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Coloring Comparability-ke Graphs

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1 Introduction

Many graph problems are NP-complete for general graphs. It is natural to consider that if the graph problem is tractable for a graph class $\mathcal{F}$, it is also tractable for a class of graphs which are close to graphs in $\mathcal{F}$. $\mathcal{F}$+ke and $\mathcal{F}$-ke graphs are classes of graphs close to $\mathcal{F}$. They are the classes of graphs obtained by adding or deleting k edges from graphs in F. We can consider the complexity of several problems on such graph classes from parametric point of view. In general, problems become difficult as k increases. A problem with parameter k is called to be fixed parameter tractable if it can be solved in $f(k)|x|^c$ time, where f is an arbitrary function and $|x|$ is the size of input.

Vertex coloring problem is a very important graph problem, which is NP-complete for general graphs. Vertex coloring for parameterized graph classes are considered in [1]. It is shown in [1] that, when vertex coloring of $\mathcal{F}$ graphs is solved in polynomial time, vertex coloring of $\mathcal{F}$+ke graphs is fixed parameter tractable if $\mathcal{F}$ is closed under identification of nonadjacent vertices, and that vertex coloring of $\mathcal{F}$-ke graphs is fixed parameter tractable if $\mathcal{F}$ is closed under edge contraction.

In this paper, we consider vertex coloring of comparability-ke graphs. A comparability graph is an undirected graph which becomes a transitive graph if we give appropriate direction to each edge. As a comparability graph is a perfect graph, vertex coloring of comparability graphs can be solved in polynomial time [3]. In addition, comparability graphs are closed neither under identification of nonadjacent vertices nor under edge contraction. On comparability+ke graphs, vertex coloring is solved in polynomial time for $k = 1$ and NP-complete for $k \geq 2$ [4].

In this paper, we first show that vertex coloring of comparability-1e graphs is solved in polynomial time. Next, we show that vertex coloring of comparability-ke graphs can be polynomially reduced to a vertex coloring problem of comparability graphs with restrictions that given pairs of nodes should have the same color.

2 Preliminaries

Let $G = (V, E)$ be an undirected graph. Then vertex coloring problem is defined as follows.
VERTEX COLORING
Input : A graph $G = (V, E)$ and a positive integer $t \leq |V|$.
Question : Is G t-colorable? That is, is there a function $f : V \rightarrow \{1, 2, \ldots, t\}$ that satisfies $f(u) \neq f(v)$ for all $(u, v) \in E$.

The chromatic number of G, denoted as $\chi(G)$, is the smallest t for which G is t-colorable. The clique number of a graph G, denoted as $\omega(G)$, is the degree of the maximum complete subgraph of G.

For a graph class $\mathcal{F}$, let $\mathcal{F}$+ke be the class of graphs which can be obtained by adding at most k edges to an $\mathcal{F}$ graph. Similarly, let $\mathcal{F}$-ke be the class of graphs which can be obtained by removing at most k edges from an $\mathcal{F}$ graph. The *modulator* of an $\mathcal{F}$+ke graph $G = (V, E)$ is a set of edges $E_k (\subset E)$ s.t. $(V, E - E_k) \in \mathcal{F}$ and $|E_k| \leq k$. The modulator of an $\mathcal{F}$-ke graph G is a set of edges E_k s.t. $(V, E \cup E_k) \in \mathcal{F}$ and $|E_k| \leq k$. In this paper, we assume that the modulator is given. For a fixed k, the modulator of $\mathcal{F}$+ke or $\mathcal{F}$-ke graphs can be found in polynomial time provided that it can be checked in polynomial time whether a graph is in class $\mathcal{F}$ or not.

A comparability graph is an undirected graph s.t. a transitive graph can be obtained by giving appropriate orientation to each edge. A directed graph is called transitive if $(u, v) \in E$ and $(v, w) \in E$, then $(u, w) \in E$ holds. It can be recognized whether a graph is a comparability graph or not and its transitive orientation can be found in $O(\gamma|E|)$ time, where γ is the maximum degree of a vertex [2]. If a comparability graph is given, its transitive orientation is obtained in linear time [5]. A comparability graph is a perfect graph, whose clique number equals the chromatic number. The clique number and the chromatic number of a comparability graph can be computed in polynomial time.

A transitive graph can be represented by a Hasse diagram. If $(u, v) \in E$ and $(v, w) \in E$, (u, w) is omitted in a Hasse diagram. When there exists a path from u to v in a Hasse diagram, we call that u is an ancestor of v and v is a descendant of u, denoted as $u \prec v$. In this paper, for simplicity, we write a Hasse diagram as an undirected graph. All the edges are assumed to be downward edges.

For a transitive graph G, a function $f : V \rightarrow \{1, 2, \ldots, \omega(G)\}$ is called a leveling function if $f(u) < f(v)$ is satisfied for all $(u, v) \in E$. We define *levmin* and *levmax* as follows.

$$levmin(v) = \begin{cases} 1 & \text{if } v \text{ is a source} \\ \max_{(u,v)\in E} levmin(u) + 1 & \text{otherwise} \end{cases}$$

$$levmax(v) = \begin{cases} \omega(G) & \text{if } v \text{ is a sink} \\ \max_{(v,u)\in E} levmax(u) - 1 & \text{otherwise} \end{cases}$$

Then *levmin* and *levmax* are leveling functions. We call that $levmin(v)$ is the level of vertex v, and that v is in level $levmin(v)$. As a maximal path in the Hasse diagram corresponds to a maximal clique in a transitive graph, $\omega(G) = \max_{v\in V} levmin(v)$. If we color vertex v by $f(v)$ for some leveling function f, it is an optimal vertex coloring of G. In this paper, we call such coloring a levelwise coloring of G by f.

3 Coloring Comparability-ke Graphs

3.1 Coloring Comparability-1e Graphs

In this section, we consider vertex coloring of comparability-1e graphs. Let $G = (V, E)$ be a comparability-1e graph and $E_1 = \{(a, b)\}$ be the modulator of G. Let $G_c = (V, E \cup E_1)$.

First, we consider the relation between $\chi(G), \omega(G)$ and $\omega(G_c)$. If $\omega(G) = \omega(G_c)$, then $\chi(G) = \omega(G_c)$ holds. If $\omega(G) = \omega(G_c) - 1$, then $\omega(G_c) - 1 \leq \chi(G) \leq \omega(G_c)$ holds. In the latter case, there may exist an $(\omega(G_c) - 1)$-coloring of G. In the $(\omega(G_c) - 1)$-coloring, a and b have the same color.

We first show that it is not difficult to check if $\omega(G) = \omega(G_c)$ or not.

Lemma 1 *The equality $\omega(G) = \omega(G_c) - 1$ holds iff there exists no vertex v ($v \neq a, b$) such that levmin(v) is equal to levmin(a) or levmin(b) and levmin(v)+levmax(v)$-1 = \omega(G_c)$.*

Proof First, we should note that, for a vertex v, the size of the maximum clique including v is *levmin*(v) + *levmax*(v) $- 1$.
($\rightarrow$) $\omega(G) = \omega(G_c) - 1$ implies that all the maximum cliques of G' includes the modulator (a, b).

Assume that there exists a vertex v ($v \neq a$) such that *levmin*(v) = *levmin*(a) and *levmin*(v) + *levmax*(v) $- 1 = \omega(G')$. As *levmin*(v) + *levmax*(v) $- 1 = \omega(G_c)$, v is included in a maximum clique of G_c. However, as *levmin*(v) = *levmin*(a), v and a are not in the same clique. Therefore, G_c has a maximum clique which does not include (a, b). It means that $\omega(G) = \omega(G_c)$.
($\leftarrow$) If there exists no vertex v ($v \neq a, b$) such that *levmin*(v) is equal to *levmin*(a) or *levmin*(b) and *levmin*(v) + *levmax*(v) $- 1 = \omega(G_c)$, all the vertices which is in some maximum clique of G_c must be connected with a and b. Thus, the modulator (a, b) is included in all the maximum cliques of G_c. □

The condition of this lemma is checked easily using *levmin* and *levmax* of each node. Thus, it can be checked in polynomial time if the condition is satisfied. In the following, we consider the graphs satisfying the condition of Lemma 1.

Even though $\omega(G) = \omega(G_c) - 1$ holds, it is not always possible to color G with $\omega(G_c) - 1$ colors. We consider how to compute if G is $(\omega(G_c) - 1)$-colorable. To consider the coloring of G, we first obtain a transitive orientation of G_c. Let G_t be the obtained transitive graph. G_t is represented as a Hasse diagram $H = (V, E_H)$. In the following, we assume w.l.o.g. that $a \prec b$ in H. In a Hasse diagram, all the vertices in a path must have different colors. However, in this case, as we consider the coloring of G, we admit that a and b have the same color in H.

We consider to modify the graph without changing its chromatic number. Let w, x, y, z be the vertices satisfying the following conditions: $(w, x), (y, x), (w, z) \in E_H$ and w and x are in the same $(\omega(G_c) - 1)$-clique of G_c (see Fig.1). Let $H' = (V, E_H \cup \{(y, z)\})$ and let G' be the comparability-1e graph represented by H' when the modulator is added.

Lemma 2 $\chi(G') = \chi(G)$.

Proof As G' is obtained by adding edges to G, a coloring of G' is also a coloring of G. That is, $\chi(G') \geq \chi(G)$ holds. We show that if G is $(\omega(G_c) - 1)$-colorable, then G' is also $(\omega(G_c) - 1)$-colorable.

Consider a $(\omega(G_c) - 1)$ coloring of G. Let $U(v)$ be the set of colors used in v and its ancestors, and $L(v)$ be the set of colors used in v and its descendants. As $(w, x) \in E_H$, it holds that $U(w) \cap L(x) = \emptyset$. Similarly, $U(y) \cap L(x) = \emptyset$ and $U(w) \cap L(z) = \emptyset$ also

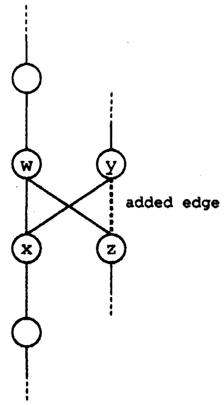

Figure 1: Add an edge to generate H'.

hold. In addition, as w and x are in the same $(\omega(G_c) - 1)$-clique of G_c, $U(w) \cup L(x) = \{1, \ldots, \omega(G_c) - 1\}$. Therefore, we can see that $U(y) \subseteq U(w)$ and $L(z) \subseteq L(x)$ hold. It follows that $U(y) \cap L(z) = \emptyset$. It means that even when (y, z) is added to H, no path of the Hasse diagram contains the same color (except two endpoints of the modulator). □

Add edges satisfying the above condition as far as possible. Let the resulting Hasse diagram be H_+ and the corresponding comparability-1e graph be G_+. Lemma2 shows that $\chi(G_+) = \chi(G)$. Let $V_a = \{v \mid a \prec v \text{ in } H_+\}$ and $V_b = \{v \mid v \prec b \text{ in } H_+\}$.

Lemma 3 *There exists an $(\omega(G_c)-1)$-coloring of G_+ iff there exists no $(\omega(G_c)-1)$-clique that does not include an endpoint of the modulator and whose all vertices are in $V_a \cup V_b$.*

Proof The vertices in V_a or V_b cannot be colored with the same color as a and b because the vertices in $V_a(V_b)$ and $a(b)$ are on the same path in H_+. If there exists an $(\omega(G_c) - 1)$-clique that does not include an endpoint of the modulator and whose all vertices are in $V_a \cup V_b$, there exists no $(\omega(G_c) - 1)$-coloring of G_+ because only $\omega(G_c) - 2$ colors can be used to color the clique.

Otherwise, we can color all the vertices using colors $\{1, 2, \ldots, \omega(G_c)-1\}$ in the following manner. Color a and b with 1. In each $(\omega(G_c) - 1)$-clique, color the vertex of the smallest level not in $V_a \cup V_b$ with 1. From the assumption, each $(\omega(G_c) - 1)$-clique includes at least one vertex colored with 1. In addition, we can show that no clique includes more than one vertices colored with 1.

Assume that a vertex d is colored with 1. Then, there exists a vertex e satisfying $(e, d) \in H_+$, $e \in V_b$ and d, e are in the same $(\omega(G_c) - 1)$-clique, and a vertex f satisfying $(e, f) \in H_+$ and $f \in V_b$. From the construction of H_+, for each vertex g s.t. $(g, d) \in H_+$, an edge (g, f) must exist in H_+. It means that any predecessor of d is included in V_b. Therefore, no two vertices in a $(\omega(G_c) - 1)$-clique can be colored with 1.

Color the other vertices with the following rule: if v is a source, color v with 2, and otherwise color v with the minimum number which is not used in the predecessors of v in H_+. We can observe that the coloring rule approves that no two vertices in a clique has the same color. In addition, we can show by induction on the level that the color of node

v with $levmin(v) = i$ is at most i. Therefore, the color of a sink is at most $\omega(G_c) - 1$. □

[Algorithm COLOR-1E]
Input: comparability-1e graph $G = (V, E)$, modulator $E_1 = \{(a, b)\}$
Output: chromatic number of G

1. Compute a transitive orientation of $G_c = (V, E \cup E_1)$ (represented by a Hasse diagram $H = (V, E_H)$) and compute $\chi(G_c)$.
2. Repeat the following until no more edge is added.
   Find the vertices w, x, y, z satisfying the following conditions: w and x are in the same $(\omega(G_c) - 1)$-clique of G_c, $(w, x) \in E_H$, $(y, x) \in E_H$ and $(w, z) \in E_H$. If there exist such vertices, add an edge (y, z) to E_H.
   Let G_+ be the obtained graph.
3. Compute $V_a = \{v \mid a \prec v \text{ in } H_+\}$ and $V_b = \{v \mid v \prec b \text{ in } H_+\}$.
4. Compute a subgraph of H_+ induced by $V_a \cup V_b - \{a, b\}$.
5. If the induced subgraph includes a path of length $\chi(G_c) - 1$, then output $\chi(G_c)$. Otherwise, output $\chi(G_c) - 1$.

Theorem 4 *Vertex coloring problem of comparability-1e graphs can be solved in polynomial time.*

3.2 Reduction to a Restricted Coloring of Comparability Graphs

In this section, we show that vertex coloring of comparability-ke graphs can be reduced to a kind of vertex coloring problem of comparability graphs. We define pair coloring problem as follows.
PAIR COLORING
Input : A graph $G = (V, E)$, a set of pairs of vertices $P \subseteq V^2$ and a positive integer k.
Output : If there exists a vertex coloring of G with k colors which colors u and v with the same color for all $(u, v) \in P$.

If there is no restriction on G, pair coloring is equivalent to vertex coloring of the graph obtained from G by identifying all the pairs of vertices in P. However, many graph classes including comparability graphs are not closed under identification of nonadjacent vertices.

Theorem 5 *Vertex coloring of comparability-ke graphs can be reduced to pair coloring of comparability graphs.*

Proof For an instance $\langle G = (V, E), k\rangle$ of vertex coloring of comparability-ke graphs, we construct an instance of pair coloring of comparability graphs $\langle G' = (V', E'), P, k'\rangle$ as follows.

Let M be the modulator of G and $G_t = (V, E_t)$ be a transitive graph obtained from $(V, E \cup M)$. Instead of G', we define a transitive graph $G'_t = (V', E'_t)$ obtained from G'. Let $B = \{w \mid (u, w) \in E_t \text{ and } (w, v) \in E_t \text{ for some } (u, v) \in M\}$. For each vertex $w \in B$, we add a vertex w'. That is, $V' = V \cup \{w' \mid w \in B\}$. Let $E'_t = \{(r, w) \mid (r, w) \in E_t, w \in$

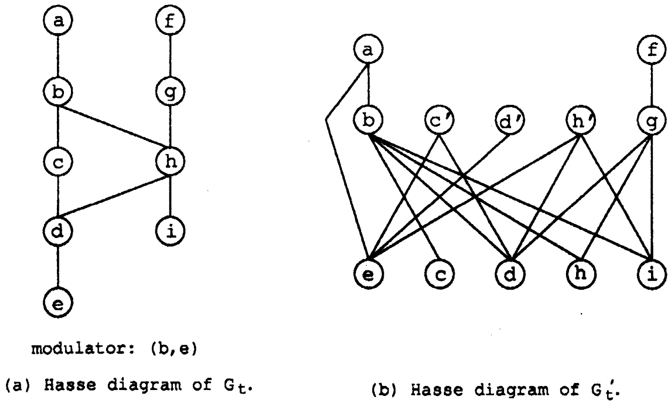

Figure 2: An example of reduction.

$B\} \cup \{(w', s) \mid (w, s) \in E_t, w \in B\} \cup \{(r, s) \mid (r, s) \in E_t - M, r \notin B, s \notin B\}$. Let the pairs of vertices be $P = \{\langle w, w' \rangle \mid w \in B\}$ and $k' = k$.

Now check that G'_t is really a transitive graph. Let $B' = \{w' \mid w \in B\}$. Note that there is no outgoing edges from vertices in B and no incoming edges from vertices in B'. Consider two edges $(a, b), (b, c) \in E'_t$. When $a, b, c \in V' - (B \cup B')$, both (a, b) and (b, c) are edges of E_t. As G_t is a transitive graph, there exists an edge $(a, c) \in E_t$, and thus $(a, c) \in E'_t$. When $c \in B$, there also exists an edge $(a, c) \in E_t$, and thus $(a, c) \in E'_t$. It is similar for the case when $a \in B'$. Fig.2 is an example of the G_t and G'_t represented as Hasse diagrams.

If we identify w and w' for all $w \in B$ on G', the obtained graph is the same as G. Therefore, it is obvious that G' has a pair coloring with k colors iff G is k-colorable. □

On the pair coloring problem, the number of pairs can be regarded as a parameter. However, as the number of pairs does not depend on the size of the modulator, this reduction is not a parameterized reduction.

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