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Geometry of Banach Spaces Predual to H^{∞} and Corona Problem

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Abstract

In this paper we present a new general approach to the multi-dimensional corona problem for the ball and polydisk. We also describe some properties of the Banach spaces predual to the spaces H^{∞} defined on these domains.

1. Corona Problem

1.1. In the present paper we discuss one of the fundamental problems in the theory of uniform algebras known as the *corona problem.* Such a problem was originally raised by S. Kakutani in 1941 and asked whether the open unit disk $\mathbb{D}\subset\mathbb{C}$ is dense in the maximal ideal space of the algebra $H^{\infty}(\mathbb{D})$ of bounded holomorphic functions on $\mathbb{D}$ with the supremum norm. This problem was answered affirmatively by L. Carleson in 1962 [C1]. Let us recall that for a uniform algebra A of continuous functions defined on a Hausdorff topological space X, the maximal ideal space $\mathcal{M}(A)$ is the set of all nonzero homomorphisms $A\to\mathbb{C}$ equipped with the weak*-topology of the dual space A^{*}, known as the *Gelfand topology.* Then $\mathcal{M}(A)$ is a compact Hausdorff space. Any function $f\in A$ can be considered as a function on $\mathcal{M}(A)$ by means of the *Gelfand transform:*

$$\hat{f}(m):=m(f),\quad m\in\mathcal{M}(A).$$

If A separates points on X, then X is naturally embedded into $\mathcal{M}(A)$, $x\mapsto\delta_{x}$, $x\in X$, where δ_{x} is the evaluation functional at x. In this case we identify X with its image in $\mathcal{M}(A)$ and by $\overline{X}\subset\mathcal{M}(A)$ denote the closure of X in $\mathcal{M}(A)$. The set $C(A):=\mathcal{M}(A)\setminus\overline{X}$ is called *corona.* Then the corona problem is to determine whether $C(A)=\emptyset$ for certain algebras A. As it was mentioned above, the celebrated Carleson corona theorem states that $C(A)=\emptyset$ for $A=H^{\infty}(\mathbb{D})$.

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Let us briefly describe further developments in the corona problems for Riemann surfaces.

A number of authors have proved the corona theorem for finite bordered Riemann surfaces, see, e.g., [JM] for the references. We will single out the proof given by Forelli [F] in which he constructed a linear projector p from $H^\infty(\mathbb{D})$ onto the subspace H^∞_G of bounded holomorphic functions invariant with respect to the action of the fundamental group G of a bordered Riemann surface S satisfying

$$p(fg) = p(f)g, \quad f \in H^\infty(\mathbb{D}), \quad g \in H^\infty_G.$$

Using this projector one can easily prove the corona theorem for S. Further developments of this idea were given by Carleson [C2], Jones and Marshall [JM] and the author [Br1], [Br2]. In particular, in [JM] the authors constructed a similar Forelli-type projector for Riemann surfaces S such that critical points of Green's function on S form an interpolating sequence for $H^\infty(S)$, and using this proved the corona theorem for such S. This class of Riemann surfaces contains, e.g., surfaces of the form $\mathbb{C} \setminus E$, where $E \subset \mathbb{R}$ is *homogeneous.* In this way the authors obtained another proof of the result of Carleson [C2]. It was also observed in [JM] that every Riemann surface from the above class is of Widom type, i.e., its topology grows slowly as measured by the Green function. Finally, by using different techniques, in [GJ] the corona theorem was proved for all Denjoy domains, i.e., domains of the form $\overline{\mathbb{C}} \setminus E$ where $E \subset \mathbb{R}$.

There are also examples of (nonplanar) Riemann surfaces for which the corona theorem fails, see, e.g., references in [JM] and [Br1]. However, the general corona problem for planar domains is still open, as is the problem in several variables for the ball and polydisk.

1.2. In this part we consider matrix-valued corona theorems which, in a sense, are nonlinear analogs of the corona theorem. First, let us recall that the corona theorem for a uniform algebra A on a topological space X such that A separates points on X is equivalent to the following statement, see, e.g., [G]:

For every collection $f_1, \dots, f_n \in A$ satisfying the corona condition

$$\max_{1 \le i \le n} |f_i(x)| \ge \delta > 0, \quad \text{for all} \quad x \in X, \tag{1.1}$$

there are functions $g_1, \dots, g_n \in A$ such that

$$\sum_{i=1}^{n} f_i g_i = 1. \tag{1.2}$$

Similarly, the matrix-valued corona problem is formulated as follows:

Let $F = (f_{ij})$, $1 \le i \le k$, $1 \le j \le n$, $k < n$, be a matrix with entries in A such that the family of all minors of order k of F satisfies the corona condition (1.1). Is there an $n \times n$ matrix $\widetilde{F}$ with entries in A whose first k rows coincide with F and such that $\det(\widetilde{F}) = 1$?

Clearly, the necessary condition of the existence of such matrices for all n is the validity of the corona theorem for A. However, it is not sufficient: some topological characteristics of $\mathcal{M}(A)$ do not allow to find such $\widetilde{F}$ for a generic F, see [L].

We will formulate now some positive results in this area for algebras H^∞ defined on Riemann surfaces.

The matrix-valued corona theorem for $H^\infty(\mathbb{D})$ was proved by Tolokonnikov, see, e.g., [Ni]. Later on using the above cited Forelli theorem, the Oka principle for holomorphic vector bundles proved by Grauert [Gr] together with Beurling type theorems for $H^\infty(\mathbb{D})$ he also proved in [T] the matrix-valued corona theorem for bordered Riemann surfaces. Recently we proved, see [Br2], the matrix-valued corona theorem for subdomains D of coverings C of bordered Riemann surfaces such that the embedding $D \hookrightarrow C$ induces an injective homomorphism of the corresponding fundamental groups. In the proof we used a new Forelli-type theorem for H^∞ on Stein manifolds, see [Br1], a generalized Oka principle for certain complex vector bundles on maximal ideal spaces of some algebras of bounded holomorphic functions on Stein manifolds, see [Br2], together with the classical techniques of $H^\infty(\mathbb{D})$. Recently we obtained also the following result, see [Br3]. Suppose that S is a Riemann surface of a finite type (that is, the rank of the first cohomology group $H^1(S,\mathbb{Z})$ is finite). Let C be a *Galois covering* of S. Then the matrix-valued corona theorem is valid for $H^\infty(C)$. Surprisingly, together with some techniques developed in [Br1] and [Br2] we used in the proof some L_2 Kodaira-Nakano vanishing theorems. Also, we conjecture that the matrix-valued corona theorem is valid for any (not necessarily Galois) covering of a Riemann surface of a finite type.

2. Banach Spaces Predual to H^∞

2.1. Our purpose is to present a general approach to the corona problems for H^∞. To this end we first describe the structure of Banach spaces predual to H^∞.

Suppose that M is a complex manifold such that the algebra $H^\infty(M)$ separates points on M. Then there is a complex Banach space $X(M)$ such that $X(M)^* = H^\infty(M)$. Such a space is constructed as follows: it is the minimal closed subspace of the dual space $H^\infty(M)^*$ containing all evaluation functionals δ_m, $m \in M$. It is easy to see that the closed unit ball of $X(M)$ is the closure in $H^\infty(M)^*$ of the set

$$\left\{ \sum_{i=1}^{k} c_{m_i}\delta_{m_i} \ : \ c_i \in \mathbb{C} \ , \ m_i \in M \ , \ k \in \mathbb{N} \ , \ \sum_{i=1}^{k} |c_i| \le 1 \right\}.$$

It is also obvious that the map $\phi_M : M \to X(M)$, $m \mapsto \delta_m$, is injective and holomorphic, that is, for every $f \in H^\infty(M)$ $(= X(M)^*)$ the function $f \circ \phi_M \in H^\infty(M)$. In what follows we identify M with its image $\phi_M(M) \subset X(M)$.

Proposition 2.1 *The family $\{\delta_m\}_{m\in M} \subset X(M)$ consists of linearly independent elements and the space generated by this family is dense in $X(M)$.*

Sketch of the Proof. The first statements follows from the fact that any finite set of points in M forms an interpolating sequence for $H^\infty(M)$. The second statement is obvious. □

Let $\mathcal{P}_2^h[X(M)]$ be the linear space of holomorphic polynomials on $X(M)$ generated by polynomials of the form

$$p(v) := f(v)g(v), \quad f, g \in H^\infty(M).$$

By definition each $f \in H^\infty(M)$ admits a continuous extension to $H^\infty(M)^*$ given by

$$f(\xi) := \xi(f), \quad f \in H^\infty(M),\ \xi \in H^\infty(M)^*.$$

Thus every polynomial from $\mathcal{P}_2^h[X(M)]$ also admits an extension to $H^\infty(M)^*$. (Without loss of generality we assume that $\mathcal{P}_2^h[X(M)]$ is already defined on $H^\infty(M)^*$.) Let 1 be the functional on $H^\infty(M)^*$ corresponding to the constant function 1. By $\mathcal{Z}(M)$ we denote the affine subspace of $H^\infty(M)^*$ defined by

$$\mathcal{Z}(M) := \{\xi \in H^\infty(M)^* \ :\ 1(\xi) = 1\}.$$

(Observe that $M \subset \mathcal{Z}(M) \cap X(M)$ and the closure in the weak*-topology on $H^\infty(M)^*$ of $\mathcal{Z}(M) \cap X(M)$ coincides with $\mathcal{Z}(M)$.)

Finally, for a bounded subset $S \subset H^\infty(M)^*$ by $h_2(S)$ we denote the polynomially convex hull of S with respect to polynomials from $\mathcal{P}_2^h[X(M)]$, that is,

$$\xi \in h_2(S) \iff |p(\xi)| \le \sup_{\mu \in S} |p(\mu)| \quad \text{for all} \quad p \in \mathcal{P}_2^h[X(M)].$$

This definition is well-defined because every $p \in \mathcal{P}_2^h[X(M)]$ is bounded on a bounded subset of $H^\infty(M)^*$ by the Banach-Alaoglu theorem. The one can easily check

Proposition 2.2 *The maximal ideal space $\mathcal{M}(H^\infty(M))$ of $H^\infty(M)$ coincides with the set $\mathcal{Z}(M) \cap h_2(M)$.* □

Let us consider another characterization of $\mathcal{M}(H^\infty(M))$. Let us introduce the norm on $\mathcal{P}_2^h[X(M)]$ by the formula

$$||p|| := \sup_{v \in B} |p(v)|, \quad p \in \mathcal{P}_2^h[X(M)],$$

where B is the unit closed ball in $X(M)$. Let $X_2(M)$ be the minimal closed subspace of the dual space $(\mathcal{P}_2^h[X(M)])^*$ (constructed with respect to this norm) generated by delta functionals δ_v, $v \in X(M)$. One can prove using the Hahn-Banach theorem that the closed unit ball of $X_2(M)$ is the closure in $(\mathcal{P}_2^h[X(M)])^*$ of the set

$$\left\{ \sum_{i=1}^{k} c_{v_i}\delta_{v_i} \ :\ c_i \in \mathbb{C}\ ,\ v_i \in B\ ,\ k \in \mathbb{N}\ ,\ \sum_{i=1}^{k} |c_i| \le 1 \right\}.$$

We identify $X(M)$ with its image in $X_2(M)$ under the map $\phi_{X(M)} : v \mapsto \delta_v$, $v \in X(M)$. Then clearly every element of $\mathcal{P}_2^h[X(M)]$ belongs to the dual space $X_2(M)^*$ and has norm ≤ 1 there. Using the Banach-Alaoglu theorem one can also check that $\mathcal{P}_2^h[X(M)]$ is dense in $X_2(M)^*$ in the weak*-topology. We will call elements of $X_2(M)^*$ *homogeneous polynomials* of degree 2 on $X(M)$. Let V be the minimal closed subspace of $X_2(M)$ containing M $(= \phi_{X(M)}(M))$.

Proposition 2.3 (a) *V is isometric to $X(M)$ and $X_2(M)^*|_V$ is isometric to $H^\infty(M)$.*

(b) *The closure of V in the weak*-topology of $(\mathcal{P}_2^h[X(M)])^*$ is isometric to $H^\infty(M)^*$.*

Proof. (a) Clearly there exists a continuous linear map from $X(M)$ into V. By $\widehat{B}$ we denote the closure of the image under this map of the unit ball in $X(M)$. Let $B(V)$ be the closed unit ball in V. By definition, $\widehat{B} \subset B(V)$. Assume that $\widehat{B}$ is a proper subset of $B(V)$. Then by the Hahn-Banach theorem there is an element $f \in X_2(M)^*$ such that $f(v_0) = 1$ for some $v_0 \in B(V)$, but $\sup_{w\in\widehat{B}} |f(w)| = s < 1$. Let $f' = f|_M$. Then the element $h := f' \cdot 1 \in \mathcal{P}_2^h[X(M)]$ and $h - f$ equals 0 on M. Thus $h = f$ on V. But by definition $||h||_V = s < 1$, a contradiction. Hence, $\widehat{B} = B(V)$ and $X(M)$ is isometric to V. The second part of (a) is then obvious.
(b) By definition, for every $f \in V^*$ there is an element $\widehat{f} \in \mathcal{P}_2^h[X(M)] \subset X_2(M)^*$ such that $\widehat{f}|_V = f$ and $||\widehat{f}||_{X_2(M)} = ||f||_V$. This gives a linear continuous projection $\mathcal{P}_2^h[X(M)] \to V^*$. Its dual map is an embedding $i : V^{**} \hookrightarrow (\mathcal{P}_2^h[X(M)])^*$ continuous in the corresponding weak*-topologies and such that the restriction of i to V coincides with the original embedding $V \hookrightarrow (\mathcal{P}_2^h[X(M)])^*$. This implies the required statement. □

Identifying V with $X(M)$ we have the following decomposition of $X_2(M)^*$.

Let $f \in X_2(M)^*$. By definition, $g := f|_{X(M)} \in H^\infty(M)$. Thus

$$f = g + (f-g), \quad g \in H^\infty(M), \quad (f-g) \in X(M)^\perp, \quad ||g|| \le ||f||, \quad ||f-g|| \le 2||f||.$$

Here $||\cdot||$ is the norm on $X_2(M)^*$. This implies the natural decomposition $X_2(M) := X(M) \oplus X(M)^0$ where $X(M)^0 \subset X_2(M)$ is such that $(X(M)^0)^\perp = H^\infty(M)$. Using this we identify the image of the map $\phi_{X(M)} : X(M) \to X_2(M)$ with the graph $\Gamma_{X(M)}$ of the map $\pi_0 \circ \phi_{X(M)} : X(M) \to X(M)^0$ where $\pi_0 : X_2(M) \to X(M)^0$ is the natural linear projection. It is easy to check that $\Gamma_{X(M)}$ is a complex Banach submanifold of $X_2(M)$ biholomorphic to $X(M)$. Let $\pi : X_2(M) \to X(M)$ be the natural projection. For any $p \in \mathcal{P}_2^h[X(M)]$ by $\widehat{p} \in X_2(M)^*$ we denote the linear functional representing p. It is easy to prove

Proposition 2.4 *$\Gamma_{X(M)}$ is the set of zeros of the family $\{p\circ\pi - \widehat{p} \;:\; p \in \mathcal{P}_2^h[X(M)]\}$ of polynomials of degree 2 on $X_2(M)^*$.* □

Let $\widetilde{\mathcal{Z}}(M) \subset \Gamma_{X(M)}$ denote $\phi_{X(M)}(\mathcal{Z}(M) \cap X(M))$. Then $\widetilde{\mathcal{Z}}(M)$ is a complex submanifold of $\Gamma_{X(M)}$ biholomorphic to $\mathcal{Z}(M) \cap X(M)$. By $\mathrm{cl}(\widetilde{\mathcal{Z}}(M))$ we denote the closure of $\widetilde{\mathcal{Z}}(M)$ in the weak*-topology of $(\mathcal{P}_2^h[X(M)])^*$. (Recall that according to our construction $X_2(M)$ is a closed subspace of $(\mathcal{P}_2^h[X(M)])^*$.) Then we have

Proposition 2.5

$$\mathcal{M}(H^\infty(M)) = \mathrm{cl}(\widetilde{\mathcal{Z}}(M)) \cap \mathcal{Z}(M).$$

Sketch of the Proof. Assume that $\xi \in \mathcal{M}(H^\infty(M))$. Then $\xi \in X(M)^{**} \subset (\mathcal{P}_2^h[X(M)])^*$ by Proposition 2.3, and $p(\pi(\xi)) - \widehat{p}(\xi) = 0$ for all $p \in \mathcal{P}_2^h[X(M)]$ (here we naturally extend $p\circ\pi$ and $\widehat{p}$ to $(\mathcal{P}_2^h[X(M)])^*$ as continuous in the weak*-topology functions). Also, by definition, $\xi \in \mathcal{Z}(M)$. Let $\{\chi_\alpha\} \subset X(M)$ be a bounded net that converges in the weak*-topology to $\pi(\xi) \subset X(M)^{**}$. Passing if necessary to a subnet we may assume without loss of generality that $\{\phi_{X(M)}(\chi_\alpha)\}$ converges to some $\chi \in \mathrm{cl}(\widetilde{\mathcal{Z}}(M))$ in the weak*-topology of $X_2(M)^{**}$ so that $\pi(\xi) = \pi(\chi)$. From here for every $p \in \mathcal{P}_2^h[X(M)]$ we have $\widehat{p}(\xi) = \widehat{p}(\chi)$. Since the weak*-topology on $X_2(M)^{**}$ is generated by elements $\widehat{p}$, we have $\xi = \chi \in \mathrm{cl}(\widetilde{\mathcal{Z}}(M))$. The converse implication of the theorem is obvious. $\square$

2.2. Let $f_1, \dots, f_n \in H^\infty(M)$ be such that

$$\max_{1\le i\le n} ||f_i||_{H^\infty(M)} \le 1 \quad \text{and} \quad \max_{1\le i\le n} |f_i(x)| \ge \delta > 0, \quad \text{for all} \quad x \in M.$$

By $K(f_1, \dots, f_n)$ we denote the set of zeros of $f_1, \dots, f_n$ in $X(M) \cap \mathcal{Z}(M)$. Then from Theorem 2.5 follows easily

Proposition 2.6 *The corona theorem is valid for $H^\infty(M)$ if and only if for every $K(f_1, \dots, f_n)$ as above one has*

$$\mathrm{cl}(\phi_{X(M)}(K(f_1, \dots, f_n))) \cap \mathcal{Z}(M) = \emptyset. \qquad \square$$

Assume now that M stands for the open unit Euclidean ball $\mathbb{B}^N$ or the open unit polydisk $\mathbb{D}^N$ in $\mathbb{C}^N$. The first step in the proof of the corona theorem for such M is the following result whose proof is based on Theorem 3.4 of the next section.

Theorem 2.7 *There is a constant $c = c(N)$ such that*

$$||\phi_{X(M)}(v) - v||_{X_2(M)} \ge c \cdot (dist(v, M))^2 \quad \text{for all} \quad v \in X(M).$$

Here

$$dist(v, M) := \inf_{z\in M} ||v - z||_{X(M)}, \quad v \in X(M).$$

In particular, this theorem implies that

$$dist(\phi_{X(M)}(K(f_1, \dots, f_n)), K(f_1, \dots, f_n)) \ge c\delta^2.$$

However, we still don't know whether this inequality leads to the required conclusion of Proposition 2.6. Let us mention that taking into account the latter inequality one can consider the statement of the corona theorem as a *nonlinear analog of the Hahn-Banach separation theorem.* Namely, if we assume for a while that $\phi_{X(M)}$ is a linear map, then the required statement $\mathrm{cl}(\phi_{X(M)}(K(f_1, \dots, f_n))) \cap \mathcal{Z}(M) = \emptyset$ follows from the classical Hahn-Banach theorem. But, unfortunately, $\phi_{X(M)}$ in our case is a holomorphic quadratic map ....

Finally, we will mention that from Theorem 2.7 it follows

Proposition 2.8 *There is a continuous plurisubharmonic function of logarithmic growth g on $X(M)$ equals $-\infty$ on M such that for some constant $\widetilde{c} = \widetilde{c}(N)$*

$$g(v) \ge \widetilde{c} + \ln(dist(v, M)) \quad \text{for all} \quad v \in X(M).$$

Let us consider now the uniform algebra of functions on the closed unit ball $B \subset X(M)$ generated by $H^\infty(M)$ and the function e^g. Let $\mathcal{M}(B)$ be the maximal ideal space of this algebra.

Problem 2.9 *Is it true that B is dense in the Gelfand topology of $\mathcal{M}(B)$?*

The positive answer in this problem leads to the solution of the corona problem for $\mathbb{B}^N$ and $\mathbb{D}^N$. Indeed, the collection of functions $e^g, f_1, \dots, f_n$ with $f_1, \dots, f_n$ as above satisfy condition (1.1) (i.e., the corona condition) on B. Then from the positive answer in Problem 2.9 it follows that there are complex polynomials $p_0, \dots p_n$ in variables $e^g, f_1, \dots, f_n$ such that

$$\left|p_0(v)e^{g(v)} + \sum_{k=1}^{n} p_k(v) f_k(v)\right| > \frac{1}{2} \quad \text{for all} \quad v \in B.$$

Since $M \subset B$ and the restriction of each p_k to M is a holomorphic polynomial in $f_1|_M, \dots f_n|_M$ only, $0 \le k \le n$, the latter inequality implies the solution of the corona problem for the family $f_1, \dots, f_n \in H^\infty(M)$.

Let us consider the general version of Problem 2.9 for B a closed ball in a Banach space V and for an arbitrary logarithmically plurisubharmonic function e^g on V such that g is of logarithmical growth. Then the only result obtained for this general setting is the following

Proposition 2.10 *Suppose that the function e^g is a norm on V. Then B is dense in the Gelfand topology of $\mathcal{M}(B)$.*

Remark 2.11 Of course, Problem 2.9 is also valid for the trivial case of $e^g \equiv const$.

3. Geometry of Banach Spaces Predual to H^∞

3.1. Let M be a Caratheodory hyperbolic complex manifold of *bounded geometry*, i.e., there exists a constant $C \ge 1$ such that every point $z \in M$ has a neighbourhood U_z admitting a biholomorphism

$$\psi_z : U_z \to \mathbb{B}^N$$

such that

$$\frac{1}{C} d_E(\psi_z(v_1), \psi_z(v_2)) \le d_M(v_1, v_2) \le C d_E(\psi_z(v_1), \psi_z(v_2)) \quad \text{for all} \quad v_1, v_2 \in U_z.$$

Here d_E is the Euclidean distance on $\mathbb{C}^N$ and d_M is the Caratheodory distance on M. Let $S_\epsilon \subset M$ be a *separated ϵ-net* with respect to d_M. This means that for every $z \in M$ there exists $s \in S$ such that $d_M(z, s) < \epsilon$ and for every distinct $s_1, s_2 \in S$, $d_M(s_1, s_2) \ge \epsilon$. Then from the definition of the bounded geometry for a sufficiently small ϵ (depending on C) it follows that

Proposition 3.1 *The restriction $H^\infty(M) \to H^\infty(M)|_{S_\epsilon}$ determines an isometric embedding of $H^\infty(M)$ into $l^\infty(S_\epsilon)$ continuous in the corresponding weak*-topologies.*

Thus this embedding determines a surjective bounded linear map $l^1(S_\epsilon) \to X(M)$ of norm 1 whose dual map coincides with the embedding of the proposition. The δ-functionals of points of S_ϵ in $l^1(S_\epsilon)$ go under this map to the corresponding δ-functionals of points of S_ϵ in $X(M)$. In particular, we obtain

Proposition 3.2 *Given $v \in X(M)$ and $\epsilon > 0$ there is a sequence $\{\alpha_t\}_{t\in S_\epsilon} \subset \mathbb{C}$ such that*

$$v = \sum_{t\in S_\epsilon} \alpha_t \delta_t \quad \text{with} \quad ||v||_{X(M)} \le \sum_{t\in S_\epsilon} |\alpha_t| \le ||v||_{X(M)} + \epsilon.$$

Remark 3.3 (1) An open and interesting question is whether $X(M)$ possesses a *Shauder basis.*
(2) A similar result can be proved for any $X(M)$ with M Caratheodory hyperbolic, where instead of S_ϵ one chooses "more dense" countable subsets of M. (We don't know whether it is possible to take here separated ϵ-nets, as well.)

3.2. In this subsection M stands for $\mathbb{B}^N$ or $\mathbb{D}^N$. Let $B(M)$ be the group of biholomorphic transformations of M. Then each $g \in B(M)$ determines an isometry $g^* : H^\infty(M) \to H^\infty(M)$, $g^*f := f \circ g$, continuous in the weak*-topology of $H^\infty(M)$. This implies existence of a linear isometry $\overline{g} : X(M) \to X(M)$ of the predual space such that $\overline{g}^* = g^*$. Identifying M with a subset of $X(M)$ as in the preceding section, we have $\overline{g}|_M = g$. The group of all such $\overline{g}$ will be also denoted by $B(M)$. Let $z_1, \dots, z_N$ be holomorphic coordinates on $\mathbb{C}^N$. By the same symbols we denote the corresponding elements of $H^\infty(M)$. Let $Z(c) \subset X(M)$, $c = (c_1, \dots, c_N) \in \mathbb{C}^N$, be the set of zeros of the equations $z_1 = c_1, \dots, z_N = c_N$. It is a complex affine subspace of $X(M)$ of codimension N. In the following result 1 is considered as the linear functional on $X(M)$ corresponding to $1 \in H^\infty(M)$.

Theorem 3.4 *Suppose that $v \in X(M)$ and $1(v) \neq 0$. Then for every $c \in 1(v) \cdot M$ there exits $\overline{g} \in B(M)$ such that $\overline{g}(v) \in Z(c)$.*

This result is the main point in the proof of Theorem 2.7.
Sketch of the Proof. We will show how to prove the result for $M = \mathbb{B}^N$. The proof for $\mathbb{D}^N$ follows easily from this case.

Recall that for every point $a \in \mathbb{B}^N$ there is a biholomorphism $\phi_a : \mathbb{B}^N \to \mathbb{B}^N$ satisfying

(1)
$$\phi_a(0) = a, \quad \phi_a(a) = 0;$$

(2)
$$\phi_a = \phi_a^{-1};$$

(3) ϕ_a has the unique fixed point.

This map can be written explicitly by

$$\phi_a(z) = \begin{cases} -z, & a = 0, \\ \dfrac{a - P_a z - (1-|a|^2)^{1/2} Q_a z}{1 - < z, a >}, & a \in \mathbb{B}^N \setminus \{0\} \end{cases}$$

where

$$P_a z = \frac{< z, a >}{< a, a >} a, \quad Q_a z = z - \frac{< z, a >}{< a, a >} a$$

and $< z, a >$ is the inner product on $\mathbb{C}^N$.

Observe that ϕ_a is defined also for $a \in \mathbb{S}^N$, the boundary of $\mathbb{B}^N$, and in this case $\phi_a \equiv a$. Suppose now that $v \in X(M)$ and $1(v) \neq 0$. Using Proposition 3.2 we present v as $\sum_{t \in S_\epsilon} \alpha_t \delta_t$ with $\sum_{t \in S_\epsilon} |\alpha_t| \leq 2||v||_{X(M)}$. Consider the linear map $z := (z_1, \ldots, z_n) : X(\mathbb{B}^N) \to \mathbb{C}^N$. Then

$$z(\overline{\phi}_a(v)) := \sum_{t \in S_\epsilon} \alpha_t z(\phi_a(t)).$$

According to the above condition this series converges uniformly for $a \in \overline{\mathbb{B}}^N$ and therefore $z \circ \overline{\phi}_.(v) : \overline{\mathbb{B}}^N \to \mathbb{C}^N$ is a continuous map. Also, for $a \in \mathbb{S}^N$ we have

$$z(\overline{\phi}_a(v)) = \sum_{t \in S_\epsilon} \alpha_t a = 1(v) \cdot a.$$

Since $1(v) \neq 0$, and $\frac{z \circ \overline{\phi}_.(v)}{1(v)} : \overline{\mathbb{B}}^N \to \mathbb{C}^N$ is identity on the boundary, by the Brower fixed point theorem the image of $z \circ \overline{\phi}_.(v)$ contains $1(v) \cdot \overline{\mathbb{B}}^N$. Therefore for every $c \in 1(v) \cdot \mathbb{B}^N$ there exists $\overline{\phi}_a$ for some $a \in \mathbb{B}^N$ such that $\overline{\phi}_a(v) \in Z(c)$. □

3.3. In this part we describe in more details the structure of the space $X(\mathbb{D}^N)$ predual to $H^\infty(\mathbb{D}^N)$. We will consider the closure $\overline{\mathbb{D}}^N$ of $\mathbb{D}^N$ as an abelian semigroup with multiplication defined for $z^i := (z_1^i, \ldots, z_N^i) \in \overline{\mathbb{D}}^N$, $i = 1, 2$, by

$$z^1 \cdot z^2 := (z_1^1 \cdot z_1^2, \ldots, z_N^1 \cdot z_N^2).$$

Next, for every $\alpha \in \overline{\mathbb{D}}^N$ by K_α we denote the map defined by

$$K_\alpha(z) := \alpha \cdot z, \quad z \in \mathbb{D}^N.$$

Then $K_\alpha^* : H^\infty(\mathbb{D}^N) \to H^\infty(\mathbb{D}^N)$, $K_\alpha^* f := f \circ K_\alpha$, is a linear operator continuous in the weak*-topology of $H^\infty(\mathbb{D}^N)$. Hence there is a linear operator $X(\mathbb{D}^N) \to X(\mathbb{D}^N)$ predual to K_α^* which coincides with K_α on $\mathbb{D}^N \subset X(\mathbb{D}^N)$. We will denote it by the same symbol K_α. Also, the set $K := \{K_\alpha\}_{\alpha \in \overline{\mathbb{D}}^N}$ is an abelian semigroup isomorphic to $\overline{\mathbb{D}}^N$. Observe, that every K_α with $\alpha \in \mathbb{D}^N$ is a compact operator and every K_α with $\alpha \in \mathbb{T}^N$, the Šilov boundary of $\mathbb{D}^N$, is an isometry. Moreover, for a sequence $\{\alpha_n\}_{1 \leq n < \infty} \subset \overline{\mathbb{D}}^N$ convergent to $\alpha \in \overline{\mathbb{D}}^N$ we have

$$\lim_{n \to \infty} K_{\alpha_n}(v) = K_\alpha(v) \quad \text{for every} \quad v \in X(\mathbb{D}^N). \tag{3.1}$$

(For $\alpha \in \mathbb{T}^N$, however, $\{K_{\alpha_n}\}$ does not converge to K_α in the operator norm.)

For every $v \in X(\mathbb{D}^N)$ by $\mathbb{T}_v$ we will denote the orbit of v with respect to the action of the group $\{K_\alpha\}_{\alpha\in\mathbb{T}^N}$. Let us determine the Cauchy integral operator on $\mathbb{T}_v$ (for $z=(z_1,\dots,z_N)\in\mathbb{D}^N$, $\xi=(\xi_1,\dots,\xi_N)\in\mathbb{T}^N$) by the formula

$$\mathcal{K}_z(v) := \left(\frac{1}{2\pi i}\right)^N \int\cdots\int_{\mathbb{T}^N} \frac{K_{\xi_1\cdots\xi_N}(v)}{(\xi_1-z_1)\cdots(\xi_N-z_N)}d\xi_1\cdots d\xi_N. \tag{3.2}$$

Proposition 3.5 *For every $v \in X(\mathbb{D}^N)$ we have*

$$\mathcal{K}_z(v) = K_z(v).$$

Sketch of the Proof. The statement is obvious for every $v := \delta_t$, $t \in \mathbb{D}^N$. Now according to Proposition 3.2 we have

$$v = \sum_{t\in S_\epsilon} \alpha_t \delta_t, \quad ||v||_{X(\mathbb{D}^N)} - \epsilon \le \sum_{t\in S_\epsilon} |\alpha_t|.$$

Hence,

$$\mathcal{K}_z(v) = \sum_{t\in S_\epsilon} \alpha_t \mathcal{K}_z(\delta_t) = \sum_{t\in S_\epsilon} \alpha_t K_z(\delta_t) = K_z(v). \quad \square$$

Proposition 3.6 *Given $v \in X(\mathbb{D}^N)$ the correspondence $z \mapsto \mathcal{K}_z(v)$ determines a continuous map $\psi_v : \overline{\mathbb{D}}^N \to X(\mathbb{D}^N)$ holomorphic on $\mathbb{D}^N$.*

Sketch of the Proof. For every $f \in H^\infty(\mathbb{D}^N)$ by Propositions 3.2, 3.5 and formula (3.2) we have

$$f(\mathcal{K}_z(v)) = \sum_{t\in S_\epsilon} \alpha_t f(zt) \in H^\infty(\mathbb{D}^N).$$

This shows that ψ_v is holomorphic on $\mathbb{D}^N$. The continuity follows easily from formula (3.1). $\square$

Let $z^i = (z_1,\dots,z_{i-1},0,z_{i+1},\dots,z_N) \in \overline{\mathbb{D}}^N$. We study the image of $K_{z^i}(X(\mathbb{D}^N))$. Let $\pi_i : \mathbb{D}^N \to \mathbb{D}^{N-1}$, $\pi_i(z) = z^i$, be the natural projection. Then as before this map gives rise to a bounded linear map $X(\mathbb{D}^N) \to X(\mathbb{D}^N)$ which coincides with π_i on $\mathbb{D}^N \subset X(\mathbb{D}^N)$. (We denote this map also by π_i.) Moreover, $\pi_i(X(\mathbb{D}^N))$ is isometric to $X(\mathbb{D}^{N-1})$ and $\pi_i : X(\mathbb{D}^N) \to \pi_i(X(\mathbb{D}^N))$ is a linear continuous projection. Set $X(\mathbb{D}_i^N) := \pi_i(X(\mathbb{D}^N))$. Clearly we have (for $0 \in \mathbb{C}^N$)

Proposition 3.7

$$K_{z^i}(X(\mathbb{D}^N)) \subset X(\mathbb{D}_i^N) \quad \text{and} \quad \bigcap_{i=1}^N X(\mathbb{D}_i^N) = X_0 := \{c\delta_0 \ : \ c \in \mathbb{C}\} \quad \square$$

Notice that $(\pi_N \circ \cdots \circ \pi_1)(X(\mathbb{D}^N)) = X_0$ and $(\pi_N \circ \cdots \circ \pi_1)(v) = 1(v)\delta_0$.

Proposition 3.8 *Suppose that for some $x, y \in \overline{\mathbb{D}}^N$ there are $v, w \in X(\mathbb{D}^N)$ such that $K_x(v) = K_y(w) \notin X(\mathbb{D}^N)\backslash(\cup_{1\le i\le N} X(\mathbb{D}_i^N))$. Then one of the sets $\psi_v(\overline{\mathbb{D}}^N), \psi_w(\overline{\mathbb{D}}^N)$ contains the other.*

Sketch of the Proof. From the assumption of the proposition we obtain that $x, y \in (\overline{\mathbb{D}}^*)^N$, $\overline{\mathbb{D}}^* := \overline{\mathbb{D}} \setminus \{0\}$, and $\psi_v(x \cdot z) = \psi_w(y \cdot z)$, $z \in \mathbb{D}^N$. Without loss of generality we may assume, e.g., that $y = \alpha \cdot x$ with $\alpha \in \overline{\mathbb{D}}^N$. Then since by the hypothesis $x \cdot \mathbb{D}^N$ is an open polydisk in $\mathbb{D}^N$, we get $\psi_v(z) = \psi_w(\alpha \cdot z)$ for all $z \in \overline{\mathbb{D}}^N$. □

Observe also that for $v \notin \cup_{1 \le i \le N} X(\mathbb{D}_i^N)$ we have

$$\psi_v((\overline{\mathbb{D}}^*)^N) = \psi_v(\overline{\mathbb{D}}^N) \setminus \cup_{1 \le i \le N} X(\mathbb{D}_i^N). \tag{3.3}$$

For every $v \in X(\mathbb{D}^N) \setminus (\cup_{1 \le i \le N} X(\mathbb{D}_i^N))$ by G_v we denote the union of all possible sets $\psi_w((\overline{\mathbb{D}}^*)^N)$ containing v. Then from (3.3) and Proposition 3.8 we get

Proposition 3.9 $X(\mathbb{D}^N) \setminus (\cup_{1 \le i \le N} X(\mathbb{D}_i^N))$ *is the disjoint union of a family of sets* G_v. □

Remark 3.10 The closure $\overline{G}_v \subset X(\mathbb{D}^N)$ of G_v coincides with the image of a holomorphic map $\Omega \to X(\mathbb{D}^N)$; here $\Omega \subset \mathbb{C}^N$ is of the form $\Omega = \Omega_1 \times \cdots \Omega_N$, where each $\Omega_i \subset \mathbb{C}$ is either $\mathbb{D}$ or $\overline{\mathbb{D}}$ or $\mathbb{C}$.

An interesting question is *for which* $v \in X(\mathbb{D}^N) \setminus (\cup_{1 \le i \le N} X(\mathbb{D}_i^N))$ *the map* $\psi_v : (\overline{\mathbb{D}}^*)^N \to X(\mathbb{D}^N)$ *is an embedding*? (Observe that for $v \in \cup_{1 \le i \le N} X(\mathbb{D}_i^N)$ the map $\psi_v : \overline{\mathbb{D}}^N \to X(\mathbb{D}^N)$ is always not injective.) Let us formulate a partial answer to this question.

Proposition 3.11 *Suppose that* $v \in X(\mathbb{D}^N) \setminus (\cup_{1 \le i \le N} X(\mathbb{D}_i^N))$ *is either presented as* $\sum_{k=1}^{l} c_k \delta_{x_k}$, $c_k \in \mathbb{C}$, $x_k \in \mathbb{D}^N$, *or does not belong to the union of the spaces* $V_i := \{v \in X(\mathbb{D}^N) \ : \ z_i(v) = 0\}$, $1 \le i \le N$. *Then the map* $\psi_v : (\overline{\mathbb{D}}^*)^N \to X(\mathbb{D}^N)$ *is an embedding.*

Sketch of the Proof. In the first case the proof follows from the fact that any finite subset of $\mathbb{D}^N$ is an interpolating sequence for $H^\infty(\mathbb{D}^N)$. In the second case we consider the linear map $z := (z_1, \ldots, z_N) : X(\mathbb{D}^N) \to \mathbb{C}^N$. Then it is easy to see that $z \circ \psi_v : \overline{\mathbb{D}}^N \to \mathbb{C}^N$ is one-to-one. □

Finally, we will show how to convert $X(\mathbb{D}^N)$ into a Banach algebra over $\mathbb{C}$ (without unit). To this end we introduce multiplication $\cdot : X(\mathbb{D}^N) \times X(\mathbb{D}^N) \to X(\mathbb{D}^N)$ by the formula

$$v \cdot w := \left(\sum_{t \in S_\epsilon} \alpha_t \delta_t \right) \left(\sum_{s \in S_\epsilon} \beta_s \delta_s \right) = \sum_{t,s \in S_\epsilon} \alpha_t \beta_s \delta_{t \cdot s}. \tag{3.4}$$

Here $v = \sum_{t \in S_\epsilon} \alpha_t \delta_t$ and $w = \sum_{s \in S_\epsilon} \beta_s \delta_s$ are some presentations of v and w as in Proposition 3.2. It is easy to check that (3.4) does not depend on the choice of presentations of v and w and so the multiplication is well-defined. Clearly, $v \cdot w = w \cdot v$ and

$$||v \cdot w||_{X(\mathbb{D}^N)} \le \sum_{t,s \in S_\epsilon} |\alpha_t| \cdot |\beta_s| \le ||v||_{X(\mathbb{D}^N)} \cdot ||w||_{X(\mathbb{D}^N)}.$$

All other axioms from the definition of a Banach algebra are trivially hold. Notice also that for every $z \in \mathbb{D}^n$ on has

$$K_z(v) := \delta_z \cdot v, \quad v \in X(\mathbb{D}^N). \tag{3.5}$$

Thus we can define a bounded linear map $K : X(\mathbb{D}^N) \to \mathcal{B}(X(\mathbb{D}^N))$, to the Banach space of bounded linear operators on $X(\mathbb{D}^N)$, by the formula

$$K(v) := \sum_{t \in S_\epsilon} \alpha_t K_t, \tag{3.6}$$

where $v = \sum_{t \in S_\epsilon} \alpha_t \delta_t$ is a presentation of v. (Note that the above formula does not depend on the choice of the presentation.) Then we have

Proposition 3.12 *The map $K : X(\mathbb{D}^N) \to \mathcal{B}(X(\mathbb{D}^N))$ is an isometric embedding and homomorphism of Banach algebras.*

Identifying $X(\mathbb{D}^N)$ with its image under K we can naturally complete it to the Banach algebra with unit, just adding to this algebra the one-dimensional vector subspace of $\mathcal{B}(X(\mathbb{D}^N))$ generated by the identity map I.

Remark 3.13 Consider the minimal closed subspace X in $\mathcal{B}(H^\infty(\mathbb{D}^N))$ generated by operators K_z^*. Then from Proposition 3.12 it follows that X is a Banach algebra isomorphic to $X(\mathbb{D}^N)$. In this way one obtains an inner description of $X(\mathbb{D}^N)$ in terms of $H^\infty(\mathbb{D}^N)$ only. Every $f \in H^\infty(\mathbb{D}^N)$ determines a continuous linear functional on $X \cong X(\mathbb{D}^N)$ defined by the formula $f(x) := [x(f)](1)$, $x \in X$, where $1 \in \mathbb{T}^N$ is the unit. (Observe that every $x(f)$ belongs to $H^\infty(\mathbb{D}^N) \cap C(\mathbb{T}^N)$.)

Now the set of nonzero complex homomorphisms of $X(\mathbb{D}^N)$ is the set of functionals $\{z_1^{\alpha_1} \cdots z_N^{\alpha_N} : \alpha_k \in \mathbb{Z}_+,\ 1 \le k \le N\}$, where $z_1, \dots, z_N$ are the coordinate functionals on $\mathbb{C}^N$. The set of maximal ideals of $X(\mathbb{D}^N)$ is then identified with $(\mathbb{Z}_+)^N$. Next, the Gelfand transform of every $v \in X(\mathbb{D}^N)$ is the bounded function f_v on $(\mathbb{Z}_+)^N$ defined by

$$f_v(\alpha_1, \dots, \alpha_N) := z_1^{\alpha_1}(v) \cdots z_N^{\alpha_N}(v), \quad (\alpha_1, \dots, \alpha_N) \in (\mathbb{Z}_+)^N.$$

The function f_v also satisfies

$$\lim_{(\max \alpha_i) \to \infty} f_v(\alpha_1, \dots, \alpha_N) = 0.$$

The Gelfand transform $F : X(\mathbb{D}^N) \to c_0((\mathbb{Z}_+)^N)$, $v \mapsto f_v$, is an injective homomorphism of Banach algebras. However, the image of $X(\mathbb{D}^N)$ under F is not closed! Also, we have

$$\lim_{n \to \infty} ||v^n||^{1/n} = \sup_\alpha |f_v(\alpha)|.$$

For every $v \in X(\mathbb{D}^N)$ the map $F \circ \psi_v : \overline{\mathbb{D}}^N \to c_0((\mathbb{Z}_+)^N)$ is holomorphic, and the map ψ_v is an embedding if and only if $F \circ \psi_v$ is. Further, let us consider the linear bounded functional l on $l_1((\mathbb{Z}_+)^N)$ defined by the formula:

$$l(\{v_\alpha\}_{\alpha \in (\mathbb{Z}_+)^N}) := \sum_\alpha v_\alpha.$$

In the next result we use the definition of the Marcinkiewicz space $M^{1/N}(\mathbb{T}^N)$. It is a quasi-Banach space of measurable functions f on $\mathbb{T}^N$ (with respect to the Lebesgue measure) satisfying

$$mes\{z \in \mathbb{T}^N \ : \ |f(z)| \geq \lambda\} \leq \frac{C}{\lambda^{1/N}}, \quad \lambda > 0.$$

The optimal for all $\lambda > 0$ constant C in such inequalities is the norm of f. Then we have

Proposition 3.14 *For each $z \in \mathbb{D}^N$, $(F \circ \psi_v)(z) \in l_1((\mathbb{Z}_+)^N)$, the function $l \circ F \circ \psi_v : \mathbb{D}^N \to \mathbb{C}$ is holomorphic and is extended to $\mathbb{T}^N$ as a function from $M^{1/N}(\mathbb{T}^N)$. The linear map $M : X(\mathbb{D}^N) \to M^{1/N}(\mathbb{T}^N)$, $v \mapsto l \circ F \circ \psi_v|_{\mathbb{T}^N}$, is continuous and injective.*

From this result we obtain another description of $X(\mathbb{D}^N)$. Consider the complex space $L^1(\mathbb{T}^N)$. Observe that the Cauchy projector $C : L^2(\mathbb{T}^N) \to H^2(\mathbb{D}^N)$ is extended to $L^1(\mathbb{T}^N)$ and then its image belongs to $M^{1/N}(\mathbb{T}^N)$.

Proposition 3.15 *The image $C(L^1(\mathbb{T}^N))$ equipped with the quotient norm is naturally isometric to $X(\mathbb{D}^N)$.*

(Similar to the above proposition results are valid also for other spaces $X(M)$ with $M \subset \mathbb{C}^N$ a bounded domain.)

Finally, we recall that the *Hadamard product* of two formal power series

$$f_i(z) = \sum_{\alpha \in (\mathbb{Z}_+)^N} a_{i,\alpha} z^\alpha, \quad i = 1, 2,$$

on $\mathbb{C}^N$ where $z^\alpha := z^{\alpha_1} \cdots z^{\alpha_N}$ for $\alpha := (\alpha_1, \dots, \alpha_N)$ is the series

$$\mathcal{H}(f_1, f_2)(z) := \sum_{\alpha \in (\mathbb{Z}_+)^N} (a_{1,\alpha} a_{2,\alpha}) z^\alpha.$$

For formal Fourier series on $\mathbb{T}^N$ the Hadamard product coincides with the convolution of series $*$.

Proposition 3.16 *The operator $M : X(\mathbb{D}^N) \to M^{1/N}(\mathbb{T}^N)$ satisfies*

$$M(v_1 \cdot v_2) = M(v_1) * M(v_2), \quad v_1, v_2 \in X(\mathbb{D}^N).$$

In particular, for $f_1, f_2 \in C(L^1(\mathbb{T}^N))$ we have $f_1 * f_2 \in C(L^1(\mathbb{T}^N))$ and

$$||f_1 * f_2||_{M^{1/N}(\mathbb{T}^N)} \leq ||f_1 * f_2|| \leq ||f_1|| \cdot ||f_2||$$

where $||\cdot||$ is the quotient norm induced from $L^1(\mathbb{T}^N)$ by C. Therefore

$$M : (X(\mathbb{D}^N), \cdot, |\cdot|_{X(\mathbb{D}^N)}) \to (C(L^1(\mathbb{T}^N)), *, ||\cdot||)$$

is an isomorphism of Banach algebras.

Moreover, for $f \in C(L^1(\mathbb{T}^N))$ we have

$$\lim_{n \to \infty} ||\underbrace{f * \cdots * f}_{n \text{ times}}||^{1/n} = \sup_\alpha |a_\alpha|$$

where a_α are Fourier coefficients of f.

Similar results are valid for functions from different Hardy spaces.

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