<table>
<thead>
<tr>
<th>項目</th>
<th>内容</th>
</tr>
</thead>
<tbody>
<tr>
<td>タイトル</td>
<td>2-2-regular graph (New Trends in Theory of Computation and Algorithm)</td>
</tr>
<tr>
<td>著者</td>
<td>Kimura, Kenji</td>
</tr>
<tr>
<td>引用</td>
<td>数理解析研究所講究録 (2006), 1489: 245-249</td>
</tr>
<tr>
<td>発行日</td>
<td>2006-05</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/58200">http://hdl.handle.net/2433/58200</a></td>
</tr>
<tr>
<td>種別</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>テキストバージョン</td>
<td>publisher</td>
</tr>
<tr>
<td>データベース</td>
<td>Kyoto University</td>
</tr>
</tbody>
</table>
2-factor in $2r$-regular graph

Department of Computer Science, The University of Electro-Communications

Abstract

Let $r$ be a positive integer such that $r \geq 2$, $G$ be a $2r$-regular graph of odd order and $G$ be connected. Then, there is some $x \in V(G)$ such that $G - x$ has a 2-factor.

1 Introduction

We consider finite undirected graphs which may have loops and multiple edges. Let $G$ be a graph. For $x \in V(G)$, we denote by $\deg_G(x)$ the degree of $x$ in $G$. The set of neighbours of $x \in V(G)$ is denoted by $N_G(x)$. If $\deg_G(x) = r$ for any $x \in V(G)$, we call the graph $r$-regular. For subsets $S$ and $T$ of $V(G)$, we denote by $e_G(S,T)$ the number of the edges joining $S$ and $T$. If $S \cap T \neq \emptyset$, the edges of $S \cap T$ are counted twice. If $S$ is a singleton $\{x\}$, we write $S = x$ instead of $S = \{x\}$. For example, we write $e_G(x,T)$ instead of $e_G(\{x\},T)$. Let $k$ be a constant. A spanning subgraph $F$ of $G$ such that $\deg_F(x) = k$ for each $x \in V(G)$ is called a $k$-factor of $G$. When no fear of confusion arises, we often identify a $k$-factor with its edge set.

Petersen proved the next theorem in 1891.

Theorem A (Petersen [1]) Every $2r$-regular graph can be decomposed into $r$ disjoint 2-factors.

This theorem implies that if $G$ is a $2r$-regular graph, then $G$ has a $k$-factor for every even integer $k$, $2 \leq k \leq 2r$.

Katerinis showed the next theorem in 1985.

Theorem B (Katerinis [2]) Let $G$ be a connected graph of even order, and let $a$, $b$, and $c$ be odd integers such that $1 \leq a < b < c$. If $G$ has both $a$-factor and $c$-factor, then $G$ has a $b$-factor.

If a $2r$-regular graph $G$ has a 1-factor, we can obtain a $(2r - 1)$-factor by excluding the 1-factor from $G$. By the 1-factor and the $(2r - 1)$-factor of $G$ and by Theorem B, $G$ has a $k$-factor for any odd integer $k$, $1 \leq k \leq 2r - 1$. Thus, by the above two
theorems, if a $2r$-regular graph $G$ has a 1-factor, then $G$ has a $k$-factor for every integer $k$, $1 \leq k \leq 2r - 1$. Note that the order of $G$ is even. For the case that the order of $G$ is odd, Katerinis proved the next theorem in 1994.

**Theorem C (Katerinis [3])** Let $G$ be a $2r$-regular, $2r$-edge-connected graph of odd order, and $k$ be an integer such that $1 \leq k \leq r$. Then for every $x \in V(G)$, the graph $G-x$ has a $k$-factor.

Let us focus our attention that the condition "$2r$-edge-connected" of Theorem C is replaced by "connected". What result can be obtained under the weakened condition? Now we will present our main theorem.

**Theorem 1** Let $r$ be a positive integer such that $r \geq 2$, $G$ be a $2r$-regular graph of odd order and $G$ be connected. Then, there is some $x \in V(G)$ such that $G-x$ has a 2-factor.

We believe that the following conjecture.

**Conjecture 1** Let $r$ be a positive integer such that $r \geq 2$, $G$ be a $2r$-regular graph of odd order, and $G$ be connected. Then for any even $k$, $2 \leq k \leq r$, there is some $x \in V(G)$ such that $G-x$ has a $k$-factor.

In order to prove Theorem 1, we use the following Tutte's Theorem. Let $G$ be a graph. For disjoint subsets $S$ and $T$ of $V(G)$, we define $\delta_G(S,T;k)$ by

$$\delta_G(S,T;k) = k|S| + \sum_{y \in T} \deg_G(y) - k|T| - h_G(S,T;k),$$

where $h_G(S,T;k)$ is the number of components $C$ of $G-(S \cup T)$ such that $k|V(C)| + e_G(V(C),T)$ is odd. These components are called odd components. We denote by $\mathcal{H}_G(S,T;k)$ the set of the odd components. That is $|\mathcal{H}_G(S,T;k)| = h_G(S,T)$. If $\delta_G(S,T;k) = \delta_G(T,S;k)$, then we say that $S$ and $T$ are symmetric.

**Theorem D (Tutte [4])** Let $G$ be a graph, and let $k$ be a positive integer. Then

1. $\delta_G(S,T;k) \equiv k|V(G)| \pmod{2}$ for each disjoint subsets $S$ and $T$ of $V(G)$, and
2. $G$ has a $k$-factor if and only if $\delta_G(S,T;k) \geq 0$ for each pair of disjoint subsets $S$ and $T$ of $V(G)$.

## 2 Proof of Theorem 1

We apply induction on $|V(G)|$. For $|V(G)| = 1$ the assertion is true. Now let $G$ be given with $|V(G)| \geq 3$, and assume that the theorem holds for graphs with fewer vertices. Assume on the contrary that $G-x$ has no 2-factor for any $x \in V(G)$. Then, there is some pair of disjoint subsets $S',T' \subseteq V(G)-x$ for every $x \in V(G)$ such that $\delta_{G-x}(S',T';2) \leq -2$ by Theorem D. Let $S = S' \cup \{x\}$, $T = T'$, and $U = G-(S \cup T)$. Then,

$$\delta_{G-x}(S-x,T;2) \leq -2. \quad (1)$$
Since $G$ is $2r$-regular,
\[ \delta_G(S, T; 2r) \geq 0 \] (2)
for each disjoint subsets $S$ and $T$ of $V(G)$. By the definition of odd component, $h_{G-x}(S-x, T; 2) = h_G(S, T; 2)$ holds. Let $h_G(S, T) = h_{G-x}(S-x, T; 2) = h_G(S, T; 2)$. Subtracting (2) from (1), we have
\[
(2 - 2r)|S| - 2 - (2 - 2r)|T| \leq -2
- (2 - 2r)|T| \leq -(2 - 2r)|S|
\]
\[ |T| \leq |S|. \] (3)
By (1) and (3),
\[ \sum_{y \in T} \deg_{G-S}(y) \leq h_G(S, T). \] (4)
On the other hand, by the definition of odd component,
\[ \sum_{y \in T} \deg_{G-S}(y) \geq e_G(T, U) \geq h_G(S, T). \] (5)
By (4) and (5),
\[ \sum_{y \in T} \deg_{G-S}(y) = h_G(S, T). \] (6)
By (1) and (6),
\[ 2|S| - 2 - 2|T| \leq -2 \]
\[ 2|S| \leq 2|T| \]
\[ |S| \leq |T|. \] (7)
By (3) and (7),
\[ |S| = |T|. \] (8)
Since $\delta_G(S, T; 2) = \delta_G(T, S; 2)$ by (8), $S$ and $T$ are symmetric. Moreover, $|U|$ is odd. By (6),
\[ e_G(T, T) + e_G(T, U) = h_G(S, T). \] (9)
By (5) and (9),
\[ e_G(T, T) = 0 \quad \text{and} \quad e_G(T, U) = h_G(S, T) \] (10)
If there is no odd component of $U$, $e_G(T, S) = 2r|T|$ holds by (9). Then, since $e_G(S \cup T, U) = 0$ holds, $G$ is disconnected. This is a contradiction. Thus, there is some odd component of $U$. Note that $e_G(V(C), T) = 1$ for each odd component $C \in \mathcal{H}_G(S, T)$. Let $\mathcal{H}_G(S, T) = \{C_1, \ldots, C_z\}$. Let $a_i, b_i \in V(C_i)$, $s_i \in S$, $t_i \in T$ for every odd component $C_i \in \mathcal{H}_G(S, T)$, $1 \leq i \leq z$, such that $N_G(a_i) \cap \{t_i\} \neq \emptyset$ and $N_G(b_i) \cap \{s_i\} \neq \emptyset$. We show that there is subgraph $H_i$ of $G$ such that $\deg_{H_i}(s_i) = \deg_{H_i}(t_i) = 1$ and $\deg_{H_i}(x_i) = 2$ for any $x \in V(C_i)$ for any odd component $C_i \in \mathcal{H}_G(S, T)$. Now, for every odd component $C_i \in \mathcal{H}_G(S, T)$ $\deg_{C_i}(x) = 2r$ for every $x \in V(C_i) - \{a_i, b_i\}$ and $\deg_{C_i}(a_i) = \deg_{C_i}(b_i) = 2r-1$. Therefore, $C_i \cup \{a_ib_i\}$ is $2r$-regular for any odd component $C_i \in \mathcal{H}_G(S, T)$. $C_i \cup \{a_ib_i\}$ has $r$ disjoint 2-factors by Theorem A in $C_i \cup \{a_ib_i\}$. Let

$F_{C_i}$ be a 2-factor including new edge $\{a_ib_i\}$ for each odd component $C_i \in \mathcal{H}_G(S,T)$ in $C_i \cup \{a_ib_i\}$. Then, $(F_{C_i} - \{a_ib_i\}) \cup \{a_it_i\} \cup \{b_is_i\}$ is the desired subgraph $H_i$ of $G$ for each odd component $C_i \in \mathcal{H}_G(S,T)$. On the other hand, there is also 2-factor $F_{C'_i}$ not to include new edge $\{a_ib_i\}$ for each odd component $C_i \in \mathcal{H}_G(S,T)$ in $C_i \cup \{a_ib_i\}$, that is, $C_i$ has a 2-factor for each odd component $C_i \in \mathcal{H}_G(S,T)$ in $C_i$.

Next, we show that there is some $x \in V(C)$ for some odd component $C \in \mathcal{H}_G(S,T)$ such that $C - x$ has a 2-factor, or there is a subgraph $H$ of $G$ including every vertices of $C - x, s_i \in S$ and $t_i \in T$ as above. Let $C$ be this odd component $C$, $s = s_i, t = t_i, a = a_i$, and $b = b_i$. By the induction hypothesis, for this odd component $C \in \mathcal{H}_G(S,T)$ there is some $x$ such that $(C \cup \{ab\}) - x$ has a 2-factor $F_C$ since $C \cup \{ab\}$ is 2r-regular and $|V(C)| < |V(G)|$.

If $F_C \cap \{ab\} \neq \emptyset$ for this odd component $C \in \mathcal{H}_G(S,T)$, $(F_C - \{ab\}) \cup \{at\} \cup \{bs\}$ is the desired subgraph $H$. Then, there is a path $P$ from $s$ to $t$ such that $C \cap P \neq \emptyset$ for this odd component $C \in \mathcal{H}_G(S,T)$. As well as this odd component $C \in \mathcal{H}_G(S,T)$, we can obtain a path $P$ for every odd component $C_i \in \mathcal{H}_G(S,T)$. Let $G'$ be a graph obtained from $G$ by contracting the path $P_i$ into a new edge $p_i$, and excluding $C_i - P_i$ in $G - x$ for every odd component $C_i \in \mathcal{H}_G(S,T)$. Let $p = p_i$ for $p_i \in C$ for some odd component $C \in \mathcal{H}_G(S,T)$. Then, graph $G'$ becomes 2r-regular graph. By Theorem A, $G'$ has a 2-factor $F'$ avoiding $p$. If $F' \cap \{p_i\} \neq \emptyset$, we can use the subgraph $H_i$ of $G$. If $F' \cap \{p_i\} = \emptyset$, we can use the 2-factor $F_{C'_i}$ in $C_i$ excluding new edge $a_ib_i$ for any odd component $C_i \in \mathcal{H}_G(S,T) - C$. Thus, $G$ has a 2-factor.

If $F_C \cap \{ab\} = \emptyset, C - x$ has a 2-factor. There is a path $P_i$ from $s$ to $t$ such that $C_i \cap P_i = \emptyset$ for each odd component $C_i \in \mathcal{H}_G(S,T) - C$. Let $G'$ be a graph obtained from $G$ by contracting the path $P_i$ into a new edge $p_i$, and excluding $C_i - P_i$ in $G - x$. Then, graph $G'$ becomes 2r-regular graph. Note that 2r-regular graph is graph obtained from 2r-regular graph by excluding an edge. Since $G' \cup \{st\}$ is 2r-regular, $G' \cup \{st\}$ has a 2-factor avoiding $st$ by Theorem A, that is, $G'$ has a 2-factor $F'$. If $F' \cap \{p_i\} \neq \emptyset,$ we can use the subgraph $H_i$ of $G$. If $F' \cap \{p_i\} = \emptyset$, we can use the 2-factor $F_{C'_i}$ in $C_i$ excluding new edge $a_ib_i$ for any odd component $C_i \in \mathcal{H}_G(S,T) - C$. Thus, $G$ has a 2-factor.

Acknowledgment

The author would like to thank Professor Shigeki Iwata for his helpful discussions and valuable suggestions.

References

