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POSITIVITY PROPERTIES OF DIRECT IMAGE BUNDLES

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ABSTRACT. This paper is based on a talk given at the conference on "Analytic geometry of Bergman kernels" in Kyoto, december 2005. It reports on some results on the curvature of vector bundles that arise as direct images of line bundles from [3]. It also gives some precisions and applications of these results.

1. INTRODUCTION

This paper is based on a talk given at the conference on "Analytic geometry of Bergman kernels" in Kyoto, december 2005. It reports on some results from [2] and [3] on the curvature of vector bundles that arise as direct images of line bundles. Since complete proofs of the main results already are published, we shall here instead focus on some precisions and applications. The precisions concern when equality holds in our estimates. The applications, or perhaps illustrations, come from two different situations. The first one concerns recent work on extremal Kähler metrics on compact manifolds. We shall then use our results to prove a variant of a recent theorem of Phong and Sturm on approximation of geodesics in the space of Kähler metrics. The second illustration was not given in my talk, but is a discussion of a comment by H Tsuji at the conference. It concerns curvature properties of the Weil-Petersson metric, and another related metric, on Teichmüller space.

The results we are discussing deal with hermitian holomorphic line bundles over the total space, X, of a holomorphic fibration. We separate two different cases, depending on whether the fibers of the fibration are compact or not. These cases are quite different, since in the non-compact case the space of holomorphic sections to our line bundle over a fiber is in general not of finite dimension.

In the non-compact case we will consider only a very special situation: the total space is a product

$$X = U_t \times \Omega_z$$

where U and Ω are domains in $\mathbb{C}^m$ and $\mathbb{C}^n$ respectively, and Ω is assumed to be pseudoconvex. We then get a trivial fibration from the projection of X to U. We will even assume that the line bundle in question is also trivial, but its hermitian structure is not. The metric is then given by a weight function

$$e^{-\phi(t,z)}$$

depending on t, i e varying from fiber to fiber. We assume that ϕ is plurisubharmonic and can be written

$$\phi = \phi_0(z) + \psi(z,t),$$

where ψ is smooth up to the boundary and bounded. This implies that the Bergman spaces

$$A_t^2(\Omega) = \{h \in H(\Omega); \int_\Omega |h|^2 e^{-\phi(t,z)} =: \|h\|_t < \infty\}$$

are all equal as spaces, but their norms depend on t. We thus get a trivial, infinite rank vector bundle, E over U, with hermitian metric $\|\cdot\|_t$. Our first result is that this bundle is positive in the sense of Nakano. In the following theorem, indices j, k denote differentiations with respect to the variables t_j and t_k, while greek indices λ, μ denote differentiation with respect to z_λ and z_μ. By $\sum \Theta_{jk} dt_j \wedge d\bar{t}_k$ we denote $E : s$ curvature form, and $\phi^t = \phi(t, \cdot)$.

Theorem 1.1. *If ϕ is strictly plurisubharmonic, the hermitian bundle $(E, \|\cdot\|_t)$ is strictly positive in the sense of Nakano.*

More precisely, if ϕ is only assumed to be strictly plurisubharmonic with respect to z for each t fixed, if $u_1, \ldots u_m$ are elements in E_t then

$$\text{(1.1)} \qquad \sum (\Theta_{j,k} u_j, u_k) \geq \int_\Omega \sum_{jk} \left(\phi_{jk} - \sum_{\lambda\mu} \phi^{\lambda\mu} \phi_{j\lambda} \bar{\phi}_{k\mu} \right) u_j \bar{u}_k e^{-\phi^t}.$$

This theorem implies that the dual bundle of E, with the dual norm, has negative curvature (in the sense of Griffiths). As a consequence, if $s(t)$ is a holomorphic section to the dual bundle, then

$$\log \|s(t)\|_t$$

is a plurisubharmonic function of t. Applying this to $s(t)$ being a point evaluation at a point in Ω that depends holomorphically on t, we find that the logarithm of the *Bergman kernel*

$$K_t(z,z) = \sup_u |u(z)|^2 / \|u\|_t^2$$

is a plurisubharmonic function in $U \times \Omega$. Once one knows this statement for a product domain a similar statement follows for more general domains. Considering infinite sequences of weight functions ϕ_ν that tend to infinity outside of a pseudoconvex subdomain D we arrive at the following theorem.

Theorem 1.2. *Let D be a pseudoconvex domain in $\mathbb{C}^m \times \mathbb{C}^n$ and let ϕ be plurisubharmonic in D. Put $D_t = \{z; (t,z) \in D\}$ and let K_t be the Bergman kernel of D_t with the weight ϕ. Then*

$$\log K_t(z,z)$$

is plurisubharmonic in D.

This theorem, which generalizes an earlier result of Maitani and Yamaguchi [6], was first proved with a slightly different method in [2]. There it is also explained how Theorem 1.2 can be viewed as a complex version of the real-variable *Prekopa's theorem* from convex analysis.

One question that turns out to be interesting is when equality holds in the estimate of Theorem 1.1, i e for which plurisubharmonic weights the curvature operator Θ may have a null vector. It is clear that this can not happen if ϕ is strictly plurisubharmonic, but quite a lot more can be said. In the next theorem we assume for simplicity that $m = 1$.

Theorem 1.3. *Assume that Ω is relatively compact in $\mathbb{C}^n$ with smooth strictly pseudoconvex boundary. Suppose that the restriction of ϕ to each fiber is strictly plurisubharmonic and smooth up to the boundary. Assume that for $t = 0$ there is some u such that $\Theta u = 0$. Then, if $\omega = \partial_z \bar{\partial}_z \psi$*

$$\frac{\partial}{\partial t} | \omega = 0$$

at $t = 0$. Moreover, if for each t in U there is some nullvector u of Θ, then ψ is pluriharmonic.

The proof of Theorem 1.1 involves an application of Hörmander's L^2-estimate for $\bar{\partial}$, and the proof of Theorem 1.3 depends on an analysis of when equality holds in Hörmander's estimate. Most likely, for bounded domains, equality never holds, and for unbounded domains (with smooth boundary) equality in Hörmander's estimate can hold only if the boundary is Levi-flat. It is easy to see that for unbounded domains with Levi-flat boundary we may get vanishing of curvature in Theorem 1.1, even if the variation of the weight function is not pluriharmonic. This occurs for instance when Ω is all of $\mathbb{C}^n$ and $\phi = \phi_0 \circ T_t + \log |T_t'|^2$, where ϕ_0 is a function of z alone, and T_t is the flow of a holomorphic vector field.

The second case of our results concerns fibrations with compact fibers, $X \xrightarrow{p} U$. The only "convexity type" assumption (to be compared with pseudoconvexity of the total space in Theorem 1.2) is then that the total space X be Kähler. We also assume given a holomorphic hermitian line bundle, L over X, which we assume has a smooth metric with semipositive curvature. The vector bundle, E, we study is now the direct image of $L \otimes K_{X/U}$ where $K_{X/U}$ is the relative canonical bundle of the fibration. Concretely, this means that the fiber E_t consists of the space of global sections to $L \otimes K_{X_t}$. The purpose of tensoring with canonical bundles before taking direct images is twofold: First, together with the semipositivity assumption on L it quarantees that E is indeed a vector bundle. Second, it gives us a canonical way to integrate sections over the fiber, and thus get a metric on E. This is done by defining, for $u = s \otimes \alpha$ (where s is a section to L and α is an $(n, 0)$-form

$$[u, u] = |s|^2 c_n \, \alpha \wedge \bar{\alpha}$$

and putting

$$\|u\|_t^2 = \int_{X_t} [u, u]. \tag{1.2}$$

We then have the following variant of Theorem 1.1.

Theorem 1.4. *Let $X \xrightarrow{p} U$ be a (non singular) Kähler fibration with compact fibers over an open set U in $\mathbb{C}^m$. Let L be a (semi)positive line bundle over X and let $E = p_*(L \otimes K_{X/U})$ be the direct image bundle described above. Then, if E is given the metric (1,2), E is (semi)positive in the sense of Nakano.*

Most likely one can prove a more general theorem, containing theorems 1.1 and 1.4, by considering fibrations having a Kähler metric which is complete on each fiber.

Just like in the planar case, Theorem 1.4 has consequences for the Bergman kernel. Define the Bergman kernel for $(L \otimes K_{X_t}$ by

$$K_t = \sum u_j \wedge \bar{u}_j,$$

if $\{u_j\}$ is an orthonormal basis for E_t. Now K is not a function anymore, but transforms like a metric on $L \otimes K_{X/U}$.

Corollary 1.5. *Assume L is (semi)positive. Then the Bergman kernel defines a (semi)positive metric on $L \otimes K_{X/U}$.*

We can think of the fibration as a way of varying smoothly the manifold X_t. Or, if the fibers are all the same, so that we have a trivial fibration, we can use it to study variations of the line bundle. Finally, even the case when the line bundle does not change with t, but only the metric on it changes, is of interest. This is precisely the situation that arises in the study of variations of Kähler metrics within one (integer) cohomology class that we will discuss in section 3.

The plan of the rest of this paper is as follows. In the next section we shall discuss Theorem 1.1, and indicate the proof of Theorem 1.3. In section 3 we study the case of Theorem 1.4 when the fibration is trivial. Finally in section 4 we apply Theorem 1.4 to the "universal curve" over Teichmüller space, and show how it relates to curvature properties of the Weil-Petersson metric.

Taking advantage of the format of these notes from the proceedings of a conference, some arguments are merely sketched. Hopefully, a more complete version will appear elsewhere.

2. Fibrations of domains in $\mathbb{C}^{1+n}$,

We will explain very briefly the proof of Theorem 1.1 with the aim of indicating the proof of Theorem 1.3. We refer to [3] for full details and for simplicity we assume $m = 1$. The main idea for the proof of Theorem 1.1 is to consider the bundle E as a subbundle of a bundle F, with fibers

$$F_t = L^2(\Omega, e^{-\phi^t}).$$

Using the definition of the Chern connection, which generalizes naturally to bundles of infinite rank, it is not hard to see that the Chern connection form on F equals the operator on F_t given by multiplication by

$$-\partial_t\psi =: -\psi_t dt.$$

Consequently the curvature form of F is given by multiplication with

$$(\partial\bar{\partial})_t\psi.$$

This operator is obviously positive as long as ψ - or equivalently ϕ - is subharmonic in the t-direction. It is also well known that curvature decreases when we pass to a subbundle. The crux of the proof is to show that this loss of positivity is not too big, if ϕ is plurisubharmonic with respect to all variables. By a formula of Griffiths the curvature of E satisfies

$$(\Theta^E u, u) = (\Theta^F u, u) - \|\pi_{E^\perp}(\bar{\partial}_z\psi_t u)\|^2.$$

Here $\pi_{E^\perp}$ is the orthogonal projection on the complement of E in F. Now note that

$$w := \pi_{E^\perp}(\bar{\partial}_z\psi_t u)$$

is the L^2-minimal solution of the $\bar{\partial}$-equation on Ω

$$\bar{\partial}_z w = \bar{\partial}_z\psi_t u.$$

We now apply the Hörmander $\bar{\partial}$-estimate to this equation. (See the proposition below.) As a result we obtain the inequality (1.1) (for $m = 1$).

Let us now study when we have equality in (1.1) for some u. As seen from the discussion above, this question reduces completely to the question when we have equality in Hörmander's estimate - the other steps in the argument are equalities.

Proposition 2.1. *Let Ω be a smoothly bounded strictly pseudoconvex domain and let ϕ be a strictly plurisubharmonic function which is smooth up to the boundary in Ω. Then, if f is a $\bar{\partial}$-closed $(0,1)$-form in Ω and w is the L^2-minimal solution to $\bar{\partial}w = f$, equality holds in Hörmander's estimate*

$$\int |w|^2 e^{-\phi} \leq \int \sum \phi^{\lambda,\mu} f_\lambda \bar{f}_\mu e^{-\phi}. \tag{2.1}$$

only for $f = 0$.

Proof. In the proof it is convenient to think of f as a form of bidegree $(n, 1)$, and w as a form of bidegree $(n, 0)$ (just my multiplying by dz). Moreover we equip Ω with the Kähler metric $\omega = i\partial\bar{\partial}\phi$. Then (2.1) just says that $\|w\| \leq \|f\|$, and amounts to saying that the smallest eigenvalue of the $\bar{\partial}$-Neumann operator, $\bar{\Box} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$, restricted to closed forms is greater than or equal to one. Conversely, if equality holds in (2.1), then f is an eigenform of $\bar{\Box}$ with eigenvalue 1; $\bar{\Box}f = f$. From this, and subelliptic estimates it follows that f is smooth up to the boundary. Moreover f lies in the domain of $\bar\partial^*$ and $w = \bar\partial^* f$. Let $\gamma = *f$, where $*$ denotes the Hodge operator defined by the metric ω, so that $\gamma \wedge \omega = f$. Combining with the fundamental Kodaira-Nakano identity we find ($\mathcal{L}$ is the Levi form of the boundary)

$$\|f\|^2 + \|\bar\partial\gamma\|^2 + \int_{\partial\Omega} \mathcal{L}_{\partial\Omega}(f, f) = \|\bar\partial^* f\|^2,$$

where, by assumption, the last term is again equal to $\|f\|^2$. Hence γ must be a holomorhic form and by the strikt pseudoconvexity f must vanish at the boundary. Thus γ vanishes at the boundary, so γ and f vanish identically. $\square$

Actually, it would probably be enough in the proposition to assume weak pseudoconvexity here; the existence of *one* boundary point of strikt pseudoconvexity is enough to conclude the argument. We have chosen to assume strict pseudoconvexity to avoid problems due to lack of global regularity of the $\bar\partial$-Neumann problem. Notice also that for unbounded domains, we may very well have equality in (2.1). The simplest example is in entire space. Equality then holds precisely when $\gamma = *f$ is holomorphic. When $\phi = |z|^2$ we can take $f = \bar\partial|z|^2$.

Returning to our vector bundle E, we now see that if the curvature of E has some non trivial null vector for $t = 0$, then $\bar\partial_z\psi_t = 0$. Applying ∂_z we find that the t-derivative of ω vanishes, so we have proved the first part of Theorem 1.3. If there are null-vectors for all t it follows that $(\partial\bar\partial)_z\phi$ is independent of t. Moreover, we see from (1.1) that $\psi_{t\bar t} = 0$. Hence $\phi - \phi_0$ is pluriharmonic.

Natural examples of when E has zero curvature even though the weight function changes in a non trivial (i e non pluriharmonic) way come from applying the flow of a holomorphic vector field to the weight. Notice that this fits nicely with the obstructions coming from points of strict pseudoconvexity above: it is intuitively reasonable that such flows do not exist near strictly pseudoconvex boundary points.

3. Variations of Kähler metrics

Here we will discuss the special case of Theorem 1.4 when the fibration is trivial. Let Z be a compact manifold and let $\tilde L$ be a positive line bundle over Z. We put $X = U \times Z$ where U is open in $\mathbb{C}$, and let L be the pull back of $\tilde L$ to X under the projection from X to Z. An arbitrary metric ϕ on L can be written

$$\phi = \phi_0 + \psi(t, z),$$

where ϕ_0 is the pullback of a metric on $\tilde L$ (i e a t-independent metric) and ψ is a *function* on X. The direct image vector bundle

$$E = p_*(L \otimes K_Z)$$

is now the trivial bundle on U with fiber equal to the space of global holomorphic sections to $L \otimes K_Z$. We equip it with the metric

$$\|u\|_t^2 = \int_Z [u,u]_{\phi_0} e^{-\psi(t,\cdot)}.$$

By the same kind of argument as in the beginning of the previous section, we find that the curvature of this metric satisfies

$$(\Theta^E u, u) \geq \int (\psi_{t\bar{t}} - |\bar{\partial}_z \psi_t|^2_\omega)[u,u]_{\phi_0} e^{-\psi}. \tag{3.1}$$

As in the previous section it follows that if u is a null vector for the curvature for, some t, then equality holds in the Hörmander estimate for the $\bar{\partial}$-equation

$$\bar{\partial} w = \bar{\partial}\psi_t \wedge u.$$

To understand when this happens we need a variant of Proposition 2.1.

Proposition 3.1. *Let A be a positive line bundle over a compact manifold Z and let f be a $\bar{\partial}$-closed, A-valued (n,q) form. Give Z the Kähler metric defined by the curvature form of A, ω. Let w be the L^2-minimal solution to the $\bar{\partial}$-equation*

$$\bar{\partial} w = f$$

Then equality holds in the Hörmander estimate

$$\|w\|^2 \leq \frac{1}{q}\|f\|^2$$

if and only if $*f$ is a holomorphic form. (Here $*$ denotes the Hodge operator with respect to ω.)

We now apply this to

$$f = \bar{\partial}\psi_t \wedge u.$$

If u is a null vector for the curvature it follows that $*f = \gamma$ is a holomorphic form. But

$$*f = \delta_V u$$

is the contraction of u with a certain vector field, V, the complex gradient of ψ_t, defined by

$$\delta_V \omega = \bar{\partial}_z \psi_t.$$

Since u and $*f$ are holomorphic, V must be holomorphic (away from the zeros of u and hence everywhere, by Riemann's theorem).

There are now two cases. Either Z has no non trivial holomorphic vector field. Then $\bar{\partial}\psi_t = 0$ and it follows much as in the previous section that

$$\frac{\partial}{\partial t}\omega = 0.$$

If Θ has a null vector for all t it follows that ω does not change.

The next case is when Z may have some holomorphic vector field. Let T_t be the associated holomorphic flow of such a field. Starting from some fixed metric ϕ_0 on L, let ω_0 be its curvature form, and put

$$\omega_t = T_t^*(\omega_0).$$

Then ω_t lies in the same cohomology class as ω_0 and is therefore the curvature of some metric $\phi = \phi_0 + \psi$, where we can take ψ to depend smoothly on t. Such a variation of the metric will give us a bundle E with zero curvature, at least if L is invariant under the flow. Conversely, we claim that if Θ has a null vector for all t we must be in this situation.

To see this we first claim that V depends holomorphically on t. This follows from a rather surprising formula: Let for any function ψ on X

$$c(\psi) := \psi_{t\bar{t}} - |\bar{\partial}_z \psi_t|^2_\omega.$$

Proposition 3.2. *Let V be the complex gradient of ψ_t. Then*

$$V_{\bar{t}} = \frac{\partial}{\partial \bar{t}} V$$

is the complex gradient of $c(\psi)$. In particular, if $c(\psi) = 0$ (or even constant), then V depends holomorphically on t.

We thus have two holomorphic fields on the total space X: V and $\partial/\partial t$. A computation shows that the complex Lie derivative of $\bar{\partial}_z \psi_t$ with respect to $V - \partial/\partial t$ vanishes. From there one gets the next proposition

Proposition 3.3. *Assume that the metric ϕ on L has semipositive curvature and that the restriction of the curvature to each fiber is strictly positive. Then, if the curvature of E has some null vector for each t it follows that*

$$(\partial\bar{\partial})_z \psi$$

is the pull back of some fixed metric form on Z under the flow associated to some holomorphic vector field on Z.

3.1. **Geodesics.** The set of all Kähler metrics on Z whose metric form lies in the cohomology class determined by the Chern class of $\tilde{L}$ is naturally idetified with the space, $\mathcal{K}$ of metrics on $\tilde{L}$ with positive curvature. This is an open subset of an affine space, and its tangent space is a space of functions on Z, see e g [5] and the references there. $\mathcal{K}$ can be given the structure of a Riemannian manifold, by defining the norm of a tangent vector ψ at a point ϕ by

$$||\psi||^2 = \int_Z |\psi|^2 dV_\phi,$$

where $dV_\phi := (i\partial\bar{\partial}\phi)_n$ is the volume element defined by ϕ. It turns out that if

$$t \longrightarrow \phi(t, z)$$

is a path in $\mathcal{K}$ and $\psi = d\phi/dt$ its tangent vector field, then $c(\psi)$ is equal to its geodesic curvature, see [8]. Here we are being a bit sloppy. When we talk about paths we are of course thinking of t as a real variable, but we can also think of t as being complex and the functions independent of $\operatorname{Im} t$. As a matter of fact, it is even more convenient to have t complex, and the path independent of the *argument* of t.

In particular, ϕ is a geodesic if $c(\psi) = 0$. With any path as above we get associated a vector bundle E on U. The inequality (3.1) for its curvature holds even if ϕ does not have positive curvature on the total space, as long as the restriction of ϕ to each copy of Z has positive curvature.

It follows that the bundle E associated to a geodesic has non-negative curvature. Moreover, the curvature is even strictly positive provided that Z has no non trivial holomorphic vector fields. In case Z does have non trivial holomorphic vector fields, the curvature of E is strictly positive unless the geodesic comes from the flow of a holomorphic field.

Let U be an annulus with inner radius 1 and outer radius e. We will consider metrics on E that depend only on $|t|$. By the above, any path in $\mathcal{K}$ with parameter interval $[0, 1]$ gives rise to such a metric on E, and if the path is a geodesic then the metric on E is at least nonnegative i- "mostly" even positive. Conversely, given any such metric on E we get a path of metrics on $\tilde{L} \otimes K_Z$ from the Bergman kernels, K_t, of E_t. By the arguments leading up to Corollary 1.5, these metrics are positive if the metric on E has positive curvature.

What we have described in the previous paragraph is very similar to the starting point of [5], with one exception. In [5] one considers the direct image of L itself, and gives it a metric by integrating against the induced volume element dV_ϕ. Here we tensor with K_Z, which gives a natural way of integrating without choosing a volume element.

We shall now state and prove a theorem corresponding to a result of Phong and Sturm, [7] in our setting. Let ϕ_0 and ϕ_1 be two choices of metrics in $\mathcal{K}$. Let ϕ_0 define a constant metric, $(\cdot, \cdot)_0$ on the restriction of E to the inner boundary of the annulus U, and let ϕ_1 define a metric on the outer boundary. These metrics can be joined by a unique flat metric in the following way (see [7]). Choose an orthonormal basis u_j for the first metric, which also diagonalizes the second metric:

$$(u_j, u_k)_1 = e^{2\lambda_j} \delta_{jk}$$

The metric on E_t equals

$$(u_j, u_k)_{\log |t|} = |t|^{2\lambda_j} \delta_{jk}.$$

Then u_j / t^{λ_j} is a holomorphic unitary frame so E is flat. Let K_t be the Bergman kernel for E_t. By inspection of the formulas, K_t defines a semi-positive metric $\phi(t)$ on $L \otimes K_Z$.

Finally, we replace L by L^k, and denote the corresponding Bergman kernels $K_t^{(k)}$. By the Boutet de Monvel-Tian-Zelditch expansion

$$K_t^{(k)} e^{-k\phi_{0/1}} / k^n = dV_\phi + O(1/k)$$

as k tends to infinity on the inner and outer boundaries. Put

$$K_t^{(k)}/k^n = e^{k\phi(t,k)}.$$

Then $k\phi(t,k)$ is a nonnegative metric on $L^k \otimes K_Z$.

Proposition 3.4. *Let ϕ^* be the supremum of all metrics with semipositive curvature ϕ on L such that*

$$\phi \leq \phi_{0/1}$$

on the inner and outer boundary respectively. Then

$$\lim \phi(t,k) = \phi^*$$

with uniform rate of convergence at most $\log k/k$.

Proof. By the Boutet de Monvel-Tian- Zelditch expansion

$$\phi(t,k) - \phi_{0/1} = O(1/k^2)$$

on the inner and outer boundary. Let ϕ be some metric with semipositive curvature on L which does not exceed ϕ_0 and ϕ_1 on the two parts of the boundary. Let $K_{k\phi}$ be the Bergman kernel determinded by $k\phi$. Then, since the metric on $E^{(k)}$ determined by $k\phi$ has nonnegative curvature it must be bigger than the flat metric determined by the boundary values $\phi_{0/1}$, so the Bergman kernels satisfy

$$K_t^{(k)} \geq K_{k\phi}.$$

Let μ be some fixed volume element on Z and put $\mu = e^{\chi}$, where χ is some metric on the canonical bundle. Using the extremal characterization of the Bergman kernel and a simple variant of the Ohsawa Takegoshi extension theorem we find that

$$K_{k\phi} \geq C e^{k\phi}\mu,$$

with a constant independent of k. Combining we see that

$$K_t^{(k)} \geq C e^{k\phi}\mu,$$

so taking supremum over all choices of ϕ

$$\phi(t,k) \geq \chi/k - \log(k^n)/k + \phi^*.$$

For the converse, note first that

$$\chi(t,k) := \phi(t,k) - \chi/k$$

is a metric on L. If ϕ^0 is an arbitrary, strictly positive metric on L then for a large enough

$$(1 - a/k)\chi(t,k) + a/k\phi^0$$

is also positive. Hence

$$\phi^* \geq (1 - a/k)\phi(t,k) + O(1/k),$$

which completes the proof. □

4. Teichmuller space

This section follows a suggestion made by H Tsuji during the Kyoto conference. Let $\mathcal{T}_g$ denote Teichmüller space, i e the space of all complex structures on a compact 2-manifold of genus $g > 1$, with two structures identified if they are related via a diffeomorphism isotopic to the identity. Each point t in $\mathcal{T}_g$ determines a compact Riemann surface of genus g, and there exists a holomorphic fibration

$$X \xrightarrow{p} \mathcal{T}_g$$

such that the fiber, X_t over each point is precisely the Riemann surface determined by that point (see [1]). The cotangent space of $\mathcal{T}_g$ at t is the space of *quadratic differentials* on X_t, i e the space of global sections to $K^2_{X_t}$. Hence the cotangent bundle of $\mathcal{T}_g$ is the direct image of the square of the relative canonical bundle

$$K^2_{X/\mathcal{T}_g}.$$

This corresponds to the setting of Theorem 1.4, with $L = K_{X/\mathcal{T}_g}$ so we must first discuss positivity of this bundle. Let us accept without proof that X is "locally Kähler", i e that any point in $\mathcal{T}_g$ has a neighbourhood U such that $p^{-1}(U)$ has *some* Kähler metric. It then follows from Corollary 1.5, with L trivial, that the Bergman kernel of $K_{X/\mathcal{T}_g}$ defines a nonnegative metric on this bundle. Actually, it follows from the proof of Theorem 1.4 in [3], that this metric is even strictly positive. Moreover, the space of all global holomorphic one-forms on a Riemann surface of genus at least 1 has no common zero set. (As pointed out to me by Ulf Persson it follows from the Riemann Roch theorem that the space of forms that vanish at any given point has smaller dimension than the space of all forms.) As a consequence, the metric defined by the Bergman kernel is non singular.

Now we can use this metric, and the recipe in Theorem 1.4, to get a metric on $p_*(K^2_{X/\mathcal{T}_g})$. By Theorem 1.4 it has strictly positive curvature and so defines a negatively curved metric on Teichmüller space.

A more popular metric on Teichmüller space is the Weil-Petersson metric. It is defined in the same way as above, but starting from a different metric on $K_{X/\mathcal{T}_g}$: the one given by the Poincaré metric on each fiber. By a theorem of Ahlfors, the Weil-Peterson metric is negatively curved. Here we shall verify that it is at least seminegative. For this we need to verify that the Poincaré metric defines a semipositively curved metric, not just on each fiber, but on all of $K_{X/\mathcal{T}_g}$. This is a special case of a theorem of Brunella, [4]. In our setting it can be proved as follows.

Locally, the fibration $X \xrightarrow{p} \mathcal{T}_g$ lifts to a fibration $\hat{X} \xrightarrow{p} \mathcal{T}_g$, where the fiber of $\hat{X}$ is the universal cover of the fiber of X, i e the disk. We claim that $\hat{X}$ is pseudocomvex. This can be seen by lifting the Bergman kernels on the fibers of X to the fibers of $\hat{X}$. These $(1,1)$-forms are naturally identified with functions on the disk. Each fiber of X is the quotient of the corresponding fiber of $\hat{X}$ under a discrete group, so the lifts of the Bergman kernels are invariant under this group. This implies that the corresponding functions are exhaustive. Since we already know they are plurisubharmonic, it follows that $\hat{X}$ is pseudoconvex. The Poincaré metrics on the fibers of X lift to the Poincaré metric on the disk, which equals the Bergman metric. Then we can apply Theorem 1.2 to see that this metric is semipositive. This implies that the Weil-Peterson metric is seminegative.

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