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Homological Methods for the Economic Equilibrium Existence Problem: Coincidence Theorem and an Analogue of Sperner's Lemma in Nikaido (1959) *

Ken Urai
Graduate School of Economics
Osaka University

Abstract

In this paper, I introduce the theorems in Professor Hukukane Nikaido's work, "Coincidence and some systems of inequalities," published in the Journal of Mathematical Society of Japan, 1959, and note the significance of his mathematical methods on the history and the future of mathematical economics. Nikaido (1959) may be considered a compilation of his works of the 1950's on economic equilibrium existence problems. It also provides, however, his further developments and attempts for mathematical methods in the theory of mathematical economics and an algebraic (algebraic topological) methods based on results of the Victoris homology theory (the earliest kind of Čech-type homology theories). From Nikaido's main mathematical results, an analogue of Sperner's lemma and a coincidence theorem, we may obtain a simple proof for Eilenberg-Montgomery's theorem for finite dimensional cases. We may also utilize such homological methods for many generalizations of fixed point arguments on multivalued mappings in relation to Lefschetz's fixed point theorem.

Keywords : Fixed point theorem, Existence of equilibrium, Čech homology theory, Victoris homology theory, Browder's fixed point theorem, Kakutani's fixed point theorem, Lefschetz's fixed point theorem.

JEL classification: C60; C62; C70; D50

1 Introduction

In this paper, I introduce the theorems in Professor Hukukane Nikaido's work, "Coincidence and some systems of inequalities," published in the Journal of Mathematical Society of Japan, 1959, and note the significance of his mathematical methods on the history and the future of mathematical economics. Nikaido (1959) may be considered a compilation of his works of the 1950's on economic equilibrium existence problems. It

*The manuscript is prepared for the special session of Nikaido Conference at Hitotsubashi University on March 18 and 19, 2006. Contents in Sections 2 - 6, except for the proof of Sperner's lemma (Lemma 4.4), arguments for class $\mathscr{B}$ (Browder type) mappings in Section 5, and several additional figures, have been taken from Chapter 6 of my Ph.D thesis (Urai, 2005).

also provides, however, his further developments and attempts for mathematical methods in the theory of mathematical economics and an algebraic (algebraic topological) methods based on results of the Vietoris homology theory (the earliest kind of Čech-type homology theories). From Nikaido's main mathematical results, an analogue of Sperner's lemma and a coincidence theorem, we may obtain a simple proof for Eilenberg-Montgomery's theorem for finite dimensional cases. We may also utilize such homological methods for many generalizations of fixed point arguments on multivalued mappings in relation to Lefschetz's fixed point theorem.

As is well-known, Professor Nikaido was a great mathematician as well as an outstanding social scientist. He had a special viewpoint on mathematical methods for the social sciences that view mathematics not as a simple tool but as a language. Therefore, for him, mathematical economics is not a simple description of the world using mathematical concepts but a study of the world through the language (or methods) of the mathematician.

With each mathematical theory is associated a different way of analyzing the world. For example, there is an important difference between the differentiable approach (research based on differential calculus) and an approach based merely on set theoretical and/or algebraic methods in mathematical economics. Since the concepts and methods of differential calculus are based on the theory of sets and/or algebra, the former includes *analytic* works that result from seeing the world as a differentiable object, and the latter include *synthetic* attempts or methods to construct models that are more appropriate to describe our real world. The results of the former are always based on the concept of differentiability so that it is more desirable to reexamine them under more primitive concepts, like finiteness, sequences, or limits under the set theoretical and/or algebraic methods.

In this sense, it is always significant for the theory of mathematical economics to use more primitive mathematical concepts together with more general or fundamental mathematical methods. Methods in mathematical economics in the 1950's and 1960's based on rigorous set theoretical arguments and general topology, e.g., Debreu (1959), Nikaido (1968), etc., have, therefore, important meaning for the history of social science as a new basic (fundamental) language for describing the society.

I introduce here some of the most general (and fundamental) theorems of Professor Nikaido from that era, an analogue of Sperner's lemma and a theorem for the coincidence of mappings (Nikaido, 1959; Lemma 1, Theorem 3). The analogue of Sperner's lemma may be considered to represent the essential part of fixed point or coincidence theorems in finite dimensional vector spaces, as does Sperner's lemma. The lemma may be useful as a proof of the theorem on coincidence points of mappings on general compact Hausdorff spaces with or without vector space structure. The result may also be directly used for economic equilibrium problems on general compact Hausdorff spaces. Arguments are based on an abstract homology theory of the Čech-type that is founded on more primitive algebraic concepts than the singular homology theory.

2 Vietoris and Čech Homology Groups

Let X be a compact Hausdorff space. $\mathit{Cover}(X)$ denotes the set of all finite open coverings of X. Remember that for each covering $\mathfrak{M}, \mathfrak{N} \in \mathit{Cover}(X)$, we write $\mathfrak{N} \preccurlyeq \mathfrak{M}$ if $\mathfrak{N}$ is a refinement of $\mathfrak{M}$ and $\mathfrak{M} \preccurlyeq^* \mathfrak{M}$ if $\mathfrak{N}$ is a star refinement of $\mathfrak{M}$ (Figure 1). It is also important to recall that for each covering $\mathfrak{M} \in \mathit{Cover}(X)$, covering $\mathfrak{N} \in \mathit{Cover}(X)$ such that $\mathfrak{N} \preccurlyeq^* \mathfrak{M}$ exists, hence relation $\preccurlyeq$ directs set $\mathit{Cover}(X)$. Since this is a crucial property, I will write down here a simple sketch of a direct proof for our special case, though the result may be seen in the literature, e.g., Tukey (1940; p.47).

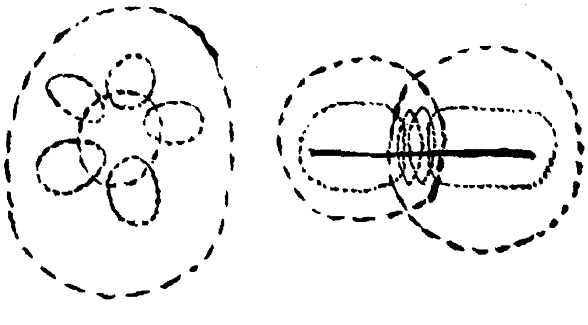

Figure 1: Star Refinements

Lemma 2.1: Let X be a compact Hausdorff space. For each covering $\mathfrak{M} \in \mathit{Cover}(X)$, a star refinement $\mathfrak{N} \in \mathit{Cover}(X)$ of $\mathfrak{M}$, $\mathfrak{N} \preccurlyeq^* \mathfrak{M}$, exists

PROOF : Suppose that X is covered by family $\mathfrak{M}\{M_1, \ldots, M_m\}$ $(m \geq 2)$. First we can see under the condition of normal space that M_1 and M_2 include closed sets C_1 and C_2 respectively, together with open sets $U_1 \subset C_1$ and $U_2 \subset C_2$ such that $X \subset U_1 \cup U_2 \cup \bigcup_{i \geq 3} M_i$. It is clear that family $\mathfrak{N}_2 = \{U_1 \cap M_2, U_2 \cap M_1, M_1 \setminus C_2, M_2 \setminus C_1\}$ satisfies $\forall N \in \mathfrak{N}_2$, the star of N in $\mathfrak{N}_2$, $St(N, \mathfrak{N}_2) = \bigcup\{N' | N \cap N' \neq \emptyset, N' \in \mathfrak{N}_2\}$ is a subset of M_1 or M_2, and $\mathfrak{N}_2 \cup \{M_3, \ldots, M_m\}$ is a covering of X. Next assume that for covering $\{M_1, \ldots, M_{n-1}\}$, family $\mathfrak{N}_{n-1}$ exists such that $\forall N \in \mathfrak{N}_{n-1}$, the star of N in $\mathfrak{N}_{n-1}$, $St(N, \mathfrak{N}_{n-1})$ is a subset of M_i for some $i = 1, \ldots, n-1$, and $\mathfrak{N}_{n-1} \cup \{M_n, M_{n+1}, \ldots, M_m\}$ is a covering of X. Then for M_n, (again under the condition of normal space,) we may chose subsets $V_n \subset D_n \subset U_n \subset C_n$ of M_n such that V_m and U_m are open, D_m and C_m are closed, and $\mathfrak{N}_{n-1} \cup \{V_n, M_{n+1}, \ldots, M_m\}$ is a covering of X (Figure 2). Define $\mathfrak{N}_n$ as $\mathfrak{N}_n = \{N \setminus C_n | N \in \mathfrak{N}_{n-1}\} \cup \{N \cap M_n \setminus D_n | N \in \mathfrak{N}_{n-1}\} \cup \{U_n\}$. It is easy to verify that $\mathfrak{N}_n$ satisfies

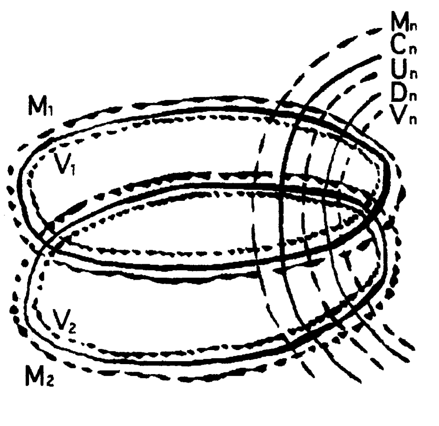

Figure 2: Construction of a Star Refinement

that $\forall N \in \mathfrak{N}_n$, the star of N in $\mathfrak{N}_n$ is a subset of M_i for some $i = 1, \ldots, n$, and $\mathfrak{N}_n \cup \{M_{n+1}, \ldots, M_m\}$ is a covering of X. Since the process may be continued to $n = m$, we may obtain a star refinement of $\mathfrak{M}$. ■

Čech Homology

The *nerve* of the covering $\mathfrak{M}$ of X, $X^c(\mathfrak{M})$, is an abstract complex such that the set of vertices of $X^c(\mathfrak{M})$ is $\mathfrak{M}$ and n-dimensional simplex $\sigma^n = M_0M_1\cdots M_n$ belongs to $X^c(\mathfrak{M})$ if and only if $\bigcap_{i=0}^n M_i \neq \emptyset$. We call an n-dimensional simplex σ^n in $X^c(\mathfrak{M})$ an *n-dimensional Čech $\mathfrak{M}$-simplex*, (or simply, *Čech simplex*, *n-dimensional Čech simplex*, *Čech $\mathfrak{M}$-simplex*, etc., as long as there is no fear of confusion). $X^c(\mathfrak{M})$ is also called the Čech $\mathfrak{M}$-complex. In the following, we assume that every Čech $\mathfrak{M}$-complex is oriented. Since $\mathfrak{M}$ is a finite covering, we may identify $X^c(\mathfrak{M})$ with a polyhedron (a *realization*) in a finite dimensional Euclidean space.

If $p : \mathfrak{N} \to \mathfrak{M}$ is a mapping such that for all $N \in \mathfrak{N}$, $N \subset p(N) \in \mathfrak{M}$, we say that p is a *projection*. It is clear that if $\mathfrak{N}$ is a refinement of $\mathfrak{M}$, then for each $N_1, N_2 \in \mathfrak{N}$, $N_1 \cap N_2 \neq \emptyset$ implies that $p(N_1) \cap p(N_2) \neq \emptyset$. Hence, the vertex mapping, projection p, induces uniquely a simplicial map $X^c(\mathfrak{N}) \ni N_1N_2\cdots N_k \mapsto p(N_1)p(N_2)\cdots p(N_k) \in X^c(\mathfrak{M})$ which is also denoted by p and called a projection.

An *n-dimensional Čech $\mathfrak{M}$-chain*, c^n, is an entity which is represented uniquely as a finite sum of Čech $\mathfrak{M}$-simplexes,

$$c^n = \sum_{i=1}^k \alpha_i\sigma_i^n, \quad (\sigma_1^n, \ldots, \sigma_k^n \in X^c(\mathfrak{M})),$$

where coefficients $\alpha_1, \ldots, \alpha_k$ are taken in a field $\boldsymbol{F}$. The set of all n-dimensional Čech $\mathfrak{M}$-chains, $C_n^c(\mathfrak{M})$, may be identified, therefore, with the vector space over $\boldsymbol{F}$ spanned by elements of the form $1\sigma^n$, where σ^n runs through the set of all n-dimensional Čech $\mathfrak{M}$-simplexes.

Let us consider the boundary operator among chains, $\partial_n : C_n^c(\mathfrak{M}) \to C_{n-1}^c(\mathfrak{M})$, for each n, as usual, i.e., the linear mapping,

$$\partial_n : M_0M_1\cdots M_n \to \sum_{i=0}^n (-1)^i M_0M_1\cdots \hat{M}_i \cdots M_n,$$

where the series of vertices with a circumflex over a vertex means the ordered array obtained from the original array by deleting the vertex with the circumflex and for all $n < 0$, it is supposed that $C_n^c(\mathfrak{M}) = 0$. Then, the set of all *n-dimensional Čech $\mathfrak{M}$-cycles*, $Z_n^c(\mathfrak{M})$, and the set of *n-dimensional Čech $\mathfrak{M}$-boundaries*, $B_n^c(\mathfrak{M})$, may be defined as usual, so that we obtain the *n-th Čech $\mathfrak{M}$-homology group*, $H_n^c(\mathfrak{M})$, for each n. For each $\mathfrak{N} \preccurlyeq \mathfrak{M}$ and dimension n, simplicial map p induces chain homomorphism $p_n^{\mathfrak{M}\mathfrak{N}}$ so that $(C_n^c(\mathfrak{M}), p_n^{\mathfrak{M}\mathfrak{N}})_{\mathfrak{M},\mathfrak{N}\in \mathit{Cover}(X)}$, $(Z_n^c(\mathfrak{M}), p_n^{\mathfrak{M}\mathfrak{N}})_{\mathfrak{M},\mathfrak{N}\in \mathit{Cover}(X)}$, and $(B_n^c(\mathfrak{M}), p_n^{\mathfrak{M}\mathfrak{N}})_{\mathfrak{M},\mathfrak{N}\in \mathit{Cover}(X)}$, form inverse systems.

Note that if $\mathfrak{N} \preccurlyeq \mathfrak{M}$, and if $p : \mathfrak{N} \to \mathfrak{M}$ and $p' : \mathfrak{N} \to \mathfrak{M}$ are projections, two simplicial maps, p and p', are contiguous, i.e., for each Čech $\mathfrak{N}$-simplex, $N_0N_1\cdots N_k$, images $p(N_0)p(N_1)\cdots p(N_k)$ and $p'(N_0)p'(N_1)\cdots p'(N_k)$ are faces of a single simplex.[1] Since two contiguous simplicial maps are *chain homotopic*,[2] p and p' induce the same homomorphism, $p_{*n}^{\mathfrak{M}\mathfrak{N}} : H_n^c(\mathfrak{N}) \to H_n^c(\mathfrak{M})$ for each n. The limit for the inverse system, $(H_n^c(\mathfrak{M}), p_{*n}^{\mathfrak{M}\mathfrak{N}})$, on the preordered family, $(\mathit{Cover}(X), \preccurlyeq)$,

$$H_n^c(X) = \varprojlim_{\mathfrak{M}} H_n^c(\mathfrak{M}),$$

is the *n-dimensional Čech Homology group.*

[1]Indeed, it is clear that the intersection $(\bigcap_{i=0}^k p(N_i)) \cap (\bigcap_{i=0}^k p'(N_i)) \supset \bigcap_{i=1}^k N_i \neq \emptyset$. Hence, the array obtained by deleting all of the second occurence for the same vertex from the series, $p(N_0)p(N_1)\cdots p(N_k)p'(N_0)p'(N_1)\cdots p'(N_k)$, is a Čech $\mathfrak{M}$-simplex.

[2]See for example Eilenberg and N.Steenrod (1952; p.164). If we are allowed to define piecewise linear extensions $\bar{p}$ and $\bar{p}'$ of p and p', respectively, it may also easy to find a homotopy bridge among $\bar{p}$ and $\bar{p}'$.

Under the definitions of the homology group and the inverse limit, an element of $H_n^c(X)$ may be considered, intuitively, as an equivalence class of a sequence of Čech cycles, $\{z^n(\mathfrak{M}) \in Z_n^c(\mathfrak{M}) : \mathfrak{M} \in \mathbf{Cover}(X)\}$, such that for each $\mathfrak{M}, \mathfrak{N} \in \mathbf{Cover}(X)$ satisfying that $\mathfrak{N} \preccurlyeq \mathfrak{M}$, we have $z^n(\mathfrak{M}) \sim p_n^{\mathfrak{M}\mathfrak{N}}(z^n(\mathfrak{N}))$, where the equivalence relation is defined relative to the class of Čech boundaries, i.e., $z^n(\mathfrak{M}) - p_n^{\mathfrak{M}\mathfrak{N}}(z^n(\mathfrak{N})) \in B_n^c(\mathfrak{M})$.[3]

Vietoris Homology

An *n-dimensional Vietoris simplex* is a collection of $n+1$ points of X, $x_0x_1\cdots x_n$. A Vietoris simplex, $\sigma = x_0x_1\cdots x_n$, is said to be an *$\mathfrak{M}$-simplex* if the set of vertices, $\{x_0, x_1, \ldots, x_n\}$, is a subset of an element of $\mathfrak{M}$. The set of all Vietoris $\mathfrak{M}$-simplexes forms a simplicial (infinite) complex (Vietoris $\mathfrak{M}$-complex) and is denoted by $X^v(\mathfrak{M})$. An *orientation* for n-dimensional Vietoris simplex $x_0x_1\cdots x_n$ is a total ordering on $\{x_0, x_1, \ldots, x_n\}$ up to even permutations. In the following we suppose that every Vietoris $\mathfrak{M}$-complex is oriented.

The set of all *n-dimensional Vietoris $\mathfrak{M}$-chain*, $C_n^v(\mathfrak{M})$, is the vector space whose elements are uniquely represented as a finite sum of n-dimensional Vietoris $\mathfrak{M}$-simplexes,

$$c^n = \sum_{i=1}^{k} \alpha_i \sigma_i^n, \quad (\sigma_1^n, \ldots, \sigma_k^n \in X^v(\mathfrak{M})),$$

where coefficients $\alpha_1, \ldots, \alpha_k$ are taken in a field $\boldsymbol{F}$. We may also consider the boundary operator among chains, $\partial_n : C_n^v(\mathfrak{M}) \to C_{n-1}^v(\mathfrak{M})$, for each n, as the linear map satisfying,

$$\partial_n : x_0x_1\cdots x_n \to \sum_{i=0}^{n}(-1)^i x_0x_1\cdots \hat{x}_i \cdots x_n,$$

where the circumflex over a vertex means the elimination as before, and it is supposed that $C_n^v(\mathfrak{M}) = 0$ for all $n < 0$. The set of all *n-dimensional Vietoris $\mathfrak{M}$-cycles*, $Z_n^v(\mathfrak{M})$, and the set of *n-dimensional Vietoris $\mathfrak{M}$-boundaries*, $B_n^v(\mathfrak{M})$, may also be defined as usual, so that we obtain the *n-th Vietoris $\mathfrak{M}$-homology group*, $H_n^v(\mathfrak{M})$, for each n.

For coverings $\mathfrak{M}, \mathfrak{N} \in \mathbf{Cover}(X)$, it is clear that $(\mathfrak{N} \preccurlyeq \mathfrak{M}) \Longrightarrow (X^v(\mathfrak{N}) \subset X^v(\mathfrak{M}))$. Denote by $h_n^{\mathfrak{M}\mathfrak{N}} : C_n^v(\mathfrak{N}) \to C_n^v(\mathfrak{M})$ the chain homomorphism induced by the above inclusion. Then, for each n, the system of vector spaces with mappings, $(C_n^v(\mathfrak{M}), h_n^{\mathfrak{M}\mathfrak{N}})_{\mathfrak{M},\mathfrak{N} \in \mathbf{Cover}(X)}$, their cycles, $(Z_n^v(\mathfrak{M}), h_n^{\mathfrak{M}\mathfrak{N}})_{\mathfrak{M},\mathfrak{N} \in \mathbf{Cover}(X)}$, and boundaries, $(B_n^v(\mathfrak{M}), h_n^{\mathfrak{M}\mathfrak{N}})_{\mathfrak{M},\mathfrak{N} \in \mathbf{Cover}(X)}$, form inverse systems. The inverse limit of the inverse system, $(Z_n^v(\mathfrak{M})/B_n^v(\mathfrak{M}), h_{*n}^{\mathfrak{M}\mathfrak{N}})_{\mathfrak{M},\mathfrak{N} \in \mathbf{Cover}(X)}$,

$$H_n^v(X) = \varprojlim_{\mathfrak{M}} H_n^v(\mathfrak{M}),$$

is the *n-dimensional (n-th) Vietoris Homology group.*

An element of $H_n^v(X)$ may be identified with an equivalence class of a sequence of n-dimensional Vietoris $\mathfrak{M}$-cycles, $\mathfrak{M} \in \mathbf{Cover}(X)$, (an n-dimensional Vietoris cycle), $\{z^n(\mathfrak{M}) \in Z_n^v(\mathfrak{M}) | \mathfrak{M} \in \mathbf{Cover}(X)\}$, such that for each $\mathfrak{M}, \mathfrak{N} \in \mathbf{Cover}(X)$ satisfying that $\mathfrak{N} \preccurlyeq \mathfrak{M}$, we have $z^n(\mathfrak{M}) \sim h_n^{\mathfrak{M}\mathfrak{N}}(z^n(\mathfrak{N}))$, where the equivalence class is taken with respect to Vietoris $\mathfrak{M}$-boundaries, i.e., $z^n(\mathfrak{M}) - h_n^{\mathfrak{M}\mathfrak{N}}(z^n(\mathfrak{N})) \in B_n^v(\mathfrak{M})$.[4]

[3]For more details of the Čech homology theory, see Eilenberg and N.Steenrod (1952). For more introductory arguments, Hocking and Young (1961; Chapter 8) is also recommended.

[4]The concept of Vietoris homology group was originally introduced by Vietoris (1927) as the first homology theory of the Čech type for metric spaces. Though the theory has been used in many researches, e.g., Eilenberg and Montgomery (1946), it has not been frequently discussed as has the more general Čech theory. The theory was extended to be applicable for cases of compact Hausdorff spaces by Begle (1950), and the result was used in Nikaido (1959) to prove an analogue of Sperner's lemma.

Vietoris and Čech Cycles

The Čech homology theory is a powerful tool to approximate the space with groups of a finite complex. The Vietoris homology theory, on the other hand, has an intuitional advantage that we may characterize the space directly by its elements (points). Fortunately, we may utilize both merits since the two homological concepts give the same homology groups (see Theorem 2.3 below).

Before proving this, let us see the following facts on equivalences of two cycles on a simplicial complex. Since a homology group is nothing but a set of equivalence classes of cycles, it is not surprising that homological arguments often depend on this type of equivalence results. Let K be a simplicial complex. Suppose that the set of vertices of K, $\boldsymbol{Vert}(K)$, is simply ordered in an arbitrary way, and let $\sigma^n = \langle a_0, a_1, \ldots, a_n\rangle$ be an n-simplex (oriented by the simple order) in K. The *product simplicial complex* of K and the unit interval denoted by $K \times \{0,1\}$ is the family of simplexes of the form $\langle (a_0,0),(a_1,0),\ldots,(a_i,0),(a_i,1),\ldots,(a_n,1)\rangle$ for each $\langle a_0, a_1, \ldots, a_n\rangle \in K$ together with all their faces (Figure 3). The subcomplex of $K \times \{0,1\}$ constructed

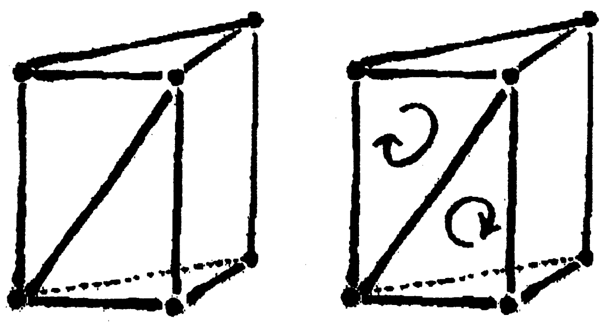

Figure 3: Prism $K \times \{0,1\}$

by all simplexes of the form $\langle (a_0,0),\ldots,(a_n,0)\rangle$ may clearly be identified with K and is called the *base* of $K \times \{0,1\}$. There also exists an isomorphism between K and the subcomplex of all simplexes of the form $\langle (a_0,1),\ldots,(a_n,1)\rangle$, which is called the *top* of $K \times \{0,1\}$. For each n-simplex $\langle \sigma^n\rangle = \langle a_0,\ldots,a_n\rangle$ of K, define an $n+1$-chain, $\Phi_n(\sigma^n)$, on product simplicial complex $K \times \{0,1\}$ as

(1)
$$\Phi_n(\sigma^n) = \sum_{j=0}^{n}(-1)^j\langle (a_0,0),\ldots,(a_j,0),(a_j,1),\ldots,(a_n,1)\rangle.$$

Extend each Φ_n to a homomorphism on $C_n(K)$ to $C_n(K \times \{0,1\})$. Then we can verify through direct calculations that for each n-chain $c^n \in K$,

(2)
$$\partial_{n+1}\Phi_n(c^n) + \Phi_{n-1}\partial_n(c^n) = c^n \times 1 - c^n \times 0 \in C_{n-1}(K \times \{0,1\}),$$

where $c^n \times 1$ (resp., $c^n \times 0$) is the chain on the top (resp. base) of $K \times \{0,1\}$ formed by replacing each vertex of each simplex of c^n by the vertex of the ordered pair with 0 (resp., 1). Hence, if z^n is a cycle on K,

(3)
$$\partial_{n+1}\Phi_n(z^n) = z^n \times 1 - z^n \times 0 \in B_n(K \times \{0,1\}),$$

i.e., we have $z^n \times 0 \sim z^n \times 1$ on $K \times \{0,1\}$. Therefore, if there exists a simplicial mapping ψ on $K \times \{0,1\}$ to a certain simplicial complex L, the next lemma holds.

Lemma 2.2: Assume that there is a simplicial mapping ψ on $K \times \{0,1\}$ to a simplicial complex L. For two images $\psi_{q+1}(z^q \times 0)$ and $\psi_{q+1}(z^q \times 1)$ in the q-th chain group $C_q(L)$ of q-cycle $z^q \in C_q(K)$ (through the induced homomorphism $\psi_{q+1} : C_{q+1}(K \times \{0,1\}) \to C_q(L)$), we have $\psi_{q+1}(z^q \times 0) \sim \psi_{q+1}(z^q \times 1)$ on L.

We now see the following fundamental result.

Theorem 2.3: (Begle 1950a) Let X be a compact Hausdorff space. The q-th Vietoris homology group, $H_q^v(X)$, is isomorphic to the corresponding Čech homology group, $H_q^c(X)$, for each q.

To show the above result, use the following two simplicial mappings.[5] Given covering $\mathfrak{M}$ in $\boldsymbol{Cover}(X)$, chose refinement $\mathfrak{N} \preccurlyeq^* \mathfrak{M}$, which is always possible for a compact Hausdorff space by Lemma 2.1. It is convenient for the discussion below to denote one of such selections for each $\mathfrak{M}$ by a fixed operator on $\boldsymbol{Cover}(X)$ as $\mathfrak{N} = {}^*\mathfrak{M}$.[6] For each $\mathfrak{M} \in \boldsymbol{Cover}(X)$ and for each $x \in X$, there are $N_x \in {}^*\mathfrak{M}$ and $M_x \in \mathfrak{M}$ such that $x \in N_x$ and $St(N_x; {}^*\mathfrak{M}) \subset M_x$. Moreover, for each $N \in {}^*\mathfrak{M}$ there is an element $x_N \in N$. Define functions $\zeta^b_{\mathfrak{M}}$ and $\varphi^b_{\mathfrak{M}}$ as

$$\zeta^b_{\mathfrak{M}} \; : \; \boldsymbol{Vert}(X^v({}^*\mathfrak{M})) = X \ni x \mapsto M_x \in \mathfrak{M} = \boldsymbol{Vert}(X^c(\mathfrak{M})) \tag{4}$$

$$\varphi^b_{\mathfrak{M}} \; : \; \boldsymbol{Vert}(X^c({}^*\mathfrak{M})) = {}^*\mathfrak{M} \ni N \mapsto x_N \in X = \boldsymbol{Vert}(X^v(\mathfrak{M})) \tag{5}$$

Under the definition of star refinement, it is easy to see that $\zeta^b_{\mathfrak{M}}$ and $\varphi^b_{\mathfrak{M}}$ are simplicial mappings. Hence, we obtain chain homomorphisms $\zeta^b_{\mathfrak{M}q} : C_q^v(\mathfrak{N}) \to C_q^c(\mathfrak{M})$ and $\varphi^b_{\mathfrak{M}q} : C_q^c(\mathfrak{M}) \to C_q^v(\mathfrak{P})$. As we see below, these mappings play essential roles in characterizing relations between Victoris and Čech homology groups. Especially, mappings $\zeta^b_{\mathfrak{M}q}$ and $\varphi^b_{\mathfrak{M}q}$ induces, respectively, isomorphisms $\zeta^b_{*q} : H_q^v(X) \to H_q^c(X)$ and $\varphi^b_{*q} : H_q^c(X) \to H_q^v(X)$ (Theorem 2.3), and $\varphi^b_{\mathfrak{M}q} \circ \zeta^b_{\mathfrak{N}q}$ ($\mathfrak{N} = {}^*\mathfrak{M}$) assures the finite dimensional character of acyclic spaces (Theorem 3.2) or locally connected spaces (Theorem 3.4).

Proof of Theorem 2.3 : Let $\gamma^q = \{\gamma^q(\mathfrak{N}) | \mathfrak{N} \in \boldsymbol{Cover}(X)\}$, (or simply, $\{\gamma^q(\mathfrak{N})\}$) be an q-dimensional Vietoris cycle. For each $\mathfrak{M} \in \boldsymbol{Cover}(X)$ and $\mathfrak{N} = {}^*\mathfrak{M}$, define $z^q(\mathfrak{M})$ as $z^q(\mathfrak{M}) = \zeta^b_{\mathfrak{M}q}(\gamma^q(\mathfrak{N}))$. We see (1) that $z^q = \{z^q(\mathfrak{M})\}$ is a Čech cycle and (2) that the mapping $\zeta^b_{*q} : \gamma^q \mapsto z^q$ is an isomorphism on $H_q^v(X)$ to $H_q^c(X)$.

(1) Since $\zeta^b_{\mathfrak{M}q} : C_q^v(\mathfrak{N}) \to C_q^c(\mathfrak{M})$ is a chain homomorphism, all $z^q(\mathfrak{M})$ ($\mathfrak{M} \in \boldsymbol{Cover}(X)$) are cycles in $C_q^c(\mathfrak{M})$. Hence, by definition of inverse limit, all we have to show is $z^q(\mathfrak{M}_1) \sim p_q^{\mathfrak{M}_1\mathfrak{M}_2}(z^q(\mathfrak{M}_2))$ for each $\mathfrak{M}_2 \preccurlyeq \mathfrak{M}_1$. Let $\mathfrak{N}_1$ and $\mathfrak{N}_2$ be refinements of $\mathfrak{M}_1$ and $\mathfrak{M}_2$, respectively, to define mappings $\zeta^b_{\mathfrak{M}_1q}$ and $\zeta^b_{\mathfrak{M}_2q}$. By Lemma 2.1, we can take $\mathfrak{P}$ as $\mathfrak{P} \preccurlyeq^* \mathfrak{N}_1$ and $\mathfrak{P} \preccurlyeq^* \mathfrak{N}_2$. Note that since $\{\gamma^q(\mathfrak{N})\}$ is a Vietoris cycle, we have $h_q^{\mathfrak{N}_1\mathfrak{P}}(\gamma^q(\mathfrak{P})) \sim \gamma^q(\mathfrak{N}_1)$ and $h_q^{\mathfrak{N}_2\mathfrak{P}}(\gamma^q(\mathfrak{P})) \sim \gamma^q(\mathfrak{N}_2)$. Hence, $z^q(\mathfrak{M}_1) = \zeta^b_{\mathfrak{M}_1q}(\gamma^q(\mathfrak{N}_1) \sim \zeta^b_{\mathfrak{M}_1q}(h_q^{\mathfrak{N}_1\mathfrak{P}}(\gamma^q(\mathfrak{P})))$ and $p_q^{\mathfrak{M}_1\mathfrak{M}_2}(z^q(\mathfrak{M}_2)) = p_q^{\mathfrak{M}_1\mathfrak{M}_2}(\zeta^b_{\mathfrak{M}_2q}(\gamma^q(\mathfrak{N}_2)) \sim p_q^{\mathfrak{M}_1\mathfrak{M}_2}(\zeta^b_{\mathfrak{M}_2q}(h_q^{\mathfrak{N}_2\mathfrak{P}}(\gamma^q(\mathfrak{P})))$.[7] It follows that all we have to show is $\zeta^b_{\mathfrak{M}_1q}(\gamma^q(\mathfrak{P})) \sim p_q^{\mathfrak{M}_1\mathfrak{M}_2}(\zeta^b_{\mathfrak{M}_2q}(\gamma^q(\mathfrak{P}))$. Let $K = K(\gamma^q(\mathfrak{P}))$ be the complex consists of all simplexes in cycle $\gamma^q(\mathfrak{P})$ together with their faces. Then by Lemma 2.2, it is sufficient to show the existence of simplicial map ψ on $K \times \{0,1\}$ to $L = X^c(\mathfrak{M}_1)$ such that $\zeta^b_{\mathfrak{M}_1q}(\gamma^q(\mathfrak{P}))$ and $p_q^{\mathfrak{M}_1\mathfrak{M}_2}(\zeta^b_{\mathfrak{M}_2q}(\gamma^q(\mathfrak{P}))$ are images through the induced map $\psi_{q+1} : C_{q+1}(K \times \{0,1\}) \to X^c(\mathfrak{M}_1)$ of $\gamma^q(\mathfrak{P}) \times 0$ and $\gamma^q(\mathfrak{P}) \times 1$, respectively. For each $a \in \boldsymbol{Vert}(K)$, define ψ as $\psi((a,0)) = \zeta^b_{\mathfrak{M}_1}(a)$ and $\psi((a,1)) = p^{\mathfrak{M}_1\mathfrak{M}_2}\zeta^b_{\mathfrak{M}_2}(a)$. For any simplex $\langle (a_0,0), \ldots, (a_i,0), (a_i,1), \ldots, (a_k,1) \rangle$ in $K \times \{0,1\}$, we have a simplex $a_0 \cdots a_k$ of $K =$

[5] These mappings are defined by Begle (1950a).

[6] For this, Axiom of Choice is needed.

[7] In the above, inclusion mappings $h_q^{\mathfrak{N}_1\mathfrak{P}}$ and $h_q^{\mathfrak{N}_2\mathfrak{P}}$ might be abbreviated. Since including relation $C_q^v(\mathfrak{N}') \subset C_q^v(\mathfrak{N})$ for each $\mathfrak{N}' \preccurlyeq \mathfrak{N}$ is obvious, these operators will be omitted henceforth as long as there is no fear of confusions.

$K(\gamma^q(\mathfrak{P}))$, so that there exists $P \in \mathfrak{P}$, $a_0, \ldots, a_k \in P$. We have to show that $\langle\zeta^b_{\mathfrak{M}_1}(a_0), \ldots, \zeta^b_{\mathfrak{M}_1}(a_i)$, $p^{\mathfrak{M}_1\mathfrak{M}_2}\zeta^b_{\mathfrak{M}_2}(a_i), \ldots, p^{\mathfrak{M}_1\mathfrak{M}_2}\zeta^b_{\mathfrak{M}_2}(a_k)\rangle$ forms a simplex in $X^c(\mathfrak{M}_1)$. For each j, $0 \le j \le i$, since $\mathfrak{P} \preccurlyeq^* \mathfrak{N}_1 \preccurlyeq^* \mathfrak{M}_1$, each $\zeta^b_{\mathfrak{M}_1}(a_j) = M_{1a_j}$ $(0 \le j \le i)$ includes $St(N_{1a_j}, \mathfrak{N}_1)$ for a certain $N_{1a_j} \ni a_j$. Hence, P which has a_j and satisfies $St(P, \mathfrak{P}) \subset N_1$ for a certain $N_1 \in \mathfrak{N}_1$ must be a subset of $St(N_{1a_j}, \mathfrak{N}_1) \subset M_{1a_j}$. For each j, $i \le j \le k$, since $\mathfrak{P} \preccurlyeq^* \mathfrak{N}_2 \preccurlyeq^* \mathfrak{M}_2 \preccurlyeq \mathfrak{M}_1$, each $p^{\mathfrak{M}_1\mathfrak{M}_2}\zeta^b_{\mathfrak{M}_2}(a_j) = p^{\mathfrak{M}_1\mathfrak{M}_2}M_{2a_j}$ $(i \le j \le k)$ includes $St(N_{2a_j}, \mathfrak{N}_2)$ for a certain $N_{2a_j} \ni a_j$. Hence, P which has a_j and satisfies $St(P, \mathfrak{P}) \subset N_2$ for a certain $N_2 \in \mathfrak{N}_2$ must be a subset of $St(N_{2a_j}, \mathfrak{N}_2) \subset M_{2a_j}$ so that the corresponding element under projection $p_0^{\mathfrak{M}_1\mathfrak{M}_2}$ of $\mathfrak{M}_1$. Therefore, we have $\zeta^b_{\mathfrak{M}_1}(a_0) \cap \ldots \cap \zeta^b_{\mathfrak{M}_1}(a_i) \cap p^{\mathfrak{M}_1\mathfrak{M}_2}\zeta^b_{\mathfrak{M}_2}(a_i) \cap p^{\mathfrak{M}_1\mathfrak{M}_2}\zeta^b_{\mathfrak{M}_2}(a_k) \supset P \neq \emptyset$ and $\langle\zeta^b_{\mathfrak{M}_1}(a_0), \ldots, \zeta^b_{\mathfrak{M}_1}(a_i), p^{\mathfrak{M}_1\mathfrak{M}_2}\zeta^b_{\mathfrak{M}_2}(a_i), \ldots, p^{\mathfrak{M}_1\mathfrak{M}_2}\zeta^b_{\mathfrak{M}_2}(a_k)\rangle \in X^c(\mathfrak{M}_1)$, i.e., ψ is a simplicial map. By the construction of induced map ψ_q, it is also clear that $\psi_{q+1}(\gamma^q \times 0) = \zeta^b_{\mathfrak{M}_1 q}(\gamma^q)$ and $\psi_{q+1}(\gamma^q \times 1) = p_q^{\mathfrak{M}\mathfrak{N}}\zeta^b_{\mathfrak{N}q}(\gamma^q)$.

(2) We have to show that mapping $\zeta^b_{*q} : Z^v_q(X) \ni \gamma^q \mapsto z^q \in Z^c_q(X)$ is one to one and onto. We shall use three steps: (2-1) define mapping $\varphi^b_{*q} : Z^c_q(X) \to Z^v_q(X)$, (2-2) show that the composite $\varphi^b_{*q} \circ \zeta^b_{*q}$ is the identity, and (2-3) show that the composite $\zeta^b_{q*} \circ \varphi^b_{q*}$ is the identity.

(2-1) Let us define a function which gives for each $\mathfrak{M}$ and $z^q = \{z^q(\mathfrak{M})\} \in Z^c_q(X)$, the element $\varphi^b_{\mathfrak{N}q}(z^q(\mathfrak{N})) \in Z^v_q(\mathfrak{M})$, where $\mathfrak{N} = {}^*\mathfrak{M}$. Denote the relation by $\varphi^b_{*q} : Z^c_q(X) \ni z^q \mapsto \{\varphi^b_{\mathfrak{M}q}(z^q({}^*\mathfrak{M}))|\mathfrak{M} \in \mathbf{Cover}(X)\} \in \prod_{\mathfrak{M}\in\mathbf{Cover}(X)} Z^v_q(\mathfrak{M})$. We see that for each $\mathfrak{M}_2 \preccurlyeq \mathfrak{M}_1$ with $\mathfrak{N}_1 = {}^*\mathfrak{M}_1$ and $\mathfrak{N}_2 = {}^*\mathfrak{M}_2$, $\varphi^b_{\mathfrak{M}_1 q}(z^q(\mathfrak{N}_1)) \sim h_q^{\mathfrak{M}_1\mathfrak{M}_2}\varphi^b_{\mathfrak{M}_2 q}(z^q(\mathfrak{N}_2))$, so that the sequence $\{\varphi^b_{\mathfrak{M}q}(z^q({}^*\mathfrak{M}))|\mathfrak{M} \in \mathbf{Cover}(X)\}$ is a Vietoris cycle. We may assume $\mathfrak{M}_2 \preccurlyeq^* \mathfrak{N}_1 \preccurlyeq^* \mathfrak{M}_1$ without loss of generality since the existence of a common star refinement $\mathfrak{M}_3$ of $\mathfrak{N}_2$ and $\mathfrak{N}_1$ combined with assertions for $\mathfrak{M}_3 \preccurlyeq^* \mathfrak{N}_1 \preccurlyeq^* \mathfrak{M}_1$ and $\mathfrak{M}_3 \preccurlyeq^* \mathfrak{N}_2 \preccurlyeq^* \mathfrak{M}_2$ assures the results for $\mathfrak{M}_2 \preccurlyeq \mathfrak{M}_1$ through $h_q^{\mathfrak{M}_1\mathfrak{M}_3}\varphi^b_{\mathfrak{M}_3 q}(z^q({}^*\mathfrak{M}_3))$. Take a common star refinement $\mathfrak{P}$ of $\mathfrak{N}_1$ and $\mathfrak{N}_2$. Since $z^q = \{z^q(\mathfrak{M})\}$ is a Čech cycle, all we have to show is $\varphi^b_{\mathfrak{M}_1 q}(p_q^{\mathfrak{N}_1\mathfrak{P}}z^q(\mathfrak{P})) \sim h_q^{\mathfrak{M}_1\mathfrak{M}_2}\varphi^b_{\mathfrak{M}_2 q}(p_q^{\mathfrak{N}_2\mathfrak{P}}z^q(\mathfrak{P}))$. Let $K = K(z^q(\mathfrak{P}))$ be the complex formed by all simplexes in cycle $z^q(\mathfrak{P}) \in X^c_q(\mathfrak{P})$ together with their faces. By Lemma 2.2, it is sufficient for our purpose to show the existence of simplicial map ψ on $K \times \{0,1\}$ to $L = X^v(\mathfrak{M}_1)$ such that $\varphi^b_{\mathfrak{M}_1 q}(p_q^{\mathfrak{N}_1\mathfrak{P}}z^q(\mathfrak{P}))$ and $h_q^{\mathfrak{M}_1\mathfrak{M}_2}\varphi^b_{\mathfrak{M}_2 q}(p_q^{\mathfrak{N}_2\mathfrak{P}}z^q(\mathfrak{P}))$ are images through the induced map $\psi_{q+1} : C_{q+1}(K \times \{0,1\}) \to X^v(\mathfrak{M}_1)$ of $z^q(\mathfrak{P}) \times 0$ and $z^q(\mathfrak{P}) \times 1$, respectively. For each $a \in \mathbf{Vert}(K) \subset \mathfrak{P}$, define ψ as $\psi((a,0)) = \varphi^b_{\mathfrak{M}_1}(p^{\mathfrak{N}_1\mathfrak{P}}(a))$ and $\psi((a,1)) = \varphi^b_{\mathfrak{M}_2}(p^{\mathfrak{N}_2\mathfrak{P}}(a))$. For any simplex $\langle(a_0,0), \ldots, (a_i,0), (a_i,1), \ldots, (a_k,1)\rangle$ in $K \times \{0,1\}$, we have a simplex $a_0 \cdots a_k$ of $K = K(z^q(\mathfrak{P}))$, so that $a_0 \cap \cdots \cap a_k \neq \emptyset$. We have to show that $\langle\varphi^b_{\mathfrak{M}_1}(p^{\mathfrak{N}_1\mathfrak{P}}(a_0)), \ldots, \varphi^b_{\mathfrak{M}_1}(p^{\mathfrak{N}_1\mathfrak{P}}(a_i)), \varphi^b_{\mathfrak{M}_2}(p^{\mathfrak{N}_2\mathfrak{P}}(a_i)), \ldots,$ $\varphi^b_{\mathfrak{M}_2}(p^{\mathfrak{N}_2\mathfrak{P}}(a_k))\rangle$ forms a simplex in $X^v(\mathfrak{M}_1)$. Note that for each j, $0 \le j \le i$, $\mathfrak{P} \preccurlyeq^* \mathfrak{N}_1 \preccurlyeq^* \mathfrak{M}_1$, and for each j, $i \le j \le k$, $\mathfrak{P} \preccurlyeq^* \mathfrak{N}_2 \preccurlyeq^* \mathfrak{M}_2 \preccurlyeq^* \mathfrak{N}_1 \preccurlyeq^* \mathfrak{M}_1$. Since $a_0 \cap \cdots \cap a_k \neq \emptyset$, there are $N_1 \in \mathfrak{N}_1$ and $N_2 \in \mathfrak{N}_2$ such that $a_0 \cup \cdots \cup a_k \subset N_1$ and $a_0 \cup \cdots \cup a_k \subset N_2$. By definitions of φ^b and p, $St(N_1; \mathfrak{N}_1)$ and $St(N_2; \mathfrak{N}_2)$ contain all points of the form $\varphi^b_{\mathfrak{M}_1}(p^{\mathfrak{N}_1\mathfrak{P}}(a_j))$, $(0 \le j \le i)$ and $\varphi^b_{\mathfrak{M}_2}(p^{\mathfrak{N}_2\mathfrak{P}}(a_j))$, $(i \le j \le k)$. There are $M_1 \in \mathfrak{M}_1$ and $M_2\mathfrak{M}_2$ such that $St(N_1; \mathfrak{N}_1) \subset M_1$ and $St(N_2; \mathfrak{N}_2) \subset M_2$. The fact $\mathfrak{M}_2 \preccurlyeq^* \mathfrak{N}_1$ means, however, that $M_2 \subset N_1'$ for some N_1' in $\mathfrak{N}_1$. Since $N_1' \cap N_1 \supset a_0 \cup \cdots \cup a_k$, $N_1' \subset St(N_1; \mathfrak{N}_1)$, so that M' includes both $St(N_1; \mathfrak{N}_1)$ and $St(N_2; \mathfrak{N}_2)$. Hence, $\langle\varphi^b_{\mathfrak{M}_1}(p^{\mathfrak{N}_1\mathfrak{P}}(a_0)), \ldots, \varphi^b_{\mathfrak{M}_1}(p^{\mathfrak{N}_1\mathfrak{P}}(a_i)), \varphi^b_{\mathfrak{M}_2}(p^{\mathfrak{N}_2\mathfrak{P}}(a_i)), \ldots,$ $\varphi^b_{\mathfrak{M}_2}(p^{\mathfrak{N}_2\mathfrak{P}}(a_k))\rangle$ forms a simplex in $X^v(\mathfrak{M}_1)$ is a simplex in $X^v(\mathfrak{M}_1)$.

(2-2) We see for each $\mathfrak{M}$, $\mathfrak{N} = {}^*\mathfrak{M}$, $\mathfrak{P} = {}^*\mathfrak{N}$, and $\gamma^q \in C^v(X)$, $\varphi^b_{\mathfrak{M}q} \circ \zeta^b_{\mathfrak{N}q}(\gamma^q(\mathfrak{P})) \sim \gamma^q(\mathfrak{P})$, which is sufficient for the assertion $\zeta^b_{*q} \circ \varphi^b_{*q}(\gamma^q) = \gamma^q$. Let $K = K(\gamma^q(\mathfrak{P}))$ be the subcomplex of $X^v(\mathfrak{P})$ formed by simplexes of $\gamma^q(\mathfrak{P})$ and their faces. By Lemma 2.2, we may reduce the problem to show the existence of simplicial map ψ on $K \times \{0,1\}$ to $L = X^v(\mathfrak{M})$ such that $\varphi^b_{\mathfrak{M}q} \circ \zeta^b_{\mathfrak{N}q}(\gamma^q(\mathfrak{P}))$ and $\gamma^q(\mathfrak{P})$ are images under the induced map $\psi_{q+1} : C_{q+1}(K \times \{0,1\}) \to X^v(\mathfrak{M})$ of $\gamma^q(\mathfrak{P}) \times 0$ and $\gamma^q(\mathfrak{P}) \times 1$, respectively. For each $a \in \mathbf{Vert}(K) \subset X$, define ψ as $\psi((a,0)) = \varphi^b_{\mathfrak{M}} \circ \zeta^b_{\mathfrak{N}}(a)$ and $\psi((a,1)) = a$. For any simplex $\langle(a_0,0), \ldots, (a_i,0)$, $(a_i,1), \ldots, (a_k,1)\rangle$ in $K \times \{0,1\}$, we have a simplex $a_0 \cdots a_k$ of $K = K(\gamma^q(\mathfrak{P}))$, so that there is a member P of $\mathfrak{P}$ such that $a_0, \ldots, a_k \in P$. We have to show that $\langle\varphi^b_{\mathfrak{M}} \circ \zeta^b_{\mathfrak{N}}(a_0), \ldots, \varphi^b_{\mathfrak{M}} \circ \zeta^b_{\mathfrak{N}}(a_i), a_i, \ldots, a_k\rangle$ forms a simplex in $X^v(\mathfrak{M})$. Since $\mathfrak{P} \preccurlyeq^* \mathfrak{N} \preccurlyeq^* \mathfrak{M}$, there are $N \in \mathfrak{N}$ and $M \in \mathfrak{M}$ such that $St(P, \mathfrak{P}) \subset N$ and $St(N, \mathfrak{N}) \subset M$. Hence, by definitions of $\varphi^b_{\mathfrak{M}}$ and $\zeta^b_{\mathfrak{M}}$, M includes all vertices of $\langle \varphi^b_{\mathfrak{M}} \circ \zeta^b_{\mathfrak{N}}(a_0), \ldots, \varphi^b_{\mathfrak{M}} \circ \zeta^b_{\mathfrak{N}}(a_i), a_i, \ldots, a_k \rangle$.

(2-3) For each $\mathfrak{M}$, $\mathfrak{N} = {}^*\mathfrak{M}$, $\mathfrak{P} = {}^*\mathfrak{N}$, and $z^q \in C^c(X)$, we see $\zeta^b_{\mathfrak{M}q} \circ \varphi^b_{\mathfrak{N}q}(z^q(\mathfrak{P})) \sim z^q(\mathfrak{P})$. This is exactly shows $\zeta^b_{*q} \circ \varphi^b_{*q}(z^q) = z^q$. Let $K = K(z^q(\mathfrak{P}))$ be the subcomplex of $X^c(\mathfrak{P})$ formed by simplexes of $z^q(\mathfrak{P})$ and their faces. By Lemma 2.2, to show the existence of simplicial map ψ on $K \times \{0, 1\}$ to $L = X^c(\mathfrak{M})$ such that $\zeta^b_{\mathfrak{M}q} \circ \varphi^b_{\mathfrak{N}q}(z^q(\mathfrak{P}))$ and $z^q(\mathfrak{P})$ are images under the induced map $\psi_{q+1} : C_{q+1}(K \times \{0, 1\}) \to X^c(\mathfrak{M})$ of $z^q(\mathfrak{P}) \times 0$ and $z^q(\mathfrak{P}) \times 1$, respectively. For each $a \in \mathbf{Vert}(K) \subset \mathfrak{P}$, define ψ as $\psi((a, 0)) = \zeta^b_{\mathfrak{M}} \circ \varphi^b_{\mathfrak{N}}(a)$ and $\psi((a, 1)) = a$. For any simplex $\langle (a_0, 0), \ldots, (a_i, 0), (a_i, 1), \ldots, (a_k, 1) \rangle$ in $K \times \{0, 1\}$, we have a simplex $a_0 \cdots a_k$ of $K = K(z^q(\mathfrak{P}))$, so that sets $a_0, \ldots, a_k \in \mathfrak{P}$ satisfy $a_0 \cap \cdots \cap a_k \neq \emptyset$. We have to show that $\langle \zeta^b_{\mathfrak{M}} \circ \varphi^b_{\mathfrak{N}}(a_0), \ldots, \zeta^b_{\mathfrak{M}} \circ \varphi^b_{\mathfrak{N}}(a_i), a_i, \ldots, a_k \rangle$ forms a simplex in $X^c(\mathfrak{M})$. By definition of $\varphi^b_{\mathfrak{N}}$ and $\zeta^b_{\mathfrak{M}}$, vertex $\zeta^b_{\mathfrak{M}} \circ \varphi^b_{\mathfrak{N}}(a_j)$ $(0 \leq j \leq i)$ is a set in $M_j \in \mathfrak{M}$ such that for a certain $x_j \in a_j$ and its neighbourhood $N_j \in \mathfrak{N}$, $M_j \supset St(N_j; \mathfrak{N})$ holds. Since $a_0 \cap \cdots \cap a_k \neq \emptyset$, there is a set $N \in \mathfrak{N}$ such that $a_0 \cup \cdots \cup a_k \subset St(a_0; \mathfrak{P}) \subset N$. Since $(N_j; \mathfrak{N})$ includes N for each $j = 0, \ldots, i$, M_j includes N for each $j = 0, \ldots, i$. Hence $M_1 \cap \cdots \cap M_i \cap a_i \cap \cdots a_k \supset a_0 \cap \cdots \cap a_k \neq \emptyset$, so that $\langle \zeta^b_{\mathfrak{M}} \circ \varphi^b_{\mathfrak{N}}(a_0), \ldots, \zeta^b_{\mathfrak{M}} \circ \varphi^b_{\mathfrak{N}}(a_i), a_i, \ldots, a_k \rangle$ is a simplex in $X^c(\mathfrak{M})$. ■

3 Vietoris-Begle's Theorem and Local Connectedness

Vietoris-Begle Mapping

It is sometimes convenient to use the notion of *reduced* set of 0-cycles and *reduced 0-th homology groups.* Reduced 0-th homology group is obtained by considering only cycles in which the sum of coefficients is 0. For 0-th homology group $H_0(X) = Z_0(X)/B_0(X)$, the reduced homology group will be denoted by $\tilde{H}_0(X) = \tilde{Z}_0(X)/B_0(X)$, where $\tilde{Z}_0(X) = \{z \in Z_0(X) | (z = \sum \alpha_i \sigma_i) \Longrightarrow (\sum \alpha^i = 0)\}$. Topological space X is called *acyclic* under a certain homology theory, if (1) X is non-empty, (2) the homology groups $H_q(X)$ are 0 for all $q > 0$, and (3) the 0-th homology group $H_0(X)$ equals to the coefficient group $\boldsymbol{F}$ (or the 0-th reduced homology group $\tilde{H}_0(X)$ equals to 0).

Let X and Y be compact Hausdorff spaces. For Vietoris $\mathfrak{M}$-complex $X^v(\mathfrak{M})$ and subset W of X, the set of all Vietoris $\mathfrak{M}$-simplexes whose vertices are points in W forms a subcomplex of $X^v(\mathfrak{M})$ and is denoted by $X^v(\mathfrak{M}) \cap W$. Then continuous function f of X onto Y is called a *Vietoris-Begle mapping of order* n if for each covering $\mathfrak{M}$ of X and for each $y \in Y$, there is a covering $\mathfrak{P} = \mathfrak{P}(\mathfrak{M}, y)$ of X with $\mathfrak{P} \preccurlyeq \mathfrak{M}$ such that each q-dimensional $(0 \leq q \leq n)$ Vietoris $\mathfrak{P}$-cycle $z^q(\mathfrak{P}) \in X^v(\mathfrak{P}) \cap f^{-1}(y)$ bounds a $q+1$-dimensional Vietoris $\mathfrak{M}$-chain $c^{q+1}(\mathfrak{M}) \in X^v(\mathfrak{M}) \cap f^{-1}(y)$, where all 0-dimensional cycles are chosen in the reduced sense (Figure 4). Continuous function $f : X \to Y$ is said to be a *Vietoris mapping* if the compact set $f^{-1}(y)$ is acyclic for all $y \in Y$, i.e., $H^v_n(f^{-1}(y)) = 0$ for all $n > 0$ and $\tilde{H}^v_0(f^{-1}(y)) = 0$. If f is a Vietoris-Begle mapping of order n for all n, by definition of the inverse limit, f is clearly a Vietoris mapping. Converse is also true in our special settings. In this subsection, we see the following two important theorems: (1) if the coefficient group $\boldsymbol{F}$ is a field, Vietoris mapping is a Vietoris-Begle mapping of order n for all n, and (2) if $f : X \to Y$ is a Vietoris-Begle mapping of order n, there are isomorphisms between $H^v_q(X)$ and $H^v_q(Y)$ $(0 \leq q \leq n)$. In this section, we see (1). Assertion (2) is treated in the next section after the concept of Vietoris-Begle barycentric subdivision is defined.

Since coefficient group $\boldsymbol{F}$ is supposed to be a field, inverse systems of Vietoris and Čech type chains, cycles, boundaries, and homology groups are systems of vector spaces. Especially, all n-dimensional chain,

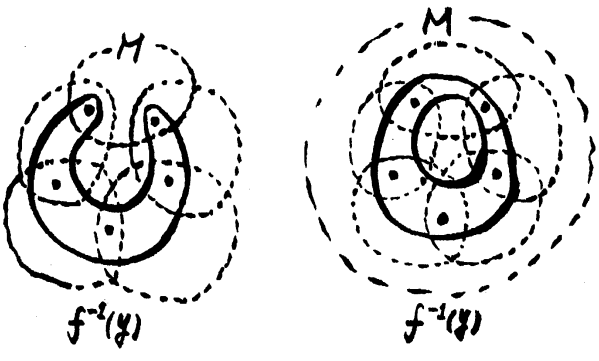

Figure 4: Vietoris-Begle Mapping of order n

cycle, and boundary groups of nerves (defining Čech homology groups) are finite dimensional. For an inverse system of finite dimensional vector spaces, we know the following result on *essential elements*.[8]

Lemma 3.1: (Essential Elements for an Inverse System of Finite Dimensional Vector Spaces) Let $(E_i, \pi_{ij})_{i,j\in I, j\geq i}$ over directed set $(I, \geq)$ be an inverse system of finite dimensional vector spaces. Then for every i there is an element $j_0 \geq i$ such that for all $j \geq j_0$, every element x_i of $\pi_{ij}(E_j) \subset E_i$ is an essential element of E_i, i.e., $x_i \in \pi_{ik}(E_k)$ for all $k \geq i$.

PROOF : The set of essential elements of E_i is the subspace $H_i = \bigcap_{j\geq i} \pi_{ij}(E_j)$. Since E_i is finite dimensional, the dimension of H_i is also finite, say n. Then there are finite elements $k_1, \ldots, k_n$ of I such that $H_i = \bigcap_{j=1}^{n} \pi_{ik_j}(E_{k_j})$. Let j_0 be an element of I such that $j_0 \geq j_k$ for each $k = 1, \ldots, n$. Then for all $j \geq j_0$, we have $\pi_{ij}(E_j) = \pi_{ij_0}(\pi_{j_0 j}(E_j)) \subset \pi_{ij_0}(E_{j_0}) = \pi_{ij_k}(\pi_{j_k j_0}(E_{j_0})) \subset \pi_{ij_k}(e_{j_k})$ for each $k = 1, \ldots, n$. Hence, for each $j \geq j_0$, $\pi_{ij}(E_j) \subset H_i = \bigcap_{j=1}^{n} \pi_{ik_j}(E_{k_j})$. ■

Since the inverse system for Čech homology group (for compact Hausdorff space X) is a system of finite dimensional vector spaces, it follows from Lemma 3.1 that for each covering $\mathfrak{M}$ of X, there is a refinement $\mathfrak{N} \preccurlyeq \mathfrak{M}_0 = {}^*\mathfrak{M}$ such that if $z^q(\mathfrak{N}) \in Z_k^c(\mathfrak{N})$ is a q-dimensional $\mathfrak{N}$-cycle of X, then $p_q^{\mathfrak{M}_0 \mathfrak{N}}(z^q(\mathfrak{N}))$ is the $\mathfrak{M}_0$-coordinate of a Čech cycle. By taking the finest $\mathfrak{N}$ for $q = 0, 1, \ldots, k$ and taking $\mathfrak{P} = {}^*\mathfrak{N}$, we have the following theorem.[9]

Theorem 3.2: (Vietoris-Begle Mapping Theorem I) Let $\mathfrak{M}$ be a covering of compact Hausdorff space X and W be a compact subset of X such that every q-dimensional Čech reduced cycle in W $(0 \leq q \leq k)$ bounds a $q+1$-dimensional Čech chain in W $(\tilde{H}_q^c(W) = 0)$.[10] Then there is a refinement $\mathfrak{P}$ of $\mathfrak{M}$ such that every q-dimensional Victoris $\mathfrak{P}$-cycle on W $(0 \leq q \leq k)$ bounds a $q+1$-dimensional Victoris $\mathfrak{M}$-chain on W. Hence, Vietoris mapping is a Vietoris-Begle mapping of order n for all n.

PROOF : Take refinements $\mathfrak{P} = {}^*\mathfrak{N}$ and $\mathfrak{N}$ of $\mathfrak{M}_0 = {}^*\mathfrak{M}$ as stated in the previous paragraph. Let $\gamma_{\mathfrak{P}}^q$ be a q-dimensional Vietoris $\mathfrak{P}$-cycle on W $(0 \leq q \leq k)$. Denote by $\zeta_{\mathfrak{N}}^b : X^v(\mathfrak{P}) \to X^c(\mathfrak{N})$ the simplicial mapping

[8]This concept of importance in the homology theory of system of groups is due to Čech (1932). See also Lefschetz (1942; p.79) and Steenrod (1936) for elementary compact coefficient groups.

[9]The assertion may be considered as a part of Vietoris-Begle's Theorem. We can see the same (though more abbreviated) argument in the proof of Theorem 2 in Begle (1950a).

[10]For notational convenience, let us define here $\tilde{H}_q^c(W)$ as $\tilde{H}_q^c(W) = H_q^c(W)$ for all $q > 0$.

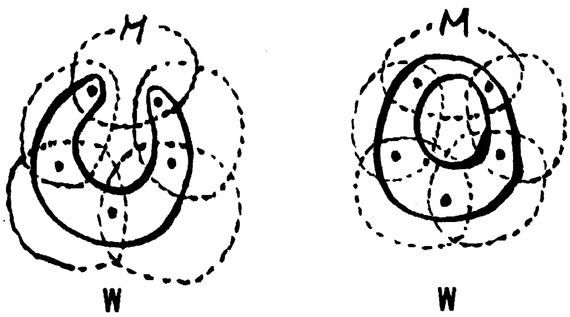

Figure 5: Cycles on Acyclic Set W

defined in the proof of Theorem 2.3. Then $\zeta^b_{\mathfrak{M}q}(\gamma^q_{\mathfrak{P}})$ is a q-dimensional Čech $\mathfrak{N}$-cycle $(0 \le q \le k)$. By definition of $\mathfrak{N}$, $p_q^{\mathfrak{M}_0 \mathfrak{N}} \zeta^b_{\mathfrak{N}}(\gamma^q_{\mathfrak{P}})$ is the $\mathfrak{M}_0$-coordinate of a Čech cycle, z^q, on W. Since $\tilde{H}^c_q(W) = 0$, this Čech cycle bounds so that $p_q^{\mathfrak{M}_0 \mathfrak{N}} \zeta^b_{\mathfrak{N}}(\gamma^q_{\mathfrak{P}}) \sim 0$ on $C^c_q(\mathfrak{M}_0)$. It follows that $\varphi^b_{\mathfrak{M}} p_q^{\mathfrak{M}_0 \mathfrak{N}} \zeta^b_{\mathfrak{N}}(\gamma^q_{\mathfrak{P}}) \sim 0$ on $W^v(\mathfrak{M}) = X^v(\mathfrak{M}) \cap W$, where $\varphi^b_{\mathfrak{M}}$ is the simplicial mapping defined in the proof of Theorem 2.3 and $X^v(\mathfrak{M}) \cap W$ denotes the subcomplex of Victoris $\mathfrak{M}$-simplexes on W. Hence, the first assertion of this theorem follows if we see $\varphi^b_{\mathfrak{M}} p_q^{\mathfrak{M}_0 \mathfrak{N}} \zeta^b_{\mathfrak{N}}(\gamma^q_{\mathfrak{P}}) \sim \gamma^q_{\mathfrak{P}}$ on $X^v(\mathfrak{M}) \cap W$. We can see it, however, by repeating completely the same argument with (2-2) in the proof of Theorem 2.3. The second assertion follows immediately from the first if we set $W = f^{-1}(y)$ for *Vietoris mapping* $f : X \to Y$ and point $y \in Y$. ∎

Locally Connected Spaces

Besides the Vietoris-Begle mapping, there is another important concept for fixed point arguments under the Čech type homology, the *local connectedness*. In the Čech type homology theory, the family of open coverings, $\mathit{Cover}(X)$, on space X is used in describing two fundamental features of topological arguments: (i) the measure of connectivity (represented by the intersection property among open sets), and (ii) the measure of convergence or approximation (as a net of refinements of coverings). All analytic concepts are changed into algabraic ones through above two channels. In the following, it is especially important to notice about the second feature, so that each covering $\mathfrak{M} \in \mathit{Cover}(X)$ is used as a sort of metric or a norm, and $\mathit{Cover}(X)$ is used as if it were the uniformity in describing the total convergence properties for space X . To emphasize that we are choosing a covering or a refinement for the second purpose, we call it *norm covering* or *norm refinement* instead of saying a covering or refinement.

The local connectedness is defined as a purely homological notion to generalize the concept of ***absolute neighborhood retracts*** frequently used under the framework of metrizable spaces. Let us consider a compact Hausdorff space Y and $\mathfrak{M} \in \mathit{Cover}(Y)$. A *realization* of simplicial complex K in $Y^v(\mathfrak{M})$ is a chain map τ. Partial realization τ' of K is a chain map defined on a subcomplex L of K such that $\mathit{Vert}(L) = \mathit{Vert}(K)$. For a norm covering $\mathfrak{N} \in \mathit{Cover}(X)$ and realization τ of K, write norm$(\tau) \le \mathfrak{N}$ if for each simplex σ of K, there is a set $N \in \mathfrak{N}$ which contains the underlying space $|\tau\sigma|$ of the chain $\tau\sigma$.[11]

[11] For a value under a homomorphism, parenthesis are abbreviated as $\tau\sigma = \tau(\sigma)$. Note also that the underlying space of chain $\tau\sigma$ is the underlying space of the corresponding complex defined by all simplexes of $\tau\sigma$ (appeared with non-zero coordinates in the formal summation).

DEFINITION 3.3: (Locally Connected Space) Topological space X is said to be *locally connected* (abbreviated by lc) if for each norm covering $\mathfrak{C} \in Cover(X)$ there is a norm refinement $\mathfrak{J} \preccurlyeq \mathfrak{C}$ satisfying the following condition: for each covering $\mathfrak{M}$, there is a refinement $\mathfrak{N}$ such that every partial realization τ' of finite complex K into $X^v(\mathfrak{N})$ with norm$(\tau') \leq \mathfrak{J}$ may be extended to a realization τ into $X^v(\mathfrak{M})$ with norm$(\tau) \leq \mathfrak{C}$.

It is clear from the definition that if X is lc, then $X \times X$ is also lc. If X is a compact Hausdorff and lc, then every closed subset of X is also lc. Moreover, compact Hausdorff lc spaces has the following strong properties.

Theorem 3.4: (Begle 1950b) If X is compact Hausdorff lc space, following (a) (b) (c) hold.

(a) There is a covering $\mathfrak{N}_0$ of X such that if z is a Vietoris cycle such that $z(\mathfrak{N}) \sim 0$ on $X^v(\mathfrak{N})$ for some $\mathfrak{N} \preccurlyeq \mathfrak{N}_0$, then $z \sim 0$.

(b) The homology groups of X are isomorphic to the corresponding groups of a finite complex.

(c) Each covering $\mathfrak{M}$ of X has a *normal refinement* $\mathfrak{M}'$, i.e., a refinement such that for each cycle $z_{\mathfrak{M}'}$ on $X^v(\mathfrak{M}') \subset X^v(\mathfrak{M})$, there is a Vietoris cycle z such that $z(\mathfrak{M}) = z_{\mathfrak{M}'}$.

Proofs are not so difficult. See Begle (1950b).

4 Nikaido's Analogue of Sperner's Lemma

In this section we see the important second half of the Vietoris-Begle mapping theorem, (2) if $f : X \to Y$ is a Vietoris-Begle mapping of order n, there are isomorphisms between $H^v_q(X)$ and $H^v_q(Y)$ $(0 \leq q \leq n)$. For this proof, we need the concept of barycentric subdivision under the framework of Victoris complexes. After the proof of Vietoris-Begle mapping theorem, we also see an extension of Sperner's lemma which was originally given by Nikaido (1959) as the first application.

Vietoris-Begle Barycentric Subdivision

Let Y be a compact Hausdorff topological space. Consider coverings $\mathfrak{N} \in Cover(Y)$ and $\mathfrak{R} \in Cover(Y)$ of Y. In the following, for Vietoris $\mathfrak{M}$-chain $c(\mathfrak{M}) \in C^v_q(\mathfrak{M})$, let us denote by $K(c(\mathfrak{M}))$ the complex of all simplexes appeared with positive coefficients in $c(\mathfrak{M})$ and by diam $|c(\mathfrak{M})| \leq \mathfrak{N}$ the fact that there is an element $N \in \mathfrak{N}$ in which all vertices of $K(c(\mathfrak{M}))$ belong. Moreover, for each q-dimensional chain $c^q \in C^v_q(\mathfrak{N})$ and $y \in Y$, we denote by $y * c$ the $(q+1)$-dimensional $\{Y\}$-chain defined as the extension of the operation $y * \langle a_0 \cdots a_k \rangle = \langle y a_0 \cdots a_k \rangle$ for each oriented k-dimensional simplex $\langle a_0 \cdots a_k \rangle$.[12] $\mathfrak{R}\mathfrak{N}$-*barycentric subdivision* of k-dimensional Vietoris $\mathfrak{R}$-simplex $\sigma^k \in X^v(\mathfrak{R})$ is chain map $Sd_q : C^v_q(\mathfrak{R}) \to C^v_q(\mathfrak{N})$, satisfying the following conditions.

(SD1) For each 0-dimensional simplex y_0 of $K(\sigma^k)$, $Sd_0(y_0) = y_0$.

(SD2) For each q-dimensional simplex $\langle y_0 \cdots y_q \rangle$ $(0 < q \leq k)$ in $K(\sigma^k)$, there exists $y \in Y$ such that $y * Sd_{q-1}(\langle y_0 \cdots \hat{y}_i \cdots y_q \rangle) \in C^v_q(\mathfrak{N})$ for each i and $Sd_q(\langle y_0 \cdots y_q \rangle) = \sum_{i=0}^q (-1)^i y * Sd_{q-1}(\langle y_0 \cdots \hat{y}_i \cdots y_q \rangle)$.

(SD3) diam $|Sd_k \sigma^k| \leq \mathfrak{N}$.

[12] Note that in the above $\{Y\} \in Cover(Y)$ is taken as a covering of Y.

Note that as long as the existence of y for each q-dimensional $\mathfrak{R}$-simplex $\langle y_0 \cdots y_q \rangle$ stated in (SD2) is assured, condition (SD1) and (SD2) may be considered as a process to construct Sd_q, $q = 0, 1, \cdots$. By mathematical induction, we can verify for each $q > 0$ that $\partial_q Sd_q(\langle y_0 \cdots y_q \rangle) = Sd_{q-1}\partial_q(\langle y_0 \cdots y_q \rangle)$, so that Sd_q constructed is indeed a chain map.

Let us consider *n-skeleton* $Y_n^v(\mathfrak{R}) \subset Y^v(\mathfrak{R})$ of $Y^v(\mathfrak{R})$, the subcomplex of all k-dimensional $(0 \leq k \leq n)$ Vietoris $\mathfrak{R}$-simplexes on Y. An *n-dimensional* $\mathfrak{R}\mathfrak{N}$*-barycentric subdivision of* Y is a chain map $\{Sd_q^{\mathfrak{R}\mathfrak{N}} : C_q^v(Y_n^v(\mathfrak{R})) \to C_q^v(\mathfrak{N})\}$ such that for each k-dimensional simplex σ^k $(0 \leq k \leq n)$, the restriction of $\{Sd_q^{\mathfrak{R}\mathfrak{N}}\}$ on the chain of subcomplex of $Y_n^v(\mathfrak{R})$ defined by σ^k is an $\mathfrak{R}\mathfrak{N}$-barycentric subdivision of σ^k.

Next, assume that there is a continuous onto map f on compact Hausdorff space X to Y. For each pair of coverings $\mathfrak{M} \in \boldsymbol{Cover}(X)$ and $\mathfrak{N} \in \boldsymbol{Cover}(Y)$ such that $\mathfrak{M} \preccurlyeq \{f^{-1}(N) | N \in \mathfrak{N}\}$, f induces simplicial map $X^v(\mathfrak{M}) \ni a_0 \cdots a_k \mapsto f(a_0) \cdots f(a_k) \in Y^v(\mathfrak{N})$ so that chain map $\{f_q : C_q^v(\mathfrak{M}) \to C_q^v(\mathfrak{N})\}$. Then as we can see in the next theorem, if f is Vietoris-Begle mapping of order n, there is a chain map $\tau = \{\tau_q\}$ on $(n+1)$-skeleton of $Y^v(\mathfrak{R})$ to $X(\mathfrak{M})$ such that $\{f_q \circ \tau_q\}$ is an $n+1$-dimensional $(\mathfrak{R}\mathfrak{N})$-barycentric subdivision of Y. Moreover, given $\mathfrak{M}$, such refinement $\mathfrak{R}$ may be taken arbitrarily small and corresponding τ's may be defined as (Vietoris homologically) unique.

Theorem 4.1: Let X and Y be compact Hausdorff spaces and let $f : X \to Y$ be a Vietoris-Begle mapping of order n. For each $\mathfrak{M} \in \boldsymbol{Cover}(X)$ and $\mathfrak{N} \in \boldsymbol{Cover}(Y)$ such that $\mathfrak{M} \preccurlyeq \{f^{-1}(N) | N \in \mathfrak{N}\}$, there exist a cover $\mathfrak{R} = \mathfrak{R}(\mathfrak{M}, \mathfrak{N}) \in \boldsymbol{Cover}(Y)$ and a chain map $\tau = \{\tau_q\}$ on $(n+1)$-skeleton of $Y^v(\mathfrak{R})$ to $X^v(\mathfrak{M})$ such that chain map $\{f_q \circ \tau_q\}$ is an n-dimensional $(\mathfrak{R}\mathfrak{N})$-barycentric subdivision of Y. Moreover, for any $\mathfrak{S} \in \boldsymbol{Cover}(Y)$, there are $\mathfrak{R}'$ and τ' satisfying the same condition with $\mathfrak{R}$ and τ such that $\mathfrak{R}' \preccurlyeq \mathfrak{S}$ and $\tau_q'(z^q) \sim \tau_q(z^q)$ in $C_q^v(\mathfrak{M})$ for all $z^q \in Z_q^v(\mathfrak{R}')$.

Above theorem shows an essential feature of the Vietoris-Begle mapping and plays crucial roles in the proof of the Vietoris-Begle mapping theorem. Before proving it, I introduce one technical lemma. In Lemma 2.2, we have seen one of the simplest kind of *prismatical relation* that may be utilized to show the equivalence between two cycles. There exists another convenient (though a little bit more complicated) method in forming prisms. Denote by $\{0, 1, I\}$ the one dimensional abstract complex formed by two 0-dimensional simplices 0 and 1 together with 1-dimensional simplex I whose boundaries are 0 and 1 under relation $\partial_1(I) = 1 - 0$. For simplicial complex K, the *product complex* of K and $\{0, 1, I\}$ denoted by $K \times \{0, 1, I\}$ is the family of simplexes of the form $\sigma \times 0$, $\sigma \times 1$, and $\sigma \times I$, where σ runs through all simplexes in K. Boundary relations on $K \times \{0, 1, I\}$ are defined as $\partial(\sigma \times 0) = (\partial\sigma) \times 0$, $\partial(\sigma \times 1) = (\partial\sigma) \times 1$, and $\partial(\sigma \times I) = (\partial\sigma) \times I + (\sigma \times 1) - (\sigma \times 0)$. (See Figure 6.) It should be noted that $K \times \{0, 1, I\}$ is no longer a simplicial complex. The subcomplex of $K \times \{0, 1, I\}$ constructed by all simplexes of the form $\sigma \times 0$ may clearly be identified with K and is called the *base* of $K \times \{0, 1, I\}$. There also exists an isomorphism between K and the subcomplex of all simplexes of the form $\sigma \times 1$, which is called the *top* of $K \times \{0, 1, I\}$. Then for each cycle z on K, we have $\partial(z \times I) = (z \times 1) - (z \times 0)$, immediately, so that $z \times 1 \sim z \times 0$ in $K \times \{0, 1, I\}$. Therefore, as before (Lemma 2.2) if there exists a chain mapping θ on $K \times \{0, 1, I\}$ to a certain simplicial complex L, we have the following.

Lemma 4.2: Assume that there is a chain mapping θ on $K \times \{0, 1, I\}$ to simplicial complex L. For two images $\theta_{q+1}(z^q \times 0)$ and $\theta_{q+1}(z^q \times 1)$ in the q-th chain group $C_q(L)$ of q-cycle $z^q \in C_q(K)$ (through the induced homomorphism $\theta_{q+1} : C_{q+1}(K \times \{0, 1, I\}) \to C_q(L)$), we have $\theta_{q+1}(z^q \times 0) \sim \theta_{q+1}(z^q \times 1)$ on L.

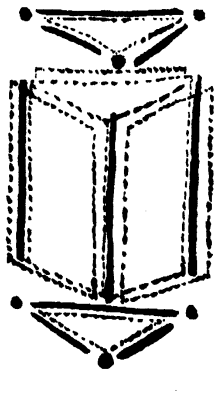

Figure 6: Prism $K \times \{0, 1, I\}$

Proof of Theorem 4.1 : We shall use four steps. Step 1 is devoted to prepare for basic tools. In Step 2, we construct $\mathfrak{R}$. Step 3 is used to define τ. Step 4 is assigned for constructions of $\mathfrak{R}'$ and τ'.

(Step1) By the definition of Vietoris-Begle mapping, there is a covering $\mathfrak{P}(\mathfrak{M}, y)$ for each $y \in Y$ and $\mathfrak{M}$. Consider closed (compact) subset $X \setminus St(f^{-1}(y); {}^*\mathfrak{P}(\mathfrak{M}, y))$. Then the image under f of $X \setminus St(f^{-1}(y); {}^*\mathfrak{P}(\mathfrak{M}, y))$ is also closed (compact) subset of the normal space Y disjointed from $\{y\}$. Given $\mathfrak{N} \in \mathit{Cover}(Y)$, chose $Q(\mathfrak{M}, \mathfrak{N}, y) \ni y$ as an element of ${}^*\mathfrak{N}$ and $\mathfrak{Q}(\mathfrak{M}, \mathfrak{N})$ as a finite subcovering of the covering $\{Q(\mathfrak{M}, \mathfrak{N}, y) | y \in Y\}$. Then covering $\mathfrak{Q}(\mathfrak{M}, \mathfrak{N})$ satisfies that if B is a subset of Y such that $B \subset Q$ for some $Q \in \mathfrak{Q}(\mathfrak{M}, \mathfrak{N})$, there is a point $y \in Y$ such that $St(y; {}^*\mathfrak{N}) \supset B$ and $St(f^{-1}(y); {}^*\mathfrak{P}(\mathfrak{M}, y)) \supset f^{-1}(B)$. In this proof we call this y the corresponding point of Y to B and use it as if it were the barycenter of points in B.

(Step 2) Hence, for each $\mathfrak{M} \in \mathit{Cover}(X)$ and $\mathfrak{N} \in \mathit{Cover}(Y)$, $\mathfrak{Q}(\mathfrak{M}, \mathfrak{N}) \in \mathit{Cover}(Y)$ satisfies that for every q-dimensional $\mathfrak{Q}(\mathfrak{M}, \mathfrak{N})$-simplex $\langle y_0 \cdots y_q \rangle$, $(0 \leq q \leq n)$, there is a point $y \in Y$ such that $y * \langle y_0 \cdots y_q \rangle$ is a ${}^*\mathfrak{N}$-simplex and $St(f^{-1}(y); {}^*\mathfrak{P}(\mathfrak{M}, y)) \supset f^{-1}(\{y_0, \ldots, y_q\})$. This suggests the possibility to obtain a sequence of refinements $\mathfrak{M}_1 \preccurlyeq \cdots \preccurlyeq \mathfrak{M}_{n+1} = \mathfrak{M}$ together with refinements $\mathfrak{N}_0 \preccurlyeq \cdots \preccurlyeq \mathfrak{N}_{n+1} = \mathfrak{N}$ such that $\mathfrak{M}_k \preccurlyeq \{f^{-1}(N) | N \in \mathfrak{N}_k\}$ for each $k = 1, \ldots, n+1$, and for each q-dimensional $\mathfrak{N}_q$-simplex $(q = 0, \ldots, n)$ $\langle y_0 \cdots y_q \rangle$, there exists $y \in Y$ such that $y * \langle y_0 \cdots y_q \rangle$ is a ${}^*\mathfrak{N}_{q+1}$-simplex and $St(f^{-1}(y); {}^*\mathfrak{P}(\mathfrak{M}_{q+1}, y)) \supset f^{-1}(\{y_0, \ldots, y_q\})$. (As we see in the next step, under the definition of barycentric subdivision (SD1)–(SD3), this property shows that for each $n+1$-dimensional $\mathfrak{N}_0$-simplex we are possible to define an $\mathfrak{N}_0\mathfrak{N}_{n+1}$-barycentric subdivision.) Indeed, given $\mathfrak{N}_{n+1} = \mathfrak{N}$ and $\mathfrak{M}_{n+1} = \mathfrak{M}$, set $\mathfrak{N}_n = \mathfrak{Q}(\mathfrak{M}_{n+1}, {}^*\mathfrak{N}_{n+1}) \preccurlyeq {}^{**}\mathfrak{N}_{n+1}$. Note that with $\mathfrak{Q}(\mathfrak{M}_{n+1}, {}^*\mathfrak{N}_{n+1})$ associates finite $y_{n+1,i}$'s such that $\mathfrak{Q}(\mathfrak{M}_{n+1}, {}^*\mathfrak{N}_{n+1})$ consists of $Q(\mathfrak{M}_{n+1}, {}^*\mathfrak{N}_{n+1}, y_{n+1,i})$'s. Let $\mathfrak{M}_n$ be a common refinement of coverings ${}^*\mathfrak{P}(\mathfrak{M}_{n+1}, y_{n+1,i})$'s and $\{f^{-1}(N) | N \in \mathfrak{N}_n\}$. Set $\mathfrak{N}_{n-1} = \mathfrak{Q}(\mathfrak{M}_n, {}^*\mathfrak{N}_n)$. Repeat the process until we obtain $\mathfrak{N}_0$. Define $\mathfrak{R}$ as $\mathfrak{R} = \mathfrak{R}(\mathfrak{M}, \mathfrak{N}) = \mathfrak{N}_0$.

(Step 3) Let us define τ_q $(0 \leq q \leq n)$ on chains of $Y^v(\mathfrak{R}) = Y^v(\mathfrak{N}_0)$ to $X^v(\mathfrak{M})$. Consider a 0-dimensional Vietoris $\mathfrak{R}$-simplex, σ^0, of $Y^v(\mathfrak{R})$. σ^0 may be identified with a point y_0 in Y. Define $\tau(\sigma^0)$ as 0-dimensional Victorias $\mathfrak{M}_0$-simplex ξ^0 of $X^v(\mathfrak{M}_0)$ which may be identified with an arbitrary point $x_0 \in f^{-1}(y_0) \subset X$. Then we have $f_0 \circ \tau_0(\sigma^0) = \sigma^0 = Sd_0(\sigma^0)$, so that we obtain τ_0 by linearly extending it. Next, consider k-dimensional Victoris $\mathfrak{R}$-simplex, σ^k, of $Y^v(\mathfrak{R})$ $(0 < k \leq n+1)$. Suppose that for each $(k-1)$-dimensional $\mathfrak{R}$-simplex σ^{k-1}, $\tau_{k-1}(\sigma^{k-1})$ is already defined and satisfies that $f_{k-1} \circ \tau_{k-1}(\sigma^{k-1})$ is a $\mathfrak{R}\,{}^*\mathfrak{N}_{k-1}$-barycentric subdivision of σ^{k-1} together with the relation of chain map, $\partial_{k-2} \circ \tau_{k-1} = \tau_{k-2} \circ \partial_{k-1}$, where τ_{k-2} for $k = 1$ is defined to be 0-map. In the following, we see that we may define $\tau_k(\sigma^k)$ so as to satisfy that $\partial_{k-1} \circ \tau_k = \tau_{k-1} \circ \partial_k$ and $f_k\tau_k\sigma^k$ is a $\mathfrak{R}\,{}^*\mathfrak{N}_k$-barycentric subdivision of σ^k for each k-dimensional Victoris $\mathfrak{R}$-simplex σ^k. Then by the mathematical induction, we may extend the definition of τ_k until it is finally defined on all of the $(n+1)$-skeleton of $Y(\mathfrak{R})$. Since $\partial_k\sigma^k$ is an $\mathfrak{R}$-chain, $\tau_{k-1}\partial_k\sigma^k$ is already defined and is a $\mathfrak{M}_k$-cycle since $\partial_{k-1}\tau_{k-1}\partial_k\sigma^k = \tau_{k-2}\partial_{k-1}\partial_k\sigma^k = 0$. By assumption $f_{k-1}\tau_{k-1}\partial_k\sigma^k = f_{k-1}\tau_{k-1}\sum_{i=0}^k(-1)^i\sigma_i^{k-1} = \sum_{i=0}^k(-1)^i f_{k-1}\tau_{k-1}\sigma_i^{k-1}$ belongs to $C_{k-1}^v({}^{**}\mathfrak{N}_{k-1})$, where σ_i^{k-1}'s are $k+1$ $(k-1)$-dimensional face of σ^k, and $f_{k-1}\tau_{k-1}\sigma_i^{k-1}$ is a $\mathfrak{R}\,{}^*\mathfrak{N}_{k-1}$-barycentric subdivision of σ_i^{k-1} for each i. It follows that all vertices of the ${}^{**}\mathfrak{N}_{k-1}$-chain, $f_{k-1}\tau_{k-1}\partial_k\sigma^k = f_{k-1}\tau_{k-1}\sum_{i=0}^k(-1)^i\sigma_i^k = \sum_{i=0}^k(-1)^k f_{k-1}\tau_{k-1}\sigma_i^k$, belongs to $St(R_0; {}^{**}\mathfrak{N}_{k-1}) \subset St({}^{**}N_{k-1}; {}^{**}\mathfrak{N}_{k-1})$ for an $R_0 \in \mathfrak{R}$ having all vertices of σ^k as its elements and ${}^{**}N_{k-1} \in {}^{**}\mathfrak{N}_{k-1}$ such that $R_0 \subset {}^{**}N_{k-1}$. Since there exists ${}^*N_{k-1} \in {}^*\mathfrak{N}_{k-1}$ such that $St({}^{**}N_{k-1}; {}^{**}\mathfrak{N}_{k-1}) \subset {}^*N_{k-1}$, we have $\operatorname{diam}|f_{k-1}\tau_{k-1}\partial_k\sigma^k| \le {}^*\mathfrak{N}_{k-1}$. Then $\mathfrak{N}_{k-1} = \mathfrak{Q}(\mathfrak{M}_k, {}^*\mathfrak{N}_k)$ implies that there is corresponding point $y = y_{k,i} \in Y$, $Q(\mathfrak{M}_k, {}^*\mathfrak{N}_k, y_{k,i}) \in \mathfrak{Q}(\mathfrak{M}_k, {}^*\mathfrak{N}_k)$, to $|f_{k-1}\tau_{k-1}\partial_k\sigma^k|$ satisfying the following two relations.[13]

$$St(y; {}^{**}\mathfrak{N}_k) \supset |f_{k-1}\tau_{k-1}\partial_k\sigma^k| \tag{6}$$

$$St(f^{-1}(y); {}^*\mathfrak{P}(\mathfrak{M}_k, y)) \supset f^{-1}(|f_{k-1}\tau_{k-1}\partial_k\sigma^k|) \supset |\tau_{k-1}\partial_k\sigma^k| \tag{7}$$

Denote by z^{k-1} the cycle $\tau_{k-1}\partial_k\sigma^k \in Z_{k-1}^v(\mathfrak{M}_{k-1})$ and let $x_1, \ldots, x_\ell$ be vertices of $K(z^{k-1})$. Note that by (7), there are finite $x_1', \ldots, x_\ell' \in f^{-1}(y)$ and ${}^*P_1, \ldots, {}^*P_\ell \in {}^*\mathfrak{P}(\mathfrak{M}_k, y)$ such that $x_1' \in {}^*P_1, \ldots, x_\ell' \in {}^*P_\ell$ and $x_1 \in {}^*P_1, \ldots, x_\ell \in {}^*P_\ell$. By defining mapping μ on $\mathbf{Vert}(K(z^{k-1}) \times \{0,1\})$ to X as $\mu(x_i, 0) = x_i$ for each vertex $(x_i, 0)$ in the base of $K(z^{k-1}) \times \{0,1\}$ and $\mu(x_i, 1) = x_i'$ for each vertex $(x_i, 1)$ in the top of $K(z^{k-1}) \times \{0,1\}$. It is easy to check that μ is a simplicial map. Indeed, if $((a_0, 0), \ldots, (a_i, 0), (a_i, 1), \ldots, (a_m, 1))$ is a simplex in $K(z^{k-1}) \times \{0,1\}$, then $((a_0, 0), \ldots, (a_m, 1))$ is a simplex in $K(z^{k-1})$, so that there exists element $M_{k-1} \in \mathfrak{M}_{k-1}$ such that $a_0, \ldots, a_m \in M_{k-1}$. Since a_i is equal to some x_j, and both $(x_j, 0)$ and $(x_j, 1)$ are in ${}^*P_j$, all vertices in $(a_0, \ldots, a_i, \mu(a_i, 1), \ldots, \mu(a_m, 1))$ belong to $St(M_{k-1}, {}^*\mathfrak{P}(\mathfrak{M}_k, y))$. By considering the fact that $\mathfrak{M}_{k-1} \preccurlyeq {}^*\mathfrak{P}(\mathfrak{M}_k, y)$, they belong to an element of $\mathfrak{P}(\mathfrak{M}_k, y)$, so that μ maps $K(z^{k-1})$ simplicially to $X^v(\mathfrak{P}(\mathfrak{M}_k, y))$. Let us use μ to define $\tau_k(\sigma^k)$ as follows: Set $\xi_1^k = \mu(\Phi_k(z^{k-1}))$, where Φ_k is the prismatic chain homotopy defined in equations (1)–(3). By (3), we have $\partial_k(\mu\Phi_k z^{k-1}) = \mu(z^{k-1} \times 1) - \mu(z^{k-1} \times 0) = \mu(z^{k-1} \times 1) - z^{k-1}$. Since $\mu(z^{k-1} \times 1)$ is a cycle on $X^v(\mathfrak{P}(\mathfrak{M}_k, y)) \cap f^{-1}(y)$, there is a chain ξ_2^k on $X^v(\mathfrak{P}(\mathfrak{M}_k, y)) \cap f^{-1}(y)$ such that $\partial_k\xi_2^k = \mu(z^{k-1} \times 1)$. Then if we set $\tau_k(\sigma^k) = \xi_2^k - \xi_1^k$, we have $\partial_k\tau_k\sigma^k = z^{k-1} = \tau_{k-1}\partial_{k-1}\sigma^k$, so that τ_k satisfies the condition for chain map. Moreover, since $f_k(\tau_k\sigma^k) = f_k(\xi_2^k - \xi_1^k) = f_k(\xi_2^k) - f_k(\mu(\Phi_k(z^{k-1})))$, we may also rewrite it as $f_k(\xi_2^k) - \hat{\mu}(\hat{\Phi}_k(f_{k-1}z^{k-1})) = f_k(\xi_2^k) - \hat{\mu}(\hat{\Phi}_k(f_{k-1}\tau_{k-1}\partial_k\sigma^k)) = f_k(\xi_2^k) - \hat{\mu}(\hat{\Phi}_k(\mathrm{Sd}_{k-1}\partial_k\sigma^k))$, where $\hat{\Phi}$ is the prismatic chain homotopy on complex $K(f_{k-1}(z^{k-1}))$ to $K(f_{k-1}(z^{k-1})) \times \{0,1\}$ and $\hat{\mu}$ is defined on $K(f_{k-1}(z^{k-1}))$ in exactly the same way as μ, i.e., $\hat{\mu}(f(x_i), 0) = f(x_i)$ and $\hat{\mu}(f(x_i), 1) = f(x_i') = y$. Since $St(y; {}^{**}\mathfrak{N}_k) \supset |f_{k-1}\tau_{k-1}\partial_k\sigma^k|$, μ is a simplicial map on $K(f_{k-1}(z^{k-1})) \times \{0,1\}$ to $Y^v({}^*\mathfrak{N}_k)$. Moreover, $f_k(\tau_k\sigma^k)$ is clearly the join of y with $\mathrm{Sd}_{k-1}\partial_k\sigma^k$ with $\operatorname{diam}|\mathrm{Sd}_k\sigma^k| \le {}^*\mathfrak{N}_k$.

(Step 4) Take $\mathfrak{M}_1' \preccurlyeq \cdots \preccurlyeq \mathfrak{M}_{n+1}'$ and $\mathfrak{N}_0' \preccurlyeq \cdots \preccurlyeq \mathfrak{N}_{n+1}'$ in the same way as $\mathfrak{M}_1 \preccurlyeq \cdots \preccurlyeq \mathfrak{M}_{n+1}$ and $\mathfrak{N}_0 \preccurlyeq \cdots \preccurlyeq \mathfrak{N}_{n+1}$ except for the process to define $\mathfrak{N}_k$ $(k \le n)$. Let us define $\mathfrak{N}_k'$ as a common refinement of $\mathfrak{Q}(\mathfrak{M}_{k+1}', {}^*\mathfrak{N}_{k+1}')$, ${}^*\mathfrak{N}_k$, and $\mathfrak{S}$ for each $k \le n$. Define $\mathfrak{R}'$ as $\mathfrak{N}_0'$ and τ_k' $(0 \le k \le n+1)$ in exactly the same way as τ_k. We now check for each $\mathfrak{R}'$-cycle z^n, $\tau_n(z^n) = \tau_n'(z^n)$. For this purpose, it is sufficient by Lemma 4.2 to show mapping θ to $X^v(\mathfrak{M})$ such that for each $\sigma^k \times 0$, $\theta(\sigma^k \times 0) = \tau_k(\sigma^k)$, and for each $\sigma^k \times 1$, $\theta(\sigma^k \times 1) = \tau_k'(\sigma^k)$, $(0 \le k \le n)$, may be extended as a chain mapping on $K(z^n) \times \{0, 1, I\}$. On the

[13]For Vietoris $\mathfrak{P}$-chain c, $|c|$ denotes the set of all vertices of simplexes appeared in c with positive coefficients.

base and top of $K(z^n) \times \{0,1,I\}$, θ clearly defines chain maps since we have $\partial_k(\theta_k \sigma^k \times 0) = \partial_k(\tau_k(\sigma^k)) = \tau_{k-1}(\partial_k \sigma^k) = \theta_{k-1}(\partial_k \sigma^k \times 0)$ and $\partial_k(\theta_k \sigma^k \times 1) = \partial_k(\tau'_k(\sigma^k)) = \tau'_{k-1}(\partial_k \sigma^k) = \theta_{k-1}(\partial_k \sigma^k \times 1)$.

Let us consider a 0-dimensional simplex σ^0 in $K(z^n)$ and $\sigma^0 \times I \in K(z^n) \times \{0,1,I\}$. By definition (in Step 3) $f_0 \tau_0 \sigma^0 = f_0 \tau'_0 \sigma^0 = \sigma^0$ and both $\tau_0(\sigma^0)$ and $\tau'_0(\sigma^0)$ are points in $f^{-1}(\sigma^0) = f^{-1}(|f_0 \tau_0 \sigma^0|) = f^{-1}(|f_0 \tau'_0 \sigma^0|) \supset |\tau_0 \sigma^0| \cup |\tau'_0 \sigma^0|$. Then it is automatically satisfied that there exists y ($y = \sigma^0$) such that

$$St(y; \mathfrak{N}_1) \supset |\sigma^0| \text{ and}$$
$$St(f^{-1}(y); {}^*\mathfrak{P}(\mathfrak{M}_1, y)) \supset f^{-1}(\sigma^0).$$

Note that $\theta\partial(\sigma^0 \times I) = \tau(\sigma^0) - \tau'(\sigma^0)$. Hence, we have $St(f^{-1}(y); {}^*\mathfrak{P}) \supset |\theta\partial(\sigma^0 \times I)|$ (Figure 7). Let

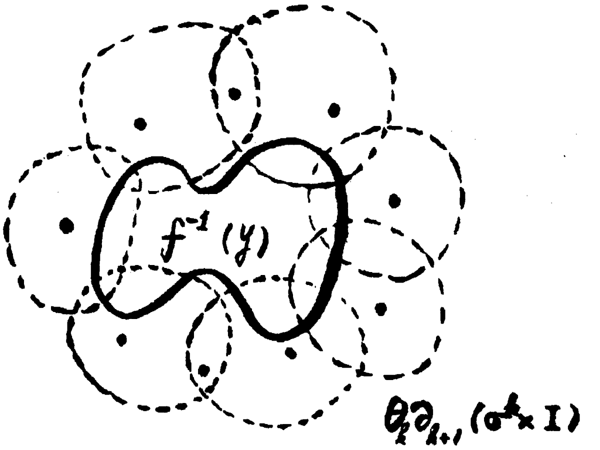

Figure 7: y and $\theta\partial(\sigma^k \times I)$

us consider simplicial complex $K = K(\tau(\sigma^0) - \tau'(\sigma^0))$ and mapping $\omega : \boldsymbol{Vert}(K \times \{0,1\})$ to X such that $\omega(a,0) = a$ and $\omega(a,1) = y^a$, where y^a is an element of $f^{-1}(y)$ satisfying $\{a, y^a\} \subset {}^*P$ for some ${}^*P \in {}^*\mathfrak{P}$. Such y^a exists since $St(f^{-1}(y); {}^*\mathfrak{P}) \supset |\theta\partial(\sigma^0 \times I)|$. Then ω is a simplicial map on $K \times \{0,1\}$ to $X^v(\mathfrak{P})$. As before, let us define ξ_1^1 as $\xi_1^1 = \omega(\Phi(\tau_0\sigma^0 - \tau'_0\sigma^0))$, where Φ denotes the prismatic chain homotopy. Note that $\partial\xi_1^1 = \omega((\tau_0\sigma^0 - \tau'_0\sigma^0) \times 1) - (\tau_0\sigma^0 - \tau'_0\sigma^0)$. Now $\omega((\tau_0\sigma^0 - \tau'_0\sigma^0) \times 1)$ is a 0-cycle (by the previous equation) on $X^v(\mathfrak{P}) \cap f^{-1}(y)$, there is a 1-chain ξ_2^1 on $X^v(\mathfrak{M}_1) \cap f^{-1}(y)$ such that $\partial\xi_2^1 = \omega((\tau_0\sigma^0 - \tau'_0\sigma^0) \times 1)$. Define $\theta(\sigma^0 \times I)$ to be $\xi_2^1 - \xi_1^1$. Then θ satisfies the condition of chain map $\partial\theta = \theta\partial$ for $\sigma^0 \times I$ for each 0-dimensional σ^0. Clearly, $f|\xi_2^1 - \xi_1^1|$ is the join of y and $\sigma^0 = y$, so that $\operatorname{diam} f|\xi_2^1 - \xi_1^1| \le {}^*\mathfrak{N}_1$

Next assume that $\theta(\sigma^m \times I)$ is defined for each $m \le k$ in such a way that $\partial\theta = \partial\theta$, $\theta(\sigma^m \times I) \in \mathfrak{M}_{m+1}$, and $\operatorname{diam} f|\theta(\sigma^m \times I)| \le {}^*\mathfrak{N}_{m+1}$. Let σ^k be a k-dimensional simplex of $K(z^n)$. Then $\theta(\partial(\sigma^k \times I))$ is already defined. Since $\theta(\partial(\sigma^k \times I)) = \theta((\partial\sigma^k) \times I) + \theta(\sigma^k \times 1) - \theta(\sigma^k \times 0)$, we have $f|\theta(\partial(\sigma^k \times I))| \subset f|\theta(\partial\sigma^k)| \cup f|\tau_k(\sigma^k)| \cup f|\tau'_k \sigma^k|$. By considering facts, $\operatorname{diam} f|\tau_k(\sigma^k)| \le {}^*\mathfrak{N}_k$ and $\operatorname{diam} f|\tau'_k(\sigma^k)| \le {}^*\mathfrak{N}'_k \preccurlyeq {}^*\mathfrak{N}$, we have $St(R'; \mathfrak{N}_k)$ contains $f|\tau_k(\sigma^k)|$ and $f|\tau'_k(\sigma^k)|$, where R' denotes an element of $\mathfrak{R}'$ to which all vertices of σ^k belong. It is also true by assumption that for each $(k-1)$-dimensional face σ^{k-1} of σ^k, $\operatorname{diam} f|\theta(\sigma^{k-1} \times I)| \le {}^*\mathfrak{N}_k$, so that we have $\operatorname{diam} f|\theta\partial(\sigma^k \times I)| \le \mathfrak{N}_k = \mathbf{\Omega}(\mathfrak{M}_{k+1}, {}^*\mathfrak{N}_{k+1})$. Hence, we have a point y such that $Q(\mathfrak{M}_{k+1}, {}^*\mathfrak{N}_{k+1}, y) \in \mathbf{\Omega}(\mathfrak{M}_{k+1}, {}^*\mathfrak{N}_{k+1})$,

$$St(y; {}^*\mathfrak{N}_{k+1}) \supset f|\theta\partial(\sigma^k \times I)| \text{ and}$$

$$St(f^{-1}(y); {}^*\mathfrak{P}(\mathfrak{M}_{k+1}, y)) \supset f^{-1}f|\theta\partial(\sigma^k \times I)|.$$

Hence, we have $St(f^{-1}(y); {}^*\mathfrak{P}(\mathfrak{M}_{k+1}, y)) \supset |\theta\partial(\sigma^k \times I)|$. (See Figure 7.) Consider again simplicial complex $K = K(\theta\partial(\sigma^k \times I))$ and mapping $\omega : \boldsymbol{Vert}(K \times I)$ to X, we may define $\theta(\sigma^k \times I)$ in exactly the same way as before until $k = n$ in such a way that $\partial\theta(\sigma^k \times I) = \theta\partial(\sigma^k \times I)$, $\theta(\sigma^k \times I) \in \mathfrak{M}_{k+1}$, and $\operatorname{diam} f|\theta(\sigma^k \times I)| \leq {}^*\mathfrak{N}_{k+1}$. ■

Vietoris-Begle Mapping Theorem

Let X and Y be two compact Hausdorff spaces and $f : X \to Y$ a continuous mapping. For each covering $N \in \boldsymbol{Cover}(Y)$, $\mathfrak{M}(\mathfrak{N}) = \{f^{-1}(N) | N \in \mathfrak{N}\}$ is a covering of X. It is clear that f maps each $\mathfrak{M}(\mathfrak{N})$-simplex to $\mathfrak{N}$-simplex so that induces a simplicial mapping on $X^v(\mathfrak{M}(\mathfrak{N}))$ to $Y^v(\mathfrak{N})$ and chain mapping $\{f_q^{\mathfrak{N}}\}$. Given q-dimensional Vietoris cycle $\gamma^q = \{\gamma^q(\mathfrak{M}) | \mathfrak{M} \in \boldsymbol{Cover}(X)\}$ of X, define $f_q(\gamma^q)$ as the q-dimensional Vietoris cycle of Y, $\{f_q^{\mathfrak{N}}(\gamma^q(\mathfrak{M}(\mathfrak{N}))) | \mathfrak{N} \in \boldsymbol{Cover}(Y)\}$. The mapping of γ^q to $f_q(\gamma^q)$ clearly induces a homomorphism. The next theorem shows that f_q indeed induces an isomorphism (Figure 8).

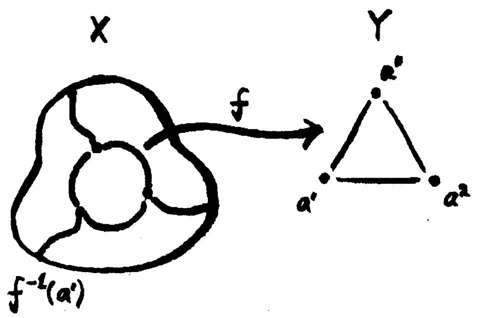

Figure 8: Isomorphism under Vietoris Begle Mapping of order n

Theorem 4.3: (Vietoris Begle Mapping Theorem II: Begle 1950a) Let X and Y be compact Hausdorff spaces. If $f : X \to Y$ is a Vietoris-Begle mapping of order n, there is an isomorphisms between $H_q^v(X)$ and $H_q^v(Y)$ for each $q = 0, 1, \ldots, n$.

PROOF : We shall use three steps to prove the assertion. In Step 1, we construct n-dimensional Vietoris cycle $\{\gamma^n(\mathfrak{M})\}$ of X from $\{z^n(\mathfrak{N})\}$ of Y. By using it, we see in Step 2, the homomorphism induced by f between $H_q^v(X)$ and $H_q^v(Y)$ for each $q = 0, 1, \ldots, n$ is onto. The homomorphism is seen to be one to one in Step 3.

(Step 1) With each $\mathfrak{M} \in \boldsymbol{Cover}(X)$ associate covering $\mathfrak{N}(\mathfrak{M}) \in \boldsymbol{Cover}(Y)$ such that $\mathfrak{M} \preccurlyeq \{f^{-1}(N) | N \in \mathfrak{N}(\mathfrak{M})\}$. If $\mathfrak{M} = \{f^{-1}(N) | N \in \mathfrak{N}\}$ for some $\mathfrak{N}$, it is always assumed that $\mathfrak{N}(\mathfrak{M})$ is equal to one of such $\mathfrak{N}$. Let $z^n = \{z^n(\mathfrak{N}) | \mathfrak{N} \in \boldsymbol{Cover}(X)\}$ (or simply $\{z^n(\mathfrak{N})\}$) be an n-dimensional Vietoris cycle of Y. For each covering $\mathfrak{M} \in \boldsymbol{Cover}(X)$, define $\gamma^n(\mathfrak{M})$ as $\gamma^n(\mathfrak{M}) = \tau_n(z^n(\mathfrak{R}(\mathfrak{M}, \mathfrak{N}(\mathfrak{M}))))$, where $\tau = \{\tau_n\}$ and $\mathfrak{R}(\mathfrak{M}, \mathfrak{N})$ are the chain mapping and the covering defined in Theorem 4.1.

We see that $\gamma^n = \{\gamma^n(\mathfrak{M})\}$ is an n-dimensional Vietoris cycle. Since every $\gamma^n(\mathfrak{M})$ that is an image of the cycle, $\tau_n(z^n(\mathfrak{R}(\mathfrak{M}, \mathfrak{N}(\mathfrak{M}))))$, is obviously an n-dimensional Victoris $\mathfrak{M}$-cycle, all we have to show is

$\gamma^n(\mathfrak{M}) \sim h_n^{\mathfrak{M}\mathfrak{M}''}(\gamma^n(\mathfrak{M}''))$ for each pair $\mathfrak{M}'' \preccurlyeq \mathfrak{M}$. That is,

$$\tau_n(z^n(\mathfrak{R}(\mathfrak{M},\mathfrak{N}(\mathfrak{M})))) \sim h_n^{\mathfrak{M}\mathfrak{M}''}(\tau_n''(z^n(\mathfrak{R}(\mathfrak{M}'', \mathfrak{N}(\mathfrak{M}'')))))$$

for each $\mathfrak{M}'' \preccurlyeq \mathfrak{M}$, where τ'' is the chain mapping associated with $\mathfrak{R}(\mathfrak{M}'',\mathfrak{N}(\mathfrak{M}''))$. For a while, denote $\mathfrak{R}(\mathfrak{M}'',\mathfrak{N}(\mathfrak{M}''))$ by $\mathfrak{R}''$ and $\mathfrak{R}(\mathfrak{M},\mathfrak{N}(\mathfrak{M}))$ by $\mathfrak{R}$. If we omit inclusion map h_n, we have to show $\tau_n(z^n(\mathfrak{R})) \sim \tau_n''(z^n(\mathfrak{R}''))$.

In Step 4 of the proof of second assertion in Theorem 4.1, we may chose

$$\mathfrak{M}_1' \preccurlyeq \cdots \preccurlyeq \mathfrak{M}_{n+1}' \text{ and } \mathfrak{N}_0' \preccurlyeq \cdots \preccurlyeq \mathfrak{N}_{n+1}'$$

as common refinements not only of serieses $\{\mathfrak{M}_k\}$ and $\{\mathfrak{N}_k\}$ constructing τ (in Step 3) for $\mathfrak{M}$ and $\mathfrak{N}$ but also of another streams $\{\mathfrak{M}''_k\}$ and $\{\mathfrak{N}''_k\}$ combined with chain map τ'' for $\mathfrak{M}''$ and $\mathfrak{N}''$ satisfying the same condition with $\mathfrak{M}$ and $\mathfrak{N}$. Since the construction of τ' is independent of τ and τ'', by repeating the same argument (to construct θ' instead of θ), we can see $\tau_n'(z^n) \sim \tau_n(z^n)$ and $\tau_n'(z^n) \sim \tau_n''(z^n)$ in $C_n^v(\mathfrak{M})$ for all $z^n \in Z_n^v(\mathfrak{R}')$.

That is, there exists common refinement $\mathfrak{R}'$ of $\mathfrak{R} = \mathfrak{R}(\mathfrak{M},\mathfrak{N}(\mathfrak{M}))$ and $\mathfrak{R}'' = \mathfrak{R}(\mathfrak{M}'',\mathfrak{N}(\mathfrak{M}''))$ together with chain map τ' such that $\tau'(z^n(\mathfrak{R}')) \sim \tau(z^n(\mathfrak{R}'))$ and $\tau'(z^n(\mathfrak{R}')) \sim \tau''(z^n(\mathfrak{R}'))$, where τ and τ'' are the chain map associated respectively with $\mathfrak{R}$ and $\mathfrak{R}''$. Hence we have $\tau(z^n(\mathfrak{R}')) \sim \tau''(z^n(\mathfrak{R}'))$. Since z^n is a Vietoris cycle, we know $h_n^{\mathfrak{R}\mathfrak{R}'}(z^n(\mathfrak{R}')) \sim z^n(\mathfrak{R})$ and $h_n^{\mathfrak{R}''\mathfrak{R}'}(z^n(\mathfrak{R}')) \sim z^n(\mathfrak{R}'')$, so that we have $\tau(z^n(\mathfrak{R})) \sim \tau''(z^n(\mathfrak{R}''))$.

(Step 2) We see that f induces an onto mapping. Let z^n be an n-dimensional Vietoris cycle of X and $\gamma^n = \{\tau_n(z^n(\mathfrak{R}(\mathfrak{M},\mathfrak{N}(\mathfrak{M}))))\}$ the n-dimensional Vietoris cycle of Y corresponding to z^n. Let us verify that $f_q(\gamma^n) \sim z^n$. Given $\mathfrak{N} \in \mathbf{Cover}(Y)$, let $\mathfrak{M}$ be the covering $\{f^{-1}(N) | N \in \mathfrak{N}\}$. Then $\gamma^n(\mathfrak{M}) = \tau(z^n(\mathfrak{R}))$, where $\mathfrak{R} = \mathfrak{R}(\mathfrak{M},\mathfrak{N}(\mathfrak{M}))$. It follows that the $\mathfrak{N}$-th coordinate of $f_n(\gamma^n)$, $f_n^{\mathfrak{M}}(\gamma^n(\mathfrak{M}))$, is equal to $f_n^{\mathfrak{M}}\tau_n z^n(\mathfrak{R}(\mathfrak{M},\mathfrak{N}(\mathfrak{M})))$. Note that $\mathfrak{N}(\mathfrak{M})$ may not equal to $\mathfrak{N}$. Since $f_n^{\mathfrak{M}}\tau_n z^n(\mathfrak{R}(\mathfrak{M},\mathfrak{N}(\mathfrak{M})))$ is an $(\mathfrak{R}\mathfrak{N}(\mathfrak{M}))$-barycentric subdivision of $z^n(\mathfrak{R}(\mathfrak{M},\mathfrak{N}(\mathfrak{M})))$, $z^n(\mathfrak{R}) \sim \mathrm{Sd}_n\, z^n(\mathfrak{R}) = f_n^{\mathfrak{M}}(\tau_n z^n(\mathfrak{R}(\mathfrak{M},\mathfrak{N}(\mathfrak{M}))))$ $= f_n^{\mathfrak{M}}(\gamma^n(\mathfrak{M}))$ on $Y^v(\mathfrak{N})$ (as well as on $Y^v(\mathfrak{N}(\mathfrak{M}))$). Moreover, since z^n is a Vietoris cycle, we have $z^n(\mathfrak{R}) \sim z^n(\mathfrak{N})$. It follows that $z^n(\mathfrak{N}) \sim f_n^{\mathfrak{M}}(\gamma^n(\mathfrak{M}))$ on $Y^v(\mathfrak{N})$.

(Step 3) Let us confirm the mapping induced by f is one to one. Since f clearly induces a homomorphism, it is sufficient to show that $f_n(\gamma^n) \sim 0$ means $\gamma^n \sim 0$ for each n-dimensional Vietoris cycle γ^n of X. Given $\mathfrak{M} \in \mathbf{Cover}(X)$, chose $\mathfrak{N} = \mathfrak{N}(\mathfrak{M})$ and $\mathfrak{R} = \mathfrak{R}(\mathfrak{M},\mathfrak{N}(\mathfrak{M}))$ as before. Let $\mathfrak{U} = \{f^{-1}(R) | R \in \mathfrak{R}\}$. Moreover let us recall sequence $\{\mathfrak{M}_k\}$ of refinements of $\mathfrak{M}$ defined in the proof of Theorem 4.1 and $\mathfrak{V}$ a common refinement of $\mathfrak{U}$ and all $\mathfrak{M}_k$'s.

Since γ^n is an n-dimensional Vietoris cycle, $\gamma^n(\mathfrak{V}) \sim \gamma^n(\mathfrak{U})$ on $X^v(\mathfrak{U})$. Then we have $f_n^{\mathfrak{R}}\gamma^n(\mathfrak{V}) \sim f_n^{\mathfrak{R}}\gamma^n(\mathfrak{U})$ on $Y^v(\mathfrak{R})$. But if $f_n(\gamma^n) \sim 0$, $\mathfrak{R}$-th coordinate of $f_n(\gamma^n)$, $f_n^{\mathfrak{R}}\gamma^n(\mathfrak{M}(\mathfrak{R})) = f_n^{\mathfrak{R}}\gamma^n(\mathfrak{U})$, satisfies $f_n^{\mathfrak{R}}\gamma^n(\mathfrak{U}) \sim 0$ on $Y^v(\mathfrak{R})$. Hence, we have $f_n^{\mathfrak{R}}(\gamma^n(\mathfrak{V})) \sim 0$, so that $\tau_n(f_n^{\mathfrak{R}}(\gamma^n(\mathfrak{V}))) \sim 0$, where $\tau = \{\tau_n\}$ is the chain map associated with $\mathfrak{R} = \mathfrak{R}(\mathfrak{M},\mathfrak{N})$. Now it is possible to show $\tau_n(f_n^{\mathfrak{R}}(\gamma^n(\mathfrak{V}))) \sim \gamma^n(\mathfrak{V})$ on $X^v(\mathfrak{M})$. Indeed, let us consider $K = K(\gamma^n(\mathfrak{V}))$ and the product cell-complex $K \times \{0, 1, I\}$ together with chain map θ defined on the base and top of $K \times \{0, 1, I\}$ to $X^v(\mathfrak{M})$ as $\theta(\sigma^k \times 0) = \sigma^k$ and $\theta(\sigma^k \times 1) = \tau_k f_k \sigma^k$ for each simplex σ^k of K. We may extend θ as a chain map on $K \times \{0, 1, I\}$ in exactly the same way with the process stated in the proof of Theorem 4.1. (In Step 4, substitute $\tau_k f_k \sigma^k$ for $\tau_k \sigma^k$ and σ^k for $\tau_k'(\sigma^k)$.) Then we have $\tau_n(f_n^{\mathfrak{R}}(\gamma^n(\mathfrak{V}))) \sim \gamma^n(\mathfrak{V})$ on $X^v(\mathfrak{M})$, so that $\gamma^n(\mathfrak{V}) \sim 0$ since $\tau_n f_n \gamma^n(\mathfrak{V}) \sim 0$ on $X^v(\mathfrak{M})$. Since γ^n is a Vietoris cycle, $\gamma^n(\mathfrak{V}) \sim \gamma^n(\mathfrak{M})$. Thus $\gamma^n(\mathfrak{M}) \sim 0$ on $X^v(\mathfrak{M})$, so $\gamma^n \sim 0$. ∎

Analogue of Sperner's Lemma

Nikaido (1959) treats a theorem which may be considered as an extension of Sperner's lemma based on Vietoris-Begle mapping theorem. Let X and Y be compact Hausdorff spaces. Suppose that Y may be identified (under homeomorphism) with n-dimensional simplex $\langle a^0 a^1 \cdots a^n \rangle$ in Euclidean $(n+1)$-space R^{n+1}. Moreover, assume that there is continuous onto function $f : X \to Y$. For each k-dimensional face $a^{i_0} \cdots a^{i_k}$ of $a^0 \cdots a^n$, denote by $[a^{i_0} \cdots a^{i_k}]$ the set of all convex combination of points of $\{a^0, \ldots, a^k\}$. In this section, we call $f^{-1}([a^{i_0} \cdots a^{i_k}])$ a k-face of X. For point x of X, there exists the smallest dimensional face $a^{i_0} \cdots a^{i_k}$ such that $f(x) \in [a^{i_0} \cdots a^{i_k}]$, the *carrier* of $f(x)$. We also call such $f^{-1}([a^{i_0} \cdots a^{i_k}])$ the carrier of x (Figure 9).

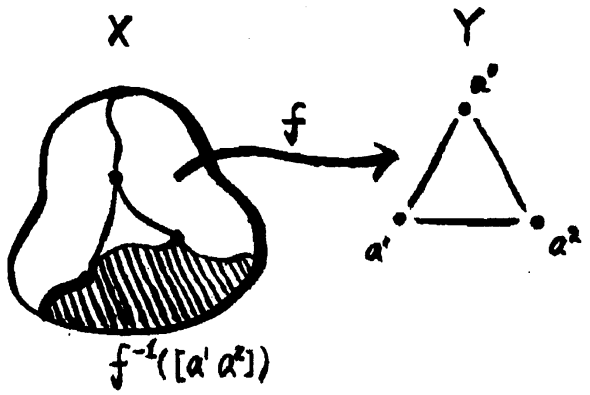

Figure 9: Faces and Carriers

Let us consider a covering $\mathfrak{M} \in \mathit{Cover}(X)$ of X and Vietoris $\mathfrak{M}$-complex $X^v(\mathfrak{M})$. Denote by $K(Y)$ the simplicial complex $K(\langle a^0 a^1 \cdots a^n \rangle)$. Suppose that there exists a chain map $\tau = \{\tau_q\}$ on chains of $K(Y)$ to chains of $X^v(\mathfrak{M})$, $\tau_q : C_q(K(Y)) \to C_q^v(\mathfrak{M})$, satisfying the following two conditions:

(T1) $\tau_k(a^{i_0} \cdots a^{i_k}) \subset f^{-1}([a^{i_0} \cdots a^{i_k}])$ for any k-face $a^{i_0} \cdots a^{i_k}$ of Y.

(T2) $\tau_0(a^i)$ is a single point for each vertex a^i of Y.

We can always construct such τ when f is a Vietoris-Begle mapping. (The same process with the construction of Vietoris-Begle barycentric subdivision in Theorem 4.1 may be utilized.) Operator τ may be considered as a generalization of the usual barycentric subdivision. If $X = Y$ and f is the identity mapping, it is clear that chain map Sd satisfies conditions (T1) and (T2).

A *vertex assignment* v is a mapping on $X = \mathit{Vert}(X^v(\mathfrak{M}))$ to $\{a^0, a^1, \ldots, a^n\} = \mathit{Vert}(K(Y))$ such that for each $x \in X$, $v(x)$ is a vertex of the carrier of $f(x)$. Obviously, v is a simplicial mapping on $X^v(\mathfrak{M})$ to $K(Y)$, so that induces a chain homomorphism which we also denoted by v or $\{v_q\}$, $v_q : C_q^v(\mathfrak{M}) \to C_q(K(Y))$. Given vertex assignment v, we call n-dimensional simplex σ^n in $X^v(\mathfrak{M})$ *regular* if $v_n(\sigma^n) = \langle a^0 a^1 \cdots a^n \rangle$ or $v_n(\sigma^n) = -\langle a^0 a^1 \cdots a^n \rangle$. It is also convenient to define a *sign* $\epsilon(\sigma^m)$ of an m-simplex of $X^v(\mathfrak{M})$ for each $m = 0, 1, \ldots, n$, as $\epsilon(\sigma^m) = 1$ if $v_m(\sigma^m) = \langle a^0 a^1 \cdots a^m \rangle$, $\epsilon(\sigma^m) = -1$ if $v_m(\sigma^m) = -\langle a^0 a^1 \cdots a^m \rangle$, and $\epsilon(\sigma^m) = 0$ otherwise. In the next lemma, we use J as an index set for all n-dimensional simplexes in

$X^v(\mathfrak{M})$.[14]

Lemma 4.4: (Nikaido 1959: Sperner's Lemma) Let $\tau_n(\langle a^0 a^1 \cdots a^n \rangle) = \sum_{j \in J} \alpha_j \sigma_j^n$, where τ denotes the chain map defined above. Then $\sum_{j \in J} \alpha_j \epsilon(\sigma_j^n) \neq 0$. Especially, there exists at least one regular simplex for an arbitrary vertex assignment.

PROOF : Note that in the above expression, $\tau_n(\langle a^0 a^1 \cdots a^n \rangle) = \sum_{j \in J} \alpha_j \sigma_j^n$, the value of τ_n, $\sum_{j \in J} \alpha_j \sigma_j^n$, is a finite sum by definition of the chain map, so that $\alpha_j = 0$ except for finitely many $j \in J$. By condition (T2), the lemma is clearly true for $n = 0$. In the following we show the lemma by using the mathematical induction over n. Let K be an index set for all $(n-1)$-dimensional simplexes in $X^v(\mathfrak{M})$. We call $(n-1)$-dimensional simplex σ^{n-1} in $X^v(\mathfrak{M})$ *regular* if $v_q(\sigma^{n-1}) = \langle a^1 \cdots a^n \rangle$ or $v_q(\sigma^{n-1}) = -\langle a^1 \cdots a^n \rangle$. Assume that the lemma is true for $n-1$, i.e., for f restricted on $f^{-1}([a^1 \cdots a^n])$ to $K(\langle a^1 \cdots a^n \rangle)$, τ restricted on chains of $K(\langle a^1 \cdots a^n \rangle)$, and an arbitrary vertex assignment v on X to $\{a^1 \cdots a^n\}$,

$$\tau_{n-1}(\langle a^1 \cdots a^n \rangle) = \sum_{k \in K} \beta_k \epsilon(\sigma_k^{n-1}) \neq 0,$$

where the summation is taken over all $k \in K$ for the sake of notational simplicity. (There is no problem since $\epsilon(\sigma_k^{n-1}) = 0$ for all $\sigma_k^{n-1} \notin X^v(\mathfrak{M}) \cap f^{-1}([a^1 \cdots a^n])$ by the definition of ϵ.) For our purpose, it is sufficient to show that

$$\sum_{j \in J} \alpha_j \epsilon(\sigma_j^n) = \sum_{k \in K} \beta_k \epsilon(\sigma_k^{n-1}).$$

(Step 1) First, let us see that

$$\sum_{j \in J} \alpha_j \epsilon(\sigma_j^n) = \sum_{j \in J} \alpha_j \sum_{k \in K} [\langle \sigma_k^{n-1} \rangle : \langle \sigma_j^n \rangle] \epsilon(\sigma_k^{n-1}),$$

where $[\cdot : \cdot]$ denotes the incidence number. Indeed, when σ_j^n is regular, there is one and only one regular $(n-1)$-face σ_k^{n-1} of σ_j^n. Let $\langle \sigma_k^{n-1} \rangle = \langle u_1 \cdots u_n \rangle$. If $[\langle \sigma_k^{n-1} \rangle : \langle \sigma_j^n \rangle] = 1$, then by using a certain point $u_0 \in X$, we may write $\langle \sigma_j^n \rangle = \langle u_0 u_1 \cdots u_n \rangle$. Hence, $v_n(\sigma_j^n) = \langle v(u_0) v(u_1) \cdots v(u_n) \rangle = \pm \langle a^0 a^1 \cdots a^n \rangle$ if and only if $v_{n-1}(\sigma_k^{n-1}) = \langle v(u_1) \cdots v(v_n) \rangle = \pm \langle a^1 \cdots a^n \rangle$. Therefore, $\epsilon(\sigma_j^n) = \epsilon(\sigma_k^{n-1})$. If $[\langle \sigma_k^{n-1} \rangle : \langle \sigma_j^n \rangle] = -1$, then we may write $\langle \sigma_j^n \rangle = -\langle u_0 u_1 \cdots u_n \rangle$. Hence, $v_n(\sigma_j^n) = -\langle v(u_0) v(u_1) \cdots v(u_n) \rangle = \pm \langle a^0 a^1 \cdots a^n \rangle$ if and only if $v_{n-1}(\sigma_k^{n-1}) = \langle v(u_1) \cdots v(v_n) \rangle = \mp \langle a^1 \cdots a^n \rangle$. Therefore, $\epsilon(\sigma_j^n) = -\epsilon(\sigma_k^{n-1})$. In each cases, we have $\epsilon(\sigma_j^n) = \sum_{k \in K} [\langle \sigma_k^{n-1} \rangle : \langle \sigma_j^n \rangle] \epsilon(\sigma_k^{n-1})$. When σ_j^n is not regular, we must show that $\sum_{k \in K} [\langle \sigma_k^{n-1} \rangle : \langle \sigma_j^n \rangle] \epsilon(\sigma_k^{n-1}) = 0$ even if σ_j^n has regular faces. Suppose that σ_i^{n-1} is a regular face of σ_j^n and let $\langle \sigma_i^{n-1} \rangle = \langle u_1 \cdots u_n \rangle$. There is a point u_0 of X such that $\boldsymbol{Vert}(\sigma_j^n) = \{u_0, u_1, \ldots, u_n\}$. Since σ_j^n is not regular, there is an m such that $v(u_0) = v(u_m)$. Let σ_k^{n-1} be the face of σ_j^n whose vertices are $\{u_0, u_1, \ldots, u_n\} \setminus \{u_m\}$. Let $\langle \sigma_k^{n-1} \rangle = \langle w_1 \cdots w_n \rangle$. Clearly, σ_j^n has exactly two regular faces, σ_i^{n-1} and σ_k^{n-1}. Then, if $[\langle \sigma_i^{n-1} \rangle : \langle \sigma_j^n \rangle] = 1$ and $[\langle \sigma_k^{n-1} \rangle : \langle \sigma_j^n \rangle] = \pm 1$, we have $\langle \sigma_j^n \rangle = \langle u_0 u_1 \cdots u_n \rangle$ and $\langle \sigma_j^n \rangle = \pm \langle u_m w_1 \cdots w_n \rangle$. Since $\langle u_0 u_1 \cdots u_n \rangle = -\langle u_m u_1 \cdots u_{m-1} u_0 u_{m+1} \cdots u_n \rangle$, we have $\langle u_m w_1 \cdots w_n \rangle = \pm \langle u_0 u_1 \cdots u_n \rangle = \mp \langle u_m u_1 \cdots u_{m-1} u_0 u_{m+1} \cdots u_n \rangle$, so that $\langle v(w_1) v(w_2) \cdots v(w_n) \rangle = \mp \langle v(u_1) \cdots v(u_{m-1}) v(u_0) v(u_{m+1}) \cdots v(u_n) \rangle = \mp \langle v(u_1) v(u_2) \cdots v(u_n) \rangle$. It follows that $\epsilon(\sigma_k^{n-1}) = \mp \epsilon(\sigma_i^{n-1})$. In exactly the same way, if $[\langle \sigma_i^{n-1} \rangle : \langle \sigma_j^n \rangle] = -1$ and $[\langle \sigma_k^{n-1} \rangle : \langle \sigma_j^n \rangle] = \pm 1$, we obtain that $\epsilon(\sigma_k^{n-1}) = \pm \epsilon(\sigma_i^{n-1})$. Therefore, we have $[\langle \sigma_i^{n-1} \rangle : \langle \sigma_j^n \rangle] \epsilon(\sigma_i^{n-1}) + [\langle \sigma_k^{n-1} \rangle : \langle \sigma_j^n \rangle] \epsilon(\sigma_k^{n-1}) = 0$ in all cases, so that $\sum_{k \in K} [\langle \sigma_k^{n-1} \rangle : \langle \sigma_j^n \rangle] \epsilon(\sigma_k^{n-1}) = 0$.

(Step 2) Next, we see that

$$\sum_{j \in J} \alpha_j \sum_{k \in K} [\langle \sigma_k^{n-1} \rangle : \langle \sigma_j^n \rangle] \epsilon(\sigma_k^{n-1}) = \sum_{k \in K} \beta_k \epsilon(\sigma_k^{n-1}).$$

[14]Recall that we treat only finite chains, so that in the formal summation all but a finite number of coefficients are 0.

Note that since $\tau_n(\langle a^0 \cdots a^n \rangle) = \sum_{j \in J} \alpha_j \sigma_j^n$, we have

$$\partial_n(\tau_n(\langle a^0 \cdots a^n \rangle)) = \partial_n(\sum_{j \in J} \alpha_j \sigma_j^n) = \sum_{j \in J} \alpha_j \partial_n(\sigma_j^n) = \sum_{j \in J} \alpha_j \sum_{k \in K} [\langle \sigma_k^{n-1} \rangle : \langle \sigma_j^n \rangle] \sigma_k^{n-1}.$$

Moreover, since $\partial \tau = \tau \partial$, we also have

$$\partial_n(\tau_n(\langle a^0 \cdots a^n \rangle)) = \tau_{n-1} \partial_n(\langle a^0 \cdots a^n \rangle) = \sum_{i=0}^{n} (-1)^i \tau_{n-1}(\langle a^0 \cdots \hat{a}^i \cdots a^n \rangle),$$

where the circumflex accent denotes the omission of vertex a^i. It follows that

$$\sum_{j \in J} \alpha_j \sum_{k \in K} [\langle \sigma_k^{n-1} \rangle : \langle \sigma_j^n \rangle] \sigma_k^{n-1} = \sum_{i=0}^{n} (-1)^i \tau_{n-1}(\langle a^0 \cdots \hat{a}^i \cdots a^n \rangle).$$

Since $\tau_{n-1}(\langle a^0 \cdots \hat{a}^i \cdots a^n \rangle) \subset f^{-1}([a^0, \cdots, \hat{a}^i, \cdots, a^n])$ (Condition (T1)), by considering the fact that each σ_k^{n-1} appearing in the formal summation $\tau_{n-1}(\langle a^0 \cdots \hat{a}^i \cdots a^n \rangle)$ except for $i = 0$ cannot be regular, the coefficient of each regular σ_k^{n-1} ($k \in K$) must equal to its coefficient in $\tau_{n-1}(\langle a^1 \cdots a^n \rangle)$, so that we must have

$$\sum_{j \in J} \alpha_j [\langle \sigma_k^{n-1} \rangle : \langle \sigma_j^n \rangle] = \beta_k$$

for each regular σ_k^{n-1} ($k \in K$). Since $\epsilon(\sigma_k^{n-1}) = 0$ for each σ_k^{n-1} that is not regular, we have

$$\sum_{j \in J} \alpha_j \sum_{k \in K} [\langle \sigma_k^{n-1} \rangle : \langle \sigma_j^n \rangle] \epsilon(\sigma_k^{n-1}) = \sum_{k \in K} \beta_k \epsilon(\sigma_k^{n-1}).$$

■

5 Eilenberg-Montgomery's Theorem

By combining Lemma 4.4 with Vietoris-Begle mapping theorem, we obtain the following coincidence theorem. Though the result may be considered as a special case of Eilenberg-Montgomery-Begle's fixed point theorem, we prove it directly and use to show a simple version of Eilenberg-Montgomery's theorem.

Theorem 5.1: (Nikaido 1959) Let X be a compact Hausdorff space and Y a set homeomorphic to finite-dimensional simplex $a^0 a^1 \cdots a^n$. Suppose that there are two continuous mappings f and θ on X to Y, one of which, say f, is a Vietoris mapping. Then there is a point $x \in X$ such that $f(x) = \theta(x)$.

PROOF : Let us identify Y with $[a^0 a^1 \cdots a^n]$. Then every point $y \in Y$ may be uniquely represented as $y = \sum_{i=0}^n y_i a^i$, where $y_i \geq 0$ for all i, and $\sum_{i=0}^n y_i = 1$. In the same way, we may represent $f(x)$ and $\theta(x)$ as $(f_0(x), \ldots, f_n(x))$ and $(\theta_0(x), \ldots, \theta_n(x))$, respectively. Denote by F_i the set $\{x \in X | f_i(x) \geq \theta_i(x)\}$. It is easy to check that for each k-face $a^{i_0} \cdots a^{i_k}$ of Y, $f^{-1}([a^{i_0} \cdots a^{i_k}]) \subset \bigcup_{j=0}^k F_{i_j}$. Then we may define vertex assignment v as $v(x) = a^i$ for a vertex a^i of the carrier of x such that $v(x) \in F_i$. Since for Vietoris mapping we may construct chain map τ in Lemma 4.4, we may obtain regular n-simplex σ^n in $X^v\mathfrak{M}$. Therefore, there is at least one $M \in \mathfrak{M}$ such that $M \cap F_i \neq \emptyset$ for all $i = 0, \ldots, n$. Now, assume that $\bigcap_{i=0}^n F_i = \emptyset$. Then the family $\{F_i^c = X \setminus F_i | i = 0, \ldots, n\}$ may be considered as a covering of X. If we apply the same argument for $\mathfrak{M}$ to $\{F_i^c = X \setminus F_i | i = 0, \ldots, n\}$, we obtain an element of $\{F_i^c = X \setminus F_i | i = 0, \ldots, n\}$ that intersects with all F_i's, which is impossible since $F_i^c \cap F_i = \emptyset$ for all i. Hence, we have $\bigcap_{i=0}^n F_i \neq \emptyset$. Now, it is easy to check that any element $x \in \bigcap_{i=0}^n F_i$ satisfies $f(x) = \theta(x)$. ■

By using Theorem 5.1, we can easily obtain the following simple version of Eilenberg-Montgomery fixed point theorem.

Theorem 5.2: (Eilenberg-Montgomery Fixed Point Theorem: Finite Dimensional) Let Y be a set homeomorphic to finite-dimensional simplex $a^0 a^1 \cdots a^n$. If $\varphi : Y \to Y$ is an acyclic valued correspondence having closed graph, then φ has a fixed point.

PROOF : Let X be the graph of φ, $G_\varphi \subset Y \times Y$. Since φ has closed graph, G_φ is a compact Hausdorff space. Consider two projections $f : X = G_\varphi \ni (x, y) \mapsto x \in Y$ and $\theta : X = G_\varphi \ni (x, y) \mapsto y \in Y$. Since φ is acyclic valued, f is a Vietoris mapping. Therefore, by Theorem 5.1, there is a point $x^* \in X = G_\varphi \subset Y \times Y$ such that $f(x^*) = \theta(x^*)$. This means, however, the first coordinate and the second coordinate of x^* are identical, i.e., x^* may be represented as (x, x). Hence, we have $(x, x) \in G_\varphi$, so that $x \in \varphi(x)$. ■

Of course, the above theorem includes Brouwer's fixed point theorem.

6 Lefschetz's Fixed Point Theorem and It's Extensions

In this section we treat compact Hausdorff lc space X. The homology groups of X are isomorphic to the corresponding groups of a finite complex (Theorem 3.4), and classical results of Lefschetz (1937) and Eilenberg and Montgomery (1946) may be shown to be extended (Begle, 1950) in such cases.

Lefschetz number of continuous mapping $f : X \to X$ is the summation of trace of homomorphisms, trace $(f_i) : H_i^v(X) \to H_i^v(X)$,

$$\sum_{i-0}^{\infty} (-1)^i \,\mathrm{trace}\,(f_i) \tag{8}$$

which is well defined since all $H_i^v(X)$ are finite dimensional and $H_i^v(X) = 0$ for all i sufficiently large. Intuitively, for every dimension i, the basis of $C_i^v(\mathfrak{M})$'s (hence, of $H_i^v(\mathfrak{M})$'s) are given by i-dimensional simplexes in $X^v(\mathfrak{M})$, so that if f maps all points in a certain simplex completely to other simplexes, the trace of linear mapping f_i should necessarily be 0 (Figure 10). The Lefschetz's fixed point theorem is

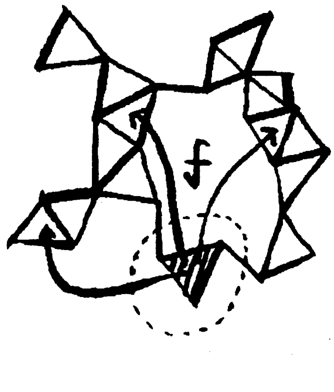

Figure 10: Lefschetz Number 0

nothing but a restatement of this intuitive observation, i.e., if there is no fixed point, the trace of all such linear functions should be equal to 0.

The purpose of this section is to relate this profound algebraic features of fixed point arguments with our fixed point theorems and methods for the general Kakutani type mappings.

Convex Structures and Mappings of the Browder Type

Before we relate Kakutani type mappings with arguments for Lefschetz's fixed point theorem, we see how methods for Browder type mappings may be recaptured through the framework of Čech type homology theory.

Let E be a Hausdorff space on which a convex strucrure, (a concept of combination among finite points with real coefficients), is defined, and let X be a non-empty compact subset which may not necessarily convex. We say that mapping $\varphi : X \to 2^X$ is of *class* $\mathscr{B}$ if φ has a fixed point free convex extension having local intersection property on $X \setminus \mathcal{F}\boldsymbol{ix}(\varphi)$. Figure 11 represents a typical situation for mapping $\varphi : X \to 2^X$

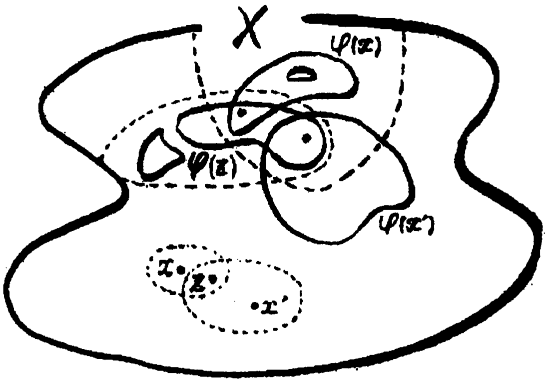

Figure 11: Mapping of class $\mathscr{B}$

of type $\mathscr{B}$, where x and x' are not in $\mathcal{F}\boldsymbol{ix}(\varphi)$. If X is convex, then a class $\mathscr{B}$ mapping is nothing but a mapping of the Browder type.

The local intersection property on $X \setminus \mathcal{F}\boldsymbol{ix}(\varphi)$ for a convex extension of mapping φ of class $\mathscr{B}$ enable us to replace the relation among open coverings of $X \setminus \mathcal{F}\boldsymbol{ix}(\varphi)$ with convex combination of points. See Figure 12, where y and y' are points in convex extensions of $\varphi(x)$ and $\varphi(x')$, respectively, satisfying the local intersection property near x and x'. If neighbourhoods of x and x' have an intersection point in $X \setminus \mathcal{F}\boldsymbol{ix}(\varphi)$, then the convex conbination of y and y' belongs to X since there is a point $z \in X \setminus \mathcal{F}\boldsymbol{ix}(\varphi)$ such that both y and y' belong to a convex extension of $\varphi(z)$.

For mapping φ such that $\mathcal{F}\boldsymbol{ix}(\varphi) = \emptyset$, then, such neighbourhoods form a covering of X and convex combination of points (y, y', etc.,) constructs a complex which may be considered as an approximation of X (See Figure 13). Clearly, the complex may also be characterized as the nerves of the covering formed by neighbourhoods of x, x', etc. Note that the partition of unity for the covering formed by neighbourhoods of points, $x, x', \ldots$, say $\alpha : X \to [0, 1]$, $\alpha' : X \to [0, 1]$, $\ldots$, gives a continuous mapping on X to the complex, say K, formed by points $y, y', \ldots$, as

$$f^{\varphi} : X \ni x \mapsto \alpha(x) + \alpha'(x) + \cdots \in |K|.$$

The continuous mapping restricted on $|K|$ to itself, however, never has a fixed point since by the property of class $\mathscr{B}$ mapping φ, $x^* \in U(x)$, $x^* \in U(x')$, $\ldots$, (neighbourhoods of x, x', $\ldots$, resp.), means y, y', $\ldots$, belong

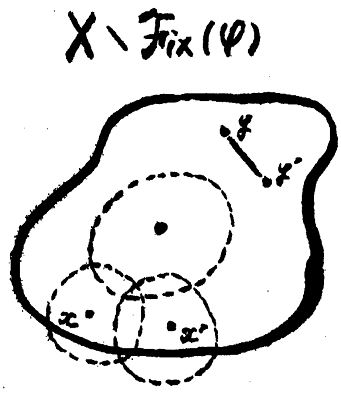

Figure 12: Intersections and Convex Combinations

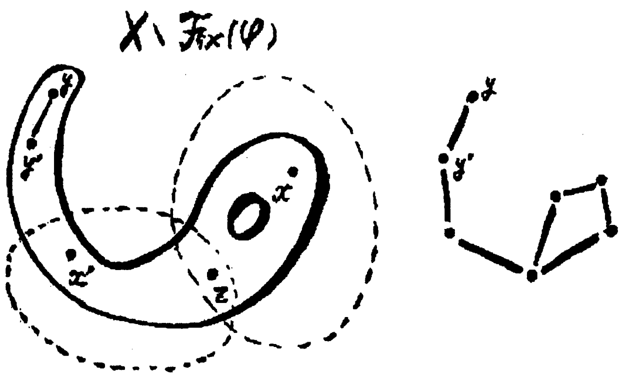

Figure 13: Realization of Čech Complex

to the fixed point free convex extension of $\varphi(x^*)$, so that x^* cannot be any convex combination among points $y, y', \ldots$. As we can see below, for such continuous mapping f^φ, the Lefschetz's fixed point arguments may be applicable, hence, for mapping φ of class *scmathB*, the trace of homology mapping $f_q^\varphi : H_q^v(|K|) \to H_q^v(|K|)$ for each $q = 0, 1, 2, \ldots$, of f^φ, (say, a certain kind of linear approximation of φ), is 0 for sufficiently fine K as long as φ has no fixed point.

Convex Structures and Mappings of Class $\mathscr{K}$

In the last part of Chapter 2 in Urai (2005), the author treated a wide class of mappings, the Kakutani type, to which we have seen that (1) the fixed point property holds, and (2) a directional structure on which the dual space representation of φ has local intersection property as long as φ has no fixed points may be definable.

Assume that on space X there is a convex structure $(D_X, h_X, \{f_A | A \in \mathscr{F}(X)\})$. We say that a mapping, $\varphi : X \to 2^X \setminus \{\emptyset\}$, is of *class* $\mathscr{K}$ if for each $x \in X$, there is a closed convex set K_x such that (1) $(x \notin \varphi(x)) \Longrightarrow (x \notin K_x)$, and (2) there is an open neighborhood U_x of x satisfying that $\forall z \in U_x$, $\varphi(z) \subset K_x$.[15] Note that for mapping φ of class $\mathscr{K}$, each neighborhood U_x of x may be chosen arbitrarily small. Of course, class $\mathscr{K}$ mapping is nothing but the Kakutani type mappings since for each mapping of the Kakutani type, for all $x \in \boldsymbol{\mathcal{F}ix}(\varphi)$, we may set K_x as $K_x = X$.

For mapping $\varphi : X \to X$ of class $\mathscr{K}$, let us define the *Lefschetz number* of φ in a generalized sense. Since X is compact and Hausdorff, for each mapping $\varphi : X \to 2^X$ of class $\mathscr{K}$, there is at least one covering $\mathfrak{M} = \{M_1, \ldots, M_m\}$ of X such that for each $i = 1, \ldots, m$, there is a convex set K_i satisfying that $(z \in M_i) \Longrightarrow \varphi(z) \subset K_i$. As stated above, $\mathfrak{M}$ may be chosen arbitrarily small, so that we may suppose that $\mathfrak{M} \preccurlyeq {}^*\mathfrak{N}_0$, where $\mathfrak{N}_0 \in \boldsymbol{Cover}(X)$ is the covering for lc space X stated in Theorem 3.4, (a). It is known that the nerve of any covering $\mathfrak{N} \preccurlyeq {}^*\mathfrak{M} \preccurlyeq {}^*\mathfrak{N}_0$ gives the finite dimensional (ordinary simplicial) homology group which is isomorphic to $H_n^v(X)$ for any dimension n. The isomorphism is induced by the composite of mappings, $\varphi^b_{\mathfrak{N}_0 n} : C_n^c({}^*\mathfrak{N}_0) \to C_n^v(\mathfrak{N}_0)$, the projection $p_n^{{}^*\mathfrak{N}_0\mathfrak{M}}$, $\zeta^b_{\mathfrak{N}} : C_n^v({}^*\mathfrak{M}) \to C_n^c(\mathfrak{M})$, and the inclusion $h_n^{{}^*\mathfrak{M}\mathfrak{N}}$ to define the mapping between cycles as $\theta_n(z) = \varphi_n^b \circ p_n \circ \zeta_n^b \circ h_n(z(\mathfrak{N}))$. (See the proof of lemma 2 in Begle (1950b).)

Let $\mathfrak{N} = \{N_1, \ldots, N_n\}$ be ${}^*\mathfrak{M}$. Take $a_1 \in N_1, \ldots, a_n \in N_n$ and $b_1 \in \varphi(a_1), \ldots, b_n \in \varphi(a_n)$ arbitrarily and denote by A and B respectively the set $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$. Denote by $K(A)$ the complex with vertices in A such that $a_{i_0} \cdots a_{i_\ell} \in K(A)$ iff $\bigcap_{j=1}^{\ell} N_{i_j} \neq \emptyset$. Clearly, $K(A)$ is isomorphic to the nerve of covering $\mathfrak{N}$, so that for an arbitrarily small refinement $\mathfrak{P}$ of ${}^*\mathfrak{N}$, there exists homomorphism θ_n between cycles defining isomorphism between homology groups,

$$\theta_n : Z_n^v(X) \to Z_n(K(A)) \tag{9}$$

for any dimension n, where $Z_n^v(X)$ denotes the set of all n-dimensional Vietoris cycles on X and $\theta_n(z) = \varphi_n^b \circ p_n \circ \zeta_n^b \circ h_n^{{}^*\mathfrak{N}\mathfrak{P}}(z(\mathfrak{P}))$.

Since $\mathfrak{N}$ is a star refinement of $\mathfrak{M}$, the complex, $K(A)$, may be considered as a subcomplex of $X^v(\mathfrak{M})$. Define an abstract complex, $K(B)$, with the set of vertices, B, as $b_{i_0} \cdots b_{i_\ell} \in K(B)$ iff $\mathrm{co}\{b_{i_j} : j = 0, \ldots, \ell\} \subset X$. Then, we may obtain a simplicial mapping $\tau : K(A) \to K(B)$ such that $\tau(a_i) = b_i$ for each $i = 1, \ldots, n$. Moreover, under convex structure on X, by taking $B' \supset B$ sufficiently large, the restriction of $f_{B'}$ on $K(B)$, we may obtain a continuous mapping r on standard realization of $K(B)$ into X. Hence, we have homomorphism $r_n \circ \tau_n \circ \theta_n : H_n^v(X) \to H_n^v(X)$ whose trace is well defined for each dimension n. Note that

[15] Since K_x is closed, we may suppose $U_x \cap K_x = \emptyset$ without loss of generality as long as $x \notin K_x$.

these mappings depend on how we chose $\mathfrak{M}$, $\mathfrak{P}$, A, B. For mapping φ of class $\mathscr{K}$, define Lefschetz number $\Lambda(\varphi)$ as the minimum of natural numbers given by such traces as,

$$\Lambda(\varphi) = \min_{\mathfrak{M},\mathfrak{P},A,B} \sum_{i=0}^{\infty} (-1)^i \operatorname{trace}(r_i \circ \tau_i \circ \theta_i). \tag{10}$$

We can verify that this number also characterize the existence of fixed points in exactly the same way as the ordinary Lefschetz number even for the wide class of mappings, $\mathscr{K}$. All we have to show is that if φ of class $\mathscr{K}$ has no fixed point, there is at least one set of $\mathfrak{M}$, A, and B under which $\operatorname{trace}(r_i \circ \tau_i \circ \theta_i) = 0$ for any dimension i. It would be a routine task, however, if we recognize the definition of θ_n (i.e., all we have to consider is $\mathfrak{P}$-simplexes which may.be taken as small as possible.)

Acyclic Valued Directional Structures and Mappings of Class $\mathscr{D}$

Arguments in the previous subsection for a generalization of Lefschetz's fixed point theorem may also be applicable to cases such that each K_x characterizing the mapping of class $\mathscr{K}$ is not convex but acyclic.

Let X be a compact Hausdorff lc space. We say that a mapping, $\varphi : X \to 2^X \setminus \{\emptyset\}$, is of *class $\mathscr{D}$* if for each $x \in X$, there exists closed acyclic set K_x such that (1) $(x \notin \varphi(x)) \Longrightarrow (x \notin K_x)$, and (2) there is an open neighborhood U_x of x satisfying that $\forall z \in U_x$, $\varphi(z) \subset K_x$. As before, since K_x is closed, we may suppose $U_x \cap K_x = \emptyset$ without loss of generality as long as $x \notin K_x$. Note also that for mapping φ of class $\mathscr{K}$, each neighborhood U_x of x may be chosen arbitrarily small. In standard cases, non-empty convex sets are acyclic, so that the discussion for class $\mathscr{D}$ mapping below may also be considered as a generalization of the previous argument for class $\mathscr{K}$ mappings (Figure 14).

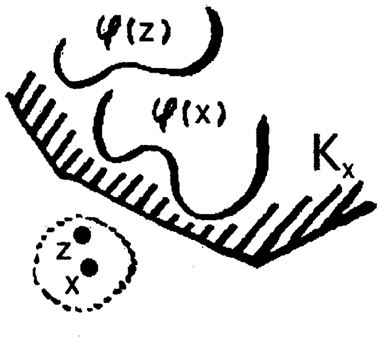

Figure 14: Mappings of Class $\mathscr{K}$ and $\mathscr{D}$

Since X is compact and Hausdorff, for mapping $\varphi : X \to 2^X$ of class $\mathscr{D}$, there is at least one covering $\mathfrak{M} = \{M_1, \ldots, M_m\}$ of X such that for each $i = 1, \ldots, m$, there exists acyclic set K_i satisfying that $(z \in M_i) \Longrightarrow (\varphi(z) \subset K_i)$. Since $\mathfrak{M}$ may be chosen arbitrarily small, we may suppose that $\mathfrak{M} \preccurlyeq \mathfrak{N}_0$, where $\mathfrak{N}_0 \in \boldsymbol{Cover}(X)$ is the covering for lc space X stated in Theorem 3.4 (a) as before. The nerve of any covering $\mathfrak{N} \preccurlyeq {}^{**}\mathfrak{M} \preccurlyeq \mathfrak{N}_0$ provides finite dimensional simplicial homology group which is isomorphic to $H^v_n(X)$ for each dimension n. The isomorphism is induced by composite of mappings, $\varphi^b_{\mathfrak{M}n} : C^c_n({}^*\mathfrak{M}) \to C^v_n(\mathfrak{M})$, projection $p_n^{\mathfrak{M}\,{}^*\mathfrak{M}}$, $\zeta^b_{{}^*\mathfrak{M}} : C^v_n({}^{**}\mathfrak{M}) \to C^c_n({}^*\mathfrak{M})$, and inclusion $h_n^{{}^{**}\mathfrak{M}\,\mathfrak{N}}$ as $\theta_n(z) = \varphi^b_n \circ p_n \circ \zeta^b_n \circ h_n(z(\mathfrak{N}))$.

Let k be the dimension of the nerve of $\mathfrak{N}$. We shall define a sequence of refinements of $\mathfrak{N}$

$$\mathfrak{M}_0 \preccurlyeq \mathfrak{M}_1 \preccurlyeq \cdots \preccurlyeq \mathfrak{M}_{k-1} \preccurlyeq \mathfrak{M}_k \preccurlyeq \mathfrak{M}_{k+1} \preccurlyeq \mathfrak{N} \preccurlyeq {}^{**}\mathfrak{M} \preccurlyeq {}^*\mathfrak{M} \preccurlyeq \mathfrak{M} \preccurlyeq \mathfrak{N}_0 \tag{11}$$

as follows: Let $\mathfrak{M}_{k+1} = \mathfrak{N}$. For ℓ such that $0 \leq \ell \leq k$, define $\mathfrak{M}_\ell$ as a refinement of $^{**}\mathfrak{M}_{\ell+1}$ such that for each compact acyclic $K_i \in \{K_1, \ldots, K_m\}$, any ℓ-dimensional Victoris $\mathfrak{M}_\ell$-cycle of K_i bounds a chain in $\mathfrak{M}_{\ell+1}$ of K_i. (This is always possible by Theorem 3.2.) Note that for each pair of $\mathfrak{M}_\ell$ and $\mathfrak{M}_{\ell+1}$ and dimension n, homomorphism $\theta_n^{\ell+1\ell} = \varphi_n^b \circ p_n \circ \zeta_n^b \circ h_n$ between $C_n^v(\mathfrak{M}_{\ell+1})$ and $C_n^v(\mathfrak{M}_\ell)$ which induces the isomorphism among homology groups exists.

Let us define a chain homomorphism $\tau = \{\tau_q\}$ on the k-skeleton of $X^v(\mathfrak{M}_0)$ to $X^v(\mathfrak{N})$. At first, denote by $\mathfrak{L} = \{L_0, L_1, \ldots, L_s\}$ the cover $^*\mathfrak{M}$. By definition of $\varphi_{\mathfrak{m}n}^b$, $\varphi_{\mathfrak{m}n}^b(L_i) = x_{L_i} \in L_i$ and there exists an $M_j \in \mathfrak{M}$ such that $St(L_i; \mathfrak{L}) \subset M_j$. Define a_i as $a_i = x_{L_i}$ and K_{a_i} as the corresponding K_j for each $i = 0, \ldots, s$. Then we have for each $x \in L_i$, $\varphi(x) \subset K_{a_i}$ for all i. With respect to a_i, fix a point $b_i \in \varphi(a_i) \subset K_{a_i}$ for each i.

For 0-dimensional simplex $\sigma^0 = \langle x^0 \rangle$ of $X^v(\mathfrak{M}_0)$, the image $\theta_0 \circ \theta_0^{n+1n} \circ \cdots \circ \theta_0^{10}(x^0)$ is by definition one of points $a_0, \ldots, a_s$, say a_i. Define $\tau_0(\sigma^0)$ as $\tau_0(\sigma^0) = b_i$ and extend it linearly on $C_0^v(\mathfrak{M}_0)$ to $C_0^v(\mathfrak{M}_0) \subset C_0^v(\mathfrak{M})$.

Next, for 1-dimensional simplex $\sigma^1 = \langle x^0 x^1 \rangle$ of $X^v(\mathfrak{M}_0)$, we may write $\tau_0\partial(\sigma^1) = \tau_0(x^0 - x^1)$ as $b_i - b_j$, where $b_i = \tau_0(x^0)$ and $b_j = \tau_0(x^1)$. Of course, $b_i - b_j$ may also be considered as an $\mathfrak{M}_0$-cycle (in the reduced sense).[16] Hence, by definition of $\mathfrak{M}_0$ relative to $\mathfrak{M}_1$, we have a $\mathfrak{M}_1$-chain c^1 such that $\partial(c^1) = b_i - b_j$

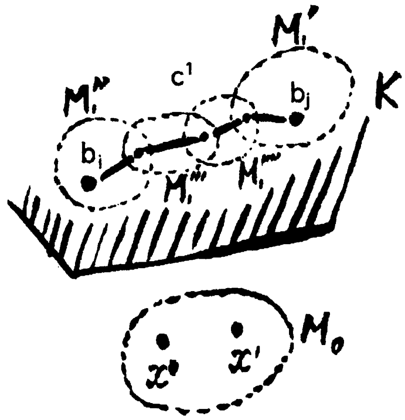

Figure 15: Class $\mathscr{D}$ mapping and $\mathfrak{M}_1$-chain

(Figure 15). Define $\tau_1(\sigma^1)$ as $\tau_1(\sigma^1) = c^1$ and extend it linearly on $C_1^v(\mathfrak{M}_0)$ to $C_1^v(\mathfrak{M}_1) \subset C_1^v(\mathfrak{M})$. Clearly, $\partial\tau_1 = \tau_0\partial$ holds.

Now, assume that for all dimension $q < \ell$, $(2 \leq \ell \leq k)$, τ_q is defined on $C_q^v(\mathfrak{M}_0)$ to $C_q^v(\mathfrak{M}_q) \subset C_q^v(\mathfrak{M})$ and $\partial\tau_q = \tau_{q-1}\partial$ holds. Then for ℓ-dimensional simplex σ^ℓ of $X^v(\mathfrak{M}_0)$, chain $c = \tau_{\ell-1}\partial(\sigma^\ell)$ is well defined. Since $\partial(c) = \partial\tau_{\ell-1}\partial(\sigma^\ell) = \tau_{\ell-2}\partial\partial(\sigma^\ell) = 0$, c is indeed $\mathfrak{M}_{\ell-1}$-cycle. Hence, by definition of $\mathfrak{M}_{\ell-1}$ relative to $\mathfrak{M}_\ell$, we have a $\mathfrak{M}_\ell$-chain c^ℓ such that $\partial(c^\ell) = c$. Define $\tau_\ell(\sigma^\ell)$ as $\tau_\ell(\sigma^\ell) = c^\ell$ and extend it linearly on $C_\ell^v(\mathfrak{M}_0)$ to $C_\ell^v(\mathfrak{M}_\ell) \subset C_1^v(\mathfrak{M})$. Clearly, $\partial\tau_\ell = \tau_{\ell-1}\partial$ holds.

Hence, by induction, we have successfully obtained the chain map $\tau = \{\tau_q\}$ on the k-skeleton of $X^v(\mathfrak{M}_0)$ to $X^v(\mathfrak{M}_{k+1}) = X^v(\mathfrak{N}) \subset X^v(\mathfrak{M})$, i.e., we have

$$\tau_q : C_q^v(\mathfrak{M}_0) \to C_q^v(\mathfrak{N}) \subset C_q^v(\mathfrak{M}) \tag{12}$$

[16] Every point of X may be considered as a 0-dimensional $\mathfrak{M}_0$-simplex. Note also that in Theorem 3.2, 0-dimensional cycles should be taken in the reduced sense.

for all $q = 0, 1, \ldots, k$. The homology groups of $X^v(\mathfrak{M}_0)$ and $X^v(\mathfrak{N})$ are isomorphic under the isomorphism induced by $\theta^{n+1n} \circ \cdots \circ \theta^{10}$. Since both of them are isomorphic to the corresponding group of a finite complex, trace (τ_q) is well defined for all q and $\sum_{i=0}^{\infty}(-1)^i \operatorname{trace}(\tau_i)$ is finite. Though definition of τ depends on $\mathfrak{M}$, $\mathfrak{N}$, and, especially, set A of all a_i's and B of all b_i's, we may define as before the minimum of such values,

$$\Lambda(\varphi) = \min_{\mathfrak{M}, \mathfrak{N}, A, B} \sum_{i=0}^{\infty} (-1)^i \operatorname{trace}(\tau_i) \tag{13}$$

as an extended Lefschetz number for mapping φ of class $\mathscr{D}$. By considering the definition of $\theta_n^{\ell+1\ell}$'s, we obtain the following extension of Lefschetz's fixed point theorem.

Theorem 6.1: (Extension of Lefschetz's Fixed Point Theorem) Let X be a compact Hausdorff lc space. Mapping φ of class $\mathscr{D}$ has a fixed point if $\Lambda(\varphi) \neq 0$.

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