal{L}$ but only on the Kähler form ω.

Definition 3. The Kähler form ω (or the associated Kähler metric) is called balanced if $\epsilon \equiv$ const.

Of course, if Ω is contractible, then the bundle $\mathcal{L}$ is trivial, so fixing a trivialization its sections can be identified with functions on Ω, and under this identification the fiber norm of a function f at a point x is given by $h(x)|f(x)|^2$ for some positive smooth function h on Ω satisfying $\omega = i\,\partial\overline{\partial}\log h$. Setting $\Phi := \log h$, we see that the space $L^2_{\text{hol}}(\mathcal{L}^*, \bigwedge^n \omega)$ reduces to $L^2_{\text{hol}}(\Omega, e^{-\Phi}\det[\partial\overline{\partial}\Phi])$, and $\epsilon(x) = e^{-\Phi(x)}K_\omega(x)$. Thus the last definition agrees with Definitions 1–2.

The function ϵ has appeared in the literature under different names. The earliest one was probably the *η-function* of Rawnsley [Raw] (later renamed to *ϵ-function* in [CGR]), defined for arbitrary Kähler manifolds; followed by the *distortion function* of Kempf [Ke] and Ji [Ji] for the special case of Abelian varieties, and of Zhang [Zha] for complex projective varieties. The metrics for which ϵ is constant were called *critical* in [Zha]; the term *balanced* was first used by Donaldson [Don], who also established the existence of such metrics on any (compact) projective Kähler manifold with constant scalar curvature.

However, very little seems to be known about the existence or uniqueness of balanced metrics on general (noncompact) manifolds or even domains in $\mathbf{C}^n$. Apart from the example above for the unit disc, the only existing examples of balanced metrics are the similar metrics on the unit ball $\mathbf{B}^n \subset \mathbf{C}^n$:

$$\Phi(z) = \alpha \log \frac{1}{1-\|z\|^2}, \qquad \alpha > n, \tag{5}$$

for which the constant on the right-hand side of (2) turns out to be

$$\frac{\Gamma(\alpha)}{\Gamma(\alpha-n)\pi^n\alpha^n}; \tag{6}$$

and the similarly defined metrics (multiples of the Bergman metric) on bounded symmetric domains. (Note however that except for the unit balls, bounded symmetric domains are never smoothly bounded nor strictly pseudoconvex).

For this reason, in this paper we will investigate the problem of existence and uniqueness of balanced metrics on smoothly bounded strictly pseudoconvex domains in $\mathbf{C}^n$. Our strategy will be to look at the boundary singularities of both sides in the equation (2). More specifically, assume that

$$e^{-\Phi} = \rho^\alpha e^g, \tag{7}$$

where ρ is a (smooth) defining function for Ω and $g \in C^\infty(\overline{\Omega})$; we will see in a moment that in order that the weighted Bergman spaces that appear be nonempty, we will need to assume that $\alpha > d$. Recall that for any function u,

$$\det[\partial\overline{\partial}\log u] = u^{-n-1}(-1)^n J[u],$$

where $J[u]$ is the Monge-Ampère determinant

$$J[u] := (-1)^n \det \begin{bmatrix} u & \partial u \\ \overline{\partial} u & \partial\overline{\partial} u \end{bmatrix}.$$

It follows that $w = e^{-\Phi}\det[\partial\overline{\partial}\Phi]$ is given by

$$w = \rho^{\alpha-n-1}\alpha^n \underbrace{J[\rho e^{g/\alpha}]\cdot e^{g\cdot(1-\frac{n+1}{\alpha})}}.$$

Note that the underbraced term is smooth up to the boundary and positive there. Thus if $\alpha - n - 1 =: m$ is a positive integer, then

$$\rho' := \rho\cdot(\alpha^n J[\rho e^{g/\alpha}]e^{g\cdot(1-\frac{n+1}{\alpha})})^{1/m}$$

is also a defining function for Ω. Recall now (see e.g. [E1]) that for any defining function ρ' and any positive integer m, we have the following generalization of Fefferman's classical result for the unweighted Bergman kernel:

$$K_{\rho'^m} = \frac{a}{\rho'^{m+n+1}} + b\log\rho'$$

where $a, b \in C^\infty(\overline{\Omega})$; further, the derivatives of a of order $\leq m+n$ as well as the derivatives of b of all orders at a point x on the boundary depend only on the jet of the boundary at x (i.e. on the jet of ρ at x). Thus in our case

$$K_w = \frac{a'}{\rho^\alpha} + b'\log\rho, \qquad a', b' \in C^\infty(\overline{\Omega}), \tag{8}$$

and

$$e^{-\Phi}K_w = a'' + b''\rho^\alpha\log\rho,$$

where the derivatives of a'' of order $< \alpha$ and the derivatives of b'' of all orders at a point $x \in \partial\Omega$ depend only on the jets of ρ and g at x. If the left-hand side of the last equation is to be constant, we therefore obtain some conditions on the behaviour of g at the boundary, from which it might hopefully be possible to construct a (formal) solution to (2) or to prove that it is unique.

With minor modifications, all this remains in force also if $\alpha - n - 1$ is not a positive integer, provided $\alpha - n - 1 > -1$, i.e. $\alpha > n$ (otherwise the space $L^2_{\mathrm{hol}}(\rho'^m)$ contains just the constant zero); the only difference is that for α not an integer (8) gets replaced by $K_w = a'\rho^{-\alpha} + b'$, and $e^{-\Phi}K_w = a'' + b''\rho^\alpha$. See [E2]. It can also be shown, by evaluating the function a'' on the boundary, that the value of the constant in (2) always has to be the same as for the unit ball, i.e. be given by (6).

Of course, in a way this approach is rather naive: as is well known, a similar treatment can be applied to solving the complex Monge-Ampère equation, and in that case the procedure breaks down at a certain stage since the solution is in fact not of the form (7) but contains also logarithmic terms of the form $\rho^{n+1}\log\rho$. However, for domains such as the disc or the ball we know that there exist solutions of the form (7) (cf. (3)), so we might at least be able to prove uniqueness of these solutions. (The solution of the Monge-Ampère equation also turns out to contain no logarithmic terms in this case, for that matter.)

In the rest of this paper, we will therefore examine the case of radial balanced metrics on the unit disc and the unit ball (i.e. those for which Φ depends only on $\|z\|$).

2. Radial balanced metrics on D

Let us start with the case of the disc. Recall that if $w(z) = F(|z|^2)$ is a radial weight on $\mathbf{D}$, then the monomials z^k form an orthogonal basis in $L^2_{\text{hol}}(\mathbf{D}, w)$, with norms

$$\|z^k\|_w^2 = \pi \int_0^1 F(t)\, t^k \, dt =: \pi c_k(w)$$

(by passing to polar coordinates). Consequently, by the familiar formula for the reproducing kernel in terms of an orthonormal basis,

$$K_w(z) = \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{t^k}{c_k(w)}, \qquad t := |z|^2. \tag{9}$$

Our goal is to find all radial solutions Φ of

$$K_{e^{-\Phi} \det[\partial\bar\partial\Phi]} = \text{const} \cdot e^{\Phi}.$$

Following the strategy outlined in the preceding section, we look for solutions in the form

$$e^{-\Phi} = (1-t)^{\alpha} \sum_{j=0}^{\infty} (1-t)^j f_j, \tag{10}$$

where $f_j \in \mathbf{R}$, $f_0 = 1$. We then compute, in turn, the weight $w = e^{-\Phi} \det[\partial\bar\partial\Phi]$, the moments $c_k(w)$, the kernel K_w, the function $e^{-\Phi} K_w$, and then check when the last is constant.

For practical computations, it is more convenient to replace the defining function $1-t$ in (10) by

$$L(t) := \log \frac{1}{t} = (1-t) + \frac{(1-t)^2}{2} + \dots.$$

This has the same boundary behaviour, and has the advantage that the moments $c_k(L^\beta)$ evaluate rather neatly: for any $\beta > -1$,

$$\int_0^1 (\log \tfrac{1}{t})^\beta \, t^n \, dt = \frac{\beta!}{(n+1)^{\beta+1}}. \tag{11}$$

(The singularity of $L(t)$ at $t = 0$ causes no problem since we are interested only in the behaviour at the boundary $t = 1$.) Thus we start from

$$e^{-\Phi} = L^\alpha \sum_{j=0}^{\infty} L^j f_j, \qquad f_0 = 1, \ \alpha > 1. \tag{12}$$

Taking logarithms yields

$$\Phi = -\alpha \log L + \sum_{j=1}^{\infty} f'_j L^j,$$

where

$$\begin{aligned} f'_m &= \sum_{k=1}^{m} \frac{(-1)^k}{k} \sum_{\substack{j_1+\dots+j_k=m \\ j_1,\dots,j_k \ge 1}} f_{j_1} \dots f_{j_k} \\ &= -f_m + (\text{a polynomial in } f_1, \dots, f_{m-1}). \end{aligned}$$

Hence

$$\begin{aligned} \partial\bar{\partial}\Phi &= \alpha \frac{e^L}{L^2} + \sum_{j=1}^{\infty} f'_j \cdot j(j-1)\, L^{j-2} e^L \\ &= \frac{\alpha}{L^2}\Big[1 + \sum_{j=1}^{\infty} f''_j L^j\Big], \end{aligned}$$

where

$$\begin{aligned} f''_m &= \frac{1}{m!} + \sum_{j=1}^{m} \frac{j(j-1) f'_j}{(m-j)!\alpha} \\ &= -\frac{m(m-1)}{\alpha} f_m + (\text{a polynomial in } f_1, \dots, f_{m-1}). \end{aligned}$$

Consequently,

$$w = e^{-\Phi} \det[\partial\bar{\partial}\Phi] = \alpha L^{\alpha-2}\Big[1 + \sum_{j=1}^{\infty} w_j L^j\Big],$$

where

$$\begin{aligned} w_m &= \sum_{j=0}^{m} f_{m-j} f''_j \qquad\qquad (f''_0 := 1) \\ &= \Big[1 - \frac{m(m-1)}{\alpha}\Big] f_m + (\text{a polynomial in } f_1, \dots, f_{m-1}). \end{aligned}$$

Thus by (11),

$$\begin{aligned} c_m(w) &= \alpha \sum_{j=0}^{\infty} w_j \frac{(\alpha-2+j)!}{(m+1)^{\alpha+j-1}} \qquad\qquad (w_0 := 1) \\ &= \frac{(\alpha-2)!\alpha}{(m+1)^{\alpha-1}} \Big[1 + \sum_{j=1}^{\infty} \frac{w'_j}{(m+1)^j}\Big], \end{aligned}$$

where

$$\begin{aligned} w'_j &= \frac{(\alpha+j-2)!}{(\alpha-2)!} w_j = (\alpha-1)_j w_j \\ &= (\alpha-1)_j \Big[1 - \frac{j(j-1)}{\alpha}\Big] f_j + (\text{a polynomial in } f_1, \dots, f_{j-1}). \end{aligned}$$

(Here $(x)_j := x(x+1)\dots(x+j-1)$ is the Pochhammer symbol.) Taking reciprocals gives

$$\frac{1}{c_m(w)} = \frac{(m+1)^{\alpha-1}}{(\alpha-2)!\alpha}\Big[1+\sum_{j=1}^{\infty}\frac{w_j''}{(m+1)^j}\Big],$$

where

$$\begin{aligned} w_n'' &= \sum_{k=1}^{n}(-1)^k \sum_{\substack{j_1+\dots j_k=n\\ j_1,\dots,j_k\ge 1}} w_{j_1}'\dots w_{j_k}' \\ &= -w_n' + (\text{a polynomial in } w_1',\dots,w_{n-1}') \\ &= -\Big[1-\frac{n(n-1)}{\alpha}\Big](\alpha-1)_n f_n + (\text{a polynomial in } f_1,\dots,f_{n-1}). \end{aligned}$$

Substituting this into (9), we get formally

$$K_w(z) = \frac{\alpha-1}{\alpha!\pi}\sum_{j=0}^{\infty} w_j'' \sum_{m=0}^{\infty}\frac{t^m}{(m+1)^{j+1-\alpha}}. \qquad (w_0'' := 1)$$

Here the double series on the right-hand side, strictly speaking, need not converge in general, but is meaningful in the sense that as j increases, the summands have weaker and weaker singularities at $t=1$. This is immediate from the standard power series estimates

$$g(t) = \sum_{j=1}^{\infty} g_j t^j, \quad g_j = O(j^{-m}),\ m\ge 2 \qquad \Longrightarrow \qquad g\in C^{m-2}(\overline{\mathbf{D}}).$$

Recall now that for $\operatorname{Re} v>0$ and $s\in\mathbf{C}$, the function

$$\Phi(t,s,v) := \sum_{m=0}^{\infty}\frac{t^m}{(m+v)^s}$$

is known as Lerch's transcendental function. Thus

$$K_w(z) = \frac{\alpha-1}{\alpha!\pi}\sum_{j=0}^{\infty} w_j''\ \Phi(t, j+1-\alpha, 1).$$

It is also known (cf. [BE], §1.11) that for $s\ne 1,2,\dots$,

$$\begin{aligned} \Phi(t,s,1) &= \frac{1}{t}\Big[(-s)!L^{s-1} + \sum_{j=0}^{\infty}\frac{(-1)^j\zeta(s-j)}{j!}L^j\Big], \qquad L\equiv \log\frac{1}{t} \\ &= \frac{1}{t}\Big[(-s)!L^{s-1} + C^\infty(\overline{\mathbf{D}}\setminus\{0\})\Big]; \end{aligned}$$

while for $s = 1, 2, \dots$,

$$\begin{aligned}\Phi(t,s,1) &= \frac{1}{t}\Big[\frac{(-1)^s}{(s-1)!}L^{s-1}\log L + \sum_{k=0}^{\infty}\frac{(-1)^k\zeta(s-k)}{k!}L^k\Big] \\ &= \frac{1}{t}\Big[\frac{(-1)^s}{(s-1)!}L^{s-1}\log L + C^\infty(\overline{\mathbf{D}}\setminus\{0\})\Big],\end{aligned}$$

where for $k = s-1$, one should substitute $1 + \frac{1}{2} + \dots + \frac{1}{s-1}$ for $\zeta(1)$.

Note that $\frac{1}{t} = e^L$. Consequently, for $\alpha \notin \mathbf{Z}$,

$$e^{-\Phi}K_w \cdot e^{-L} = \frac{\alpha-1}{\alpha\pi}\sum_{j=0}^{\infty} w_j''' L^j + L^\alpha \cdot C^\infty(\overline{\mathbf{D}}\setminus\{0\}),$$

where

$$\begin{aligned} w_k''' &= \sum_{j=0}^{k} f_{k-j}\,\frac{(\alpha-j-1)!}{(\alpha-1)!}\,w_j'' \\ &= \sum_{j=0}^{k}\frac{f_{k-j}w_j''}{(\alpha-j)_j} \\ &= f_k\cdot\underbrace{\Big[1-\frac{(\alpha-1)_k}{(\alpha-k)_k}\Big(1-\frac{k(k-1)}{\alpha}\Big)\Big]}_{=:\,\kappa_k} + (\text{a polynomial in } f_1,\dots,f_{k-1}) \end{aligned} \tag{13}$$

if $k \geq 1$. For $k = 0$ we get $w_0''' = f_0 = 1$, showing that $e^{-\Phi}K_w = \frac{\alpha-1}{\alpha\pi}$ for $t = 1$. In order that $e^{-\Phi}K_w$ be constant, we thus must have

$$w_k''' = \frac{(-1)^k}{k!}, \qquad \forall k = 1, 2, \dots. \tag{14}$$

If $\kappa_k \neq 0\ \forall k \geq 1$, then we can recursively solve (14) for $f_1, f_2, \dots$. Tracing back it is easily seen that the whole argument is reversible and thus (12) produces a function Φ for which

$$K_w = \frac{\alpha-1}{\alpha\pi}e^\Phi \qquad \text{mod } C^\infty(\overline{\mathbf{D}}). \tag{15}$$

For $\alpha \in \mathbf{Z}$, we get similarly

$$\frac{\alpha\pi}{\alpha-1}e^{-\Phi}K_w e^{-L} = \sum_{j=0}^{\alpha-1} w_j''' L^j + \sum_{j=\alpha}^{\infty} w_j''' L^j \log L + L^\alpha \cdot C^\infty(\overline{\mathbf{D}}\setminus\{0\}),$$

where w_j''' is given again by (13) for $j = 0, \dots, \alpha-1$, while for $j \geq \alpha$

$$w_j''' = \underbrace{\frac{(-1)^{j+1-\alpha}}{(j-\alpha)!(\alpha-1)!}\Big(1-\frac{j(j-1)}{\alpha}\Big)} + (\text{a polynomial in } f_1,\dots,f_{j-1}).$$

Here the underbraced expression never vanishes, except when $j = \alpha = 2$. Thus if $\kappa_1, \dots, \kappa_{\alpha-1} \neq 0$ and $\alpha \geq 3$, we can again recursively solve (14) for $f_1, f_2, \dots$ and arrive at a function Φ satisfying (15).

Unfortunately, it turns out that there is a stumbling block: the coefficients κ_1 and κ_2 always vanish. Indeed,

$$\kappa_1 = 1 - \frac{(\alpha-1)}{(\alpha-1)}(1-0) = 0,$$
$$\kappa_2 = 1 - \frac{(\alpha-1)\alpha}{(\alpha-1)(\alpha-2)}\Big(1 - \frac{2}{\alpha}\Big) = 0.$$

A brief computation also shows that, regardless of the values of f_1, f_2, always

$$w_1''' = -1, \qquad w_2''' = 1/2.$$

Thus the equations (14) are always fulfilled for $k = 1, 2$, and we arrive at the following corollary.

Corollary. *For any $\alpha > 1$, there exists an infinite family of functions Φ on $\mathbf{D}$ (with different boundary behaviours) such that $e^{-\Phi(z)} = (1-|z|^2)^\alpha e^{C^\infty(\overline{\mathbf{D}})}$ and*

$$K_{e^{-\Phi}\det[\partial\overline{\partial}\Phi]} = \frac{\alpha-1}{\alpha\pi} \cdot e^{\Phi} + C^\infty(\overline{\mathbf{D}}).$$

Proof. Let m be the largest index for which $\kappa_m = 0$; such index exists since $|\kappa_j| \sim j^{2\alpha} \to \infty$ as $j \to \infty$. Let $g_1, g_2, \dots$ be the Taylor coefficients (at $x = 0$) of the function $(\frac{1-e^{-x}}{x})^\alpha$; that is (here still $L = \log\frac{1}{t}$),

$$L^\alpha \sum_{j=0}^{\infty} g_j L^j = (1-t)^\alpha.$$

Set $f_j := g_j$ for $j = 1, \dots, m-1$. Since we know from (3) that $e^{-\Phi} = (1-t)^\alpha$ is balanced, it follows that the equations (14) for $1 \leq k \leq m-1$ must be satisfied for these values of f_j, while the equation (14) for $k = m$ must have at least the solution $f_m = g_m$. Since $\kappa_m = 0$, (14) for $k = m$ must in fact be satisfied for any value of f_m. As $\kappa_j \neq 0$ for $j > m$, once we choose f_m, the unknowns $f_{m+1}, f_{m+2}, \dots$ can then be solved uniquely from the equations (14) for $k = m+1, m+2, \dots$. Thus we have a family of solutions of (15) parameterized by $f_m \in \mathbf{R}$. □

Example. Let $e^{-\Phi} = (\log\frac{1}{t})^\alpha \equiv L^\alpha$, and let Φ' be obtained from Φ by adjusting it in a small neighbourhood of $t := |z|^2 = 0$ so as to make it smooth on $\mathbf{D}$. Since $K_w - K_{w'} \in C^\infty(\overline{\mathbf{D}})$ when $w - w'$ is supported in a compact subset of $\mathbf{D}$, we see that $K_{w'}$, $w' := e^{-\Phi'}\partial\overline{\partial}\Phi'$, will differ by a term smooth up to the boundary from K_w, where

$$w := e^{-\Phi}\partial\overline{\partial}\Phi = \alpha L^{\alpha-2} e^L.$$

Now by (11) $c_k(w) := \frac{\alpha!}{\alpha-1} k^{1-\alpha}$, whence

$$\begin{aligned}
K_w(z) &= \frac{\alpha-1}{\alpha!\pi} \sum_{k=0}^{\infty} k^{\alpha-1} t^k \\
&= \frac{\alpha-1}{\alpha\pi} t\, \Phi(t, 1-\alpha, 1) \\
&= \frac{\alpha-1}{\alpha\pi} \Big[L^{-\alpha} + \sum_{j=0}^{\infty} \frac{\zeta(1-\alpha-j)(-1)^j}{j!(\alpha-1)!} L^j \Big] \\
&= \frac{\alpha-1}{\alpha\pi} e^{\Phi'} + C^\infty(\overline{\mathbf{D}}),
\end{aligned}$$

for any $\alpha > 1$. Thus Φ' is a solution to (15) different from (3). □

3. Radial balanced metrics on $\mathbf{B}^n$

The case of the unit ball $\mathbf{B}^n$ is susceptible to the same treatment as for the disc. Namely, for any radial weight $w(z) = F(\|z\|^2)$, the monomials z^ν (with ν a multiindex) form an orthogonal basis in $L^2_{\mathrm{hol}}(\mathbf{B}^n, w)$, with norms

$$\|z^\nu\|_w^2 = \frac{\pi^n \nu!}{(|\nu|+n-1)!} c_{|\nu|}(w), \qquad \text{where } c_m(w) := \int_0^1 F(t)\, t^{n+m-1}\, dt;$$

and

$$K_w(z) = \frac{1}{\pi^n} \sum_{m=0}^{\infty} \frac{t^m}{m!} \frac{\Gamma(m+n)}{c_m(w)}, \qquad t := \|z\|^2.$$

Thus starting again with the ansatz

$$e^{-\Phi} = L^\alpha \sum_{j=0}^{\infty} L^j f_j, \qquad f_0 = 1,\ \alpha > n,\ L = \log\frac{1}{t},$$

we get in turn

$$\begin{aligned}
&\Phi = -\alpha \log L + \sum_{j=1}^{\infty} f'_j L^j, \\
&\det[\partial\overline{\partial}\Phi] = e^{nL} \frac{\alpha^n}{L^{n+1}} \Big[1 + \sum_{j=1}^{\infty} f''_j L^j \Big], \\
&w = e^{-\Phi} \det[\partial\overline{\partial}\Phi] = \alpha^n L^{\alpha-n-1} \Big[1 + \sum_{j=1}^{\infty} w_j L^j \Big], \\
&c_m(w) = \frac{\alpha^n \Gamma(\alpha-n)}{(m+n)^{\alpha-n}} \sum_{j=0}^{\infty} \frac{w'_j}{(m+n)^j}, \\
&\frac{1}{c_m(w)} = \frac{(m+n)^{\alpha-n}}{\alpha^n \Gamma(\alpha-n)} \Big[1 + \sum_{j=0}^{\infty} \frac{w''_j}{(m+n)^j} \Big], \\
&K_w(z) = \frac{1}{\pi^n \alpha^n \Gamma(\alpha-n)} \sum_{j=0}^{\infty} w''_j \Big(\frac{d}{dt} \Big)^{n-1} \Phi(t, j+n-\alpha, 1),
\end{aligned}$$

where

$$\begin{aligned}
f'_m &= \sum_{k=1}^{m} \frac{(-1)^k}{k} \sum_{\substack{j_1+\dots+j_k=m,\\ j_1,\dots,j_k\geq 1}} f_{j_1}\dots f_{j_k}\\
&= -f_m + (\text{a polynomial in } f_1,\dots,f_{m-1});\\
f''_m &= \sum_{\substack{k+l+j_1+\dots+j_{n-1}=m,\\ k,l,j_1,\dots,j_{n-1}\geq 0}} \frac{n^k}{k!} s_l r_{j_1}\dots r_{j_{n-1}},\\
&\qquad\qquad r_j := \frac{-j}{\alpha} f'_j,\ r_0 := 1,\ \ s_j := \frac{j(j-1)}{\alpha} f'_j,\ s_0 := 1,\\
&= \left[1 - \frac{m(m-n)}{\alpha}\right] f_m + (\text{a polynomial in } f_1,\dots,f_{m-1});\\
w_m &= \sum_{j=0}^{m} f_{m-j} f''_j\\
&= \left[1 - \frac{m(m-n)}{\alpha}\right] f_m + (\text{a polynomial in } f_1,\dots,f_{m-1});\\
w'_m &= (\alpha-n)_m w_m;\\
w''_m &= \sum_{k=1}^{m} (-1)^k \sum_{\substack{j_1+\dots j_k=m,\\ j_1,\dots,j_m\geq 1}} w'_{j_1}\dots w'_{j_k}\\
&= -(\alpha-n)_m \left[1 - \frac{m(m-n)}{\alpha}\right] f_m + (\text{a polynomial in } f_1,\dots,f_{m-1}).
\end{aligned}$$

Now from the formulas in the preceding section for the singularity of $\mathbf{\Phi}$ at $t=1$, it is not difficult to compute that for $j+n-\alpha \neq 1,2,\dots,$

$$\left(\frac{d}{dt}\right)^{n-1} \mathbf{\Phi}(t, j+n-\alpha, 1) = e^{nL}\left[(\alpha-n-j)! \sum_{k=0}^{n-1} q_{k,j+n-\alpha-1} L^{j+n-\alpha-1-k} + \psi\right]$$

where $\psi \in C^\infty(\overline{\mathbf{B}^n} \setminus \{0\})$ and $q_{k,\nu}$ are the numbers defined by

$$q_{k,\nu} := (-1)^{n-1} e_{n-1-k}(1,\dots,n-1) \frac{\nu!}{(\nu-k)!},$$

where $e_j(x_1,\dots,x_{n-1})$ is the elementary symmetric polynomial (i.e. the coefficient at y^j in $\prod_{l=1}^{n-1}(1+x_l y)$). Similarly for $j+n-\alpha = 1,2,\dots$ (when some log-terms appear). Thus for $\alpha \notin \mathbf{Z}$,

$$e^{-\Phi} K_w e^{-nL} = \frac{\Gamma(\alpha)}{\pi^n \alpha^n \Gamma(\alpha-n)} \sum_{j=0}^{\infty} w'''_j L^j + L^\alpha \cdot C^\infty(\overline{\mathbf{B}^n} \setminus \{0\})$$

and similarly for $\alpha \in \mathbf{Z}$ (with some log-terms), where

$$w_m''' = \sum_{\substack{j,i,l\geq 0,\ l\leq n-1,\\ j+i+l=m}} f_i \frac{\Gamma(\alpha-d-j+1)}{\Gamma(\alpha)} q_{n-1-l,j+n-\alpha-1} w_j'' \qquad (w_0'' := 1)$$

$$= \underbrace{\left[1 - \frac{(\alpha-n)_m}{(\alpha-m)_m}\left(1 - \frac{m(m-n)}{\alpha}\right)\right]}_{=:\,\kappa_m} f_m + (\text{a polynomial in } f_1,\dots,f_{m-1}).$$

For $m = 0$ this gives $w_0''' = 1$, showing that $e^{-\Phi}K_w = \frac{\Gamma(\alpha)}{\pi^n\alpha^n\Gamma(\alpha-n)}$ for $t = 1$. In order that $e^{-\Phi}K_w$ be constant, we thus must have

$$w_k''' = \frac{(-n)^k}{k!} \qquad \forall k \geq 1. \tag{16}$$

Consequently, as long as $\kappa_k \neq 0$, we can recursively solve these equations and obtain a solution Φ which is "almost balanced" in the sense that K_w has the same boundary singularity as $\frac{\Gamma(\alpha)}{\pi^n\alpha^n\Gamma(\alpha-n)}e^{\Phi}$.

This time it turns out that, however, $\kappa_k = 0$ for $k = n, n+1$, and again (16) is always fulfilled for these two values of k. Hence we arrive at the same corollary as for the disc.

Corollary. *For any $\alpha > n$, there exists an infinite family of functions Φ on $\mathbf{B}^n$ (with different boundary behaviours) such that $e^{-\Phi} = (1-\|z\|^2)^\alpha e^{C^\infty(\overline{\mathbf{B}^n})}$ and*

$$K_{e^{-\Phi}\det[\partial\bar\partial\Phi]} = \frac{\Gamma(\alpha)}{(\alpha\pi)^n\Gamma(\alpha-n)} \cdot e^{\Phi} + C^\infty(\overline{\mathbf{B}^n}).$$

4. Hypothetical consequences of existence and uniqueness of balanced metrics

The result in the previous two sections raises a lot of questions. First of all, it is unclear whether the situation we have encountered prevails also for general smoothly bounded strictly pseudoconvex domains in $\mathbf{C}^n$: the above result for $\mathbf{D}$ and $\mathbf{B}^n$ could be just an anomaly caused by "too much symmetry" of these domains. For domains with real-analytic boundaries, it should in principle be possible to carry out a similar analysis using explicit formulas for the boundary singularity of K_w (i.e. for the jets at a boundary point of the functions a', b' in (8)) provided by Kashiwara's microlocal description of the Bergman kernel; however, the resulting formulas will probably be pretty complicated.

Also, in our approach we have always looked only at the boundary singularities, so it by no means follows that we arrive at a genuine balanced metric (i.e. without the smooth error term as in (15)). In conclusion, it is thus still unclear whether there exists a balanced metric on any smoothly bounded strictly pseudoconvex domain in $\mathbf{C}^n$; and the uniqueness of such metrics remains open even on the unit disc. Nevertheless, let us conclude this paper by a brief speculation on the consequences which would follow if the existence and uniqueness of balanced metrics could be established.

Thus, from now on let us assume that the following conjecture holds.

Conjecture. *On each smoothly bounded strictly pseudoconvex domain $\Omega \subset \mathbf{C}^n$ and for each fixed $\alpha > n$, there exists a unique smooth strictly-PSH function $\Phi = \Phi_{\Omega,\alpha}$ on Ω such that*

- $K_{e^{-\Phi}\det[\partial\overline{\partial}\Phi]} = const \cdot e^{\Phi}$, *i.e.* Φ *is balanced;*
- $e^{-\Phi}/u^\alpha \to 1$ *at* $\partial\Omega$, *where* $u = u_\Omega$ *is the solution to the Monge-Ampère equation* $J[u] = 1$ *on* Ω.

The second condition is just a holomorphically-invariant version of saying that $e^{-\Phi/\alpha}$ should be commensurable to a defining function.

Our first observation is the biholomorphic invariance of balanced metrics: namely, assume that $f : \Omega' \to \Omega$ is a biholomorphic map, and let $\Phi = \Phi_{\Omega,\alpha}$. Set

$$\Phi' := \Phi \circ f + \frac{2\alpha}{n+1} \log |\operatorname{Jac} f|,$$

where $\operatorname{Jac} f$ stands for the complex Jacobian of f. In terms of the weights $w = e^{-\Phi}\det[\partial\overline{\partial}\Phi]$, this becomes

$$w' = w \circ f \cdot |\operatorname{Jac} f|^{2-2\alpha/(n+1)}.$$

Using the standard transformation formula $K_{w\circ f} = K_w \circ f \cdot |\operatorname{Jac} f|^2$ (easily proved by change of variables in integration), this implies

$$K_{w'} = K_w \circ f \cdot |\operatorname{Jac} f|^{2\alpha/(n+1)}.$$

As $e^{\Phi'} = e^{\Phi} \circ f \cdot |\operatorname{Jac} f|^{2\alpha/(n+1)}$, we thus see that Φ' is balanced. On the other hand, from the transformation formula for the solution of the Monge-Ampère equation $u' = u \circ f \cdot |\operatorname{Jac} f|^{-2/(n+1)}$ it follows that $e^{-\Phi'}/u'^\alpha = (e^{-\Phi}/u^\alpha) \circ f \to 1$ at the boundary. (Recall that f extends continuously to the boundary by Fefferman's theorem.) Consequently, $\Phi' = \Phi_{\Omega',\alpha}$.

Recall that a domain functional $\Omega \mapsto F_\Omega$ (i.e. a mapping assigning to a domain a function on it) is said to obey transformation law of weight $r \in \mathbf{R}$ (or simply to *be of weight* r) if for any biholomorphic map $f : \Omega' \to \Omega$

$$F_{\Omega'} = (F_\Omega \circ f) \cdot |\operatorname{Jac} f|^{2r/(n+1)}.$$

Thus our finding means that $\Omega \mapsto e^{\Phi_{\Omega,\alpha}}$ is a domain functional of weight α:

$$e^{\Phi'} = e^{\Phi} \circ f \cdot |\operatorname{Jac} f|^{2\alpha/(n+1)}.$$

In particular, for each $\alpha > n$, the balanced metric

$$g^{(\alpha)}_{i\overline{j}} := \frac{\partial^2 \Phi_{\Omega,\alpha}}{\partial z_i \partial \overline{z}_j}$$

is invariant under biholomorphic maps.

Recall furthermore that there is a standard procedure for fabricating new domain functionals from old ones: namely, if F_Ω is a domain functional of any weight $r \in \mathbf{R}$ such that F_Ω is never zero, then

$$\det[\partial\overline{\partial}\log F_\Omega]$$

is always a domain functional of weight $n+1$; in particular, if $r \neq 0$, then

$$\beta_F := F^{-(n+1)/r} \det[\partial\overline{\partial}\log F]$$

is a domain functional of weight 0, i.e. a biholomorphic invariant. Examples of domain functionals of this kind include the *Bergman invariant*, obtained upon taking for F the unweighted Bergman kernel $K_1 =: K$ (which is a domain functional of weight $n+1$):

$$\beta_K = K^{-1} \det[\partial\overline{\partial}\log K];$$

or the somewhat less familiar "Szegö invariant" obtained upon taking for F the invariantly defined Szegö kernel K_{Sz}, which is a domain functional of weight n:

$$\beta_{K_{\mathrm{Sz}}} = K_{\mathrm{Sz}}^{-(n+1)/n} \det[\partial\overline{\partial}\log K_{\mathrm{Sz}}].$$

We can also apply this to $F = u$, the solution of the Monge-Ampère equation, which is a domain functional of weight -1; however, now the corresponding invariant is rather trivial, since

$$\beta_u = u^{n+1} \det[\partial\overline{\partial}\log u] = (-1)^n J[u] = (-1)^n.$$

However, we do get interesting new invariants from our balanced metrics:

$$\beta_{\Phi,\alpha} := (e^{\Phi/\alpha})^{-n-1} \det[\partial\overline{\partial}\Phi] = (-\alpha)^n \, J[e^{-\Phi/\alpha}].$$

Observe that for $\alpha = n+1$, in particular,

$$\beta_{\Phi,n+1} = e^{-\Phi} \det[\partial\overline{\partial}\Phi] = w,$$

the weight function occurring in the definition of the balanced metric.

For the Bergman and Szegö kernels, there exist various ways of obtaining interesting CR-invariants from suitable "invariant" descriptions of their boundary singularity [Hi],[HK],[HKN]. It is quite conceivable that other invariants of this kind might similarly be obtained by studying the boundary behaviour of the potentials Φ_α of balanced metrics.

Remark. We remark that a very similar phenomenon as in the two Corollaries in Sections 2 and 3 occurs if one tries to solve formally the Monge-Ampère equation $J[u] = 1$ on $\mathbf{B}^n$ within the class of radial functions, i.e. looks for solutions of the form $u = L\sum_{j=0}^{\infty} L^j u_j$, $L := \log\frac{1}{t}$, $t := \|z\|^2$. Namely, there exist infinitely many formal solutions, parameterized by the value of $u_{n+1} \in \mathbf{R}$. Perhaps this indicates that the solutions to the equation (2), i.e. the potentials Φ_α, if they exist, have the same kind of logarithmic boundary singularities as the Bergman kernel or the Monge-Ampère solution (which would hardly be surprising).

REFERENCES

[BE] H. Bateman, A. Erdélyi, *Higher transcendental functions I*, McGraw-Hill, New York - Toronto - London, 1953.

[CGR] M. Cahen, S. Gutt, J. Rawnsley, *Quantization of Kähler manifolds. I: Geometric interpretation of Berezin's quantization*, J. Geom. Physics **7** (1990), 45–62.

[Don] S.K. Donaldson, *Scalar curvature and projective embeddings I*, J. Diff. Geom. **59** (2001), 479–522.

[E1] M. Engliš, *A Forelli-Rudin construction and asymptotics of weighted Bergman kernels*, J. Funct. Anal. **177** (2000), 257–281.

[E2] M. Engliš, *Toeplitz operators and weighted Bergman kernels*, forthcoming.

[Hi] K. Hirachi, *Invariant theory of the Bergman kernel of strictly pseudoconvex domains*, Sugaku Expositions **17** (2004), 151–169.

[HK] K. Hirachi, G. Komatsu, *Invariant theory of the Bergman kernel*, CR-geometry and overdetermined systems (Osaka 1994), Adv. Stud. Pure Math. 25, Math. Soc. Japan, Tokyo, 1997, pp. 167–220.

[HKN] K. Hirachi, G. Komatsu, N. Nakazawa, *CR invariants of weight five in the Bergman kernel*, Adv. Math. **143** (1999), 185–250.

[Ji] S. Ji, *Inequality for distortion function of invertible sheaves on Abelian varieties*, Duke Math. J. **58** (1989), 657–667.

[Ke] G.R. Kempf, *Metrics on invertible sheaves on abelian varieties*, Topics in algebraic geometry (Guanajuato, 1989), Aportaciones Mat. Notas Investigacion 5, Soc. Mat. Mexicana, Mexico, 1992, pp. 107–108.

[Raw] J. Rawnsley, *Coherent states and Kähler manifolds*, Quart. J. Math. Oxford (2) **28** (1977), 403–415.

[Zha] S. Zhang, *Heights and reductions of semi-stable varieties*, Comp. Math. **104** (1996), 77–105.

MATHEMATICS INSTITUTE, SILESIAN UNIVERSITY AT OPAVA, NA RYBNÍČKU 1, 74601 OPAVA, CZECH REPUBLIC and MATHEMATICS INSTITUTE, ŽITNÁ 25, 11567 PRAGUE 1, CZECH REPUBLIC

E-mail address: englis@math.cas.cz