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Algebraic cycles on Jacobian varieties

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1 Introduction

Let X be a projective smooth variety over $\mathbf{C}$. We denote by $Z_l(X)$ the $\mathbf{Q}$-vector space freely generated by all subvarieties of dimension l in X. The subspace $Z_l(X)_{\mathrm{rat}} \subset Z_l(X)$ is generated by divisors of rational functions on subvarieties of dimension $l+1$ in X, and the subspace $Z_l(X)_{\mathrm{alg}} \subset Z_l(X)$ is generated by the difference of two subvarieties which are equivalent by algebraic deformation in X. Then $Z_l(X)_{\mathrm{rat}}$ is contained in $Z_l(X)_{\mathrm{alg}}$, and $Z_l(X)_{\mathrm{alg}}$ is contained in the kernel $Z_l(X)_{\mathrm{hom}}$ of the topological cycle class map $Z_l(X) \to H_{2l}(X, \mathbf{Q})$. When $l = 0$ or $l = \dim X - 1$, we have $Z_l(X)_{\mathrm{alg}} = Z_l(X)_{\mathrm{hom}}$. But, in [4], using a Hodge-theoretic invariant, Griffiths found a nontrivial element in the quotient space $Z_1(X)_{\mathrm{hom}}/Z_1(X)_{\mathrm{alg}}$ for a quintic hypersurface X in $\mathbf{P}^4$. In this paper, we define descending filtration on $Z_l(X)$ and $Z_l(X)_{\mathrm{alg}}$ such that $\mathrm{Fil}^1 Z_l(X) = Z_l(X)_{\mathrm{hom}}$ and $\mathrm{Fil}^1 Z_l(X)_{\mathrm{alg}} = Z_l(X)_{\mathrm{alg}}$, and we find a nontrivial element in the quotient space $\mathrm{Fil}^p Z_l(X)/\mathrm{Fil}^p Z_l(X)_{\mathrm{alg}}$ for a Jacobian variety X. The space $\mathrm{Fil}^p Z_l(X)/\mathrm{Fil}^p Z_l(X)_{\mathrm{alg}}$ for a hypersurface X in $\mathbf{P}^n$ is studied by Saito [6].

Let C be a projective smooth curve over $\mathbf{C}$, and let J be the Jacobian variety of C. When we fix a point $p_0 \in C$, we have a natural morphism

$$\iota_l : \underbrace{C \times \cdots \times C}_{l} \longrightarrow J = H^0(C, \Omega^1_C)^\vee / H_1(C, \mathbf{Z});\ (p_1, \ldots, p_l) \longmapsto \Big[\omega \mapsto \sum_{i=1}^{l} \int_{p_0}^{p_i} \omega\Big].$$

The image W_l of ι_l is a subvariety of dimension l in J for $1 \le l \le g$. We denote by W_l^- the image of W_l by the multiplication by (-1) on J. Then W_l and W_l^- have the same homology class in $H_{2l}(J, \mathbf{Z})$. Here we have a natural question.

Question 1.1. $W_l - W_l^-$ is contained in $Z_l(J)_{\mathrm{alg}}$ or not?

If C is a hyperelliptic curve, then $W_l - W_l^-$ is contained in $Z_l(J)_{\mathrm{alg}}$. When C is not a hyperelliptic curve, using a Hodge-theoretic invariant, Ceresa proved the following result.

Theorem 1.2 (Ceresa [2]). *If C is a generic curve of genus g, then $W_l - W_l^-$ is not contained in $Z_l(J)_{\mathrm{alg}}$ for $1 \leq l \leq g-2$.*

In this paper, we go to a generalization of this theorem. To explain the generalization, we have to recall Beauville's result about algebraic cycles on abelian varieties. Let X be an abelian variety. We denote by $\mathbf{n} : X \to X$ the multiplication by $n \in \mathbf{Z}$ on X. We set a subspace of the $\mathbf{Q}$-vector space $\mathrm{CH}_l(X) = Z_l(X)/Z_l(X)_{\mathrm{rat}}$ by

$$\mathrm{CH}_l^{(p)}(X) = \{z \in \mathrm{CH}_l(X) \mid \mathbf{n}_* z = n^{2l+p} z \text{ for any } n \in \mathbf{Z}\}.$$

Theorem 1.3 (Beauville [1]). *There is a natural decomposition*

$$\mathrm{CH}_l(X) = \bigoplus_p \mathrm{CH}_l^{(p)}(X).$$

Using this decomposition for $[W_l] \in \mathrm{CH}_l(J)$;

$$[W_l] = \sum_p w_l^p, \quad (w_l^p \in \mathrm{CH}_l^{(p)}(J)),$$

the class of Ceresa's cycle $W_l - W_l^-$ is written by

$$[W_l - W_l^-] = \sum_p w_l^p - \sum_p (-1)^{2l+p} w_l^p = 2 \sum_{p:\mathrm{odd}} w_l^p.$$

We remark that w_l^p is contained in $\mathrm{Fil}^p \mathrm{CH}_l(J) = \mathrm{Fil}^p Z_l(J)/Z_l(J)_{\mathrm{rat}}$. Since the Hodge-theoretic invariant for $w_l^p (p \neq 1)$ is trivial, Ceresa's theorem is essentially equivalent to say that $w_l^1 \notin \mathrm{CH}_l(J)_{\mathrm{alg}} = Z_l(J)_{\mathrm{alg}}/Z_l(J)_{\mathrm{rat}}$. Here we have a generalized problem.

Question 1.4. w_l^p is contained in $\mathrm{Fil}^p \mathrm{CH}_l(J)_{\mathrm{alg}} = \mathrm{Fil}^p Z_l(J)_{\mathrm{alg}}/Z_l(J)_{\mathrm{rat}}$ or not?

We will find a curve such that $w_l^p \notin \mathrm{Fil}^p \mathrm{CH}_l(J)_{\mathrm{alg}}$. To show this, we use an algebraic invariant which is defined by using algebraic differential forms. When $p = 1$, the algebraic invariant is equal to the Griffiths' infinitesimal invariant, which is defined by the Hodge-theoretic invariant. The Griffiths' infinitesimal invariant for Ceresa's cycle $W_l - W_l^-$ is computed by Collino-Pirola [3].

This paper proceeds as follows. In Section 2, for any projective smooth variety X, we introduce the filtration on $\mathrm{CH}_l(X)$, and define the algebraic invariant for elements in $\mathrm{Fil}^p\,\mathrm{CH}_l(X)$. In Section 3, we prove a formula to compute the algebraic invariant for w_l^p, and give examples satisfying $w_l^p \notin \mathrm{Fil}^p\,\mathrm{CH}_l(J)_{\mathrm{alg}}$.

Some results in this paper is essentially same as [5], but the definition of filtration and the formulation of infinitesimal invariants are different from [5], and we give a new example.

2 Algebraic cycles and differential forms

2.1 Filtration

Let X be a projective smooth variety over $\mathbf{C}$. There exists a subfield $K \subset \mathbf{C}$ of finite transcendental degree over $\mathbf{Q}$, and a projective smooth variety X_K over K such that $X \simeq X_K \times_{\mathrm{Spec}\,K} \mathrm{Spec}\,\mathbf{C}$. We have an exact sequence

$$0 \longrightarrow \Omega^1_{K/\mathbf{Q}} \otimes \mathcal{O}_{X_K} \longrightarrow \Omega^1_{X_K/\mathbf{Q}} \longrightarrow \Omega^1_{X_K/K} \longrightarrow 0$$

of locally free $\mathcal{O}_{X_K}$-modules of finite ranks. We define filtration on $\Omega^r_{X_K/\mathbf{Q}} = \bigwedge^r \Omega^1_{X_K/\mathbf{Q}}$ by

$$\mathrm{Fil}^p\,\Omega^r_{X_K/\mathbf{Q}} = \mathrm{Image}\,(\Omega^p_{K/\mathbf{Q}} \otimes \Omega^{r-p}_{X_K/\mathbf{Q}} \longrightarrow \Omega^r_{X_K/\mathbf{Q}};\ \eta \otimes \omega \longmapsto \eta \wedge \omega),$$

and define filtration on the cohomology group by

$$\mathrm{Fil}^p\,H^i(X_K, \Omega^r_{X_K/\mathbf{Q}}) = \mathrm{Image}\,(H^i(X_K, \mathrm{Fil}^p\,\Omega^r_{X_K/\mathbf{Q}}) \longrightarrow H^i(X_K, \Omega^r_{X_K/\mathbf{Q}})).$$

Then we have $\mathrm{Gr}^p\,\Omega^r_{X_K/\mathbf{Q}} \simeq \Omega^p_{K/\mathbf{Q}} \otimes \Omega^{r-p}_{X_K/K}$, and there is a spectral sequence of K-vector spaces

$$E_1^{p,q} = H^{p+q}(X_K, \mathrm{Gr}^p\,\Omega^r_{X_K/\mathbf{Q}}) \Longrightarrow H^{p+q}(X_K, \Omega^r_{X_K/\mathbf{Q}}).$$

Proposition 2.1. *The spectral sequence degenerates at the E_2-term.*

Proof. This is proved by the same way as Lemma 2.3. in [5]. □

Let Z be a subvariety of dimension l in X_K, and let $\widetilde{Z} \to Z$ be a resolution of singularity. We set $m = \dim_K \Omega^1_{K/\mathbf{Q}}$. Then the pull-back

$$\Phi_Z : H^l(X_K, \Omega^{l+m}_{X_K/\mathbf{Q}}) \longrightarrow H^l(\widetilde{Z}, \Omega^{l+m}_{\widetilde{Z}/\mathbf{Q}}) \simeq \Omega^m_{K/\mathbf{Q}}$$

does not depend on the choice of the resolution $\widetilde{Z}$, and this induces a bilinear form

$$\Phi : \mathrm{CH}_l(X_K) \times H^l(X_K, \Omega^{l+m}_{X_K/\mathbf{Q}}) \longrightarrow \Omega^m_{K/\mathbf{Q}}$$

by $\Phi([Z], \omega) = \Phi_Z(\omega)$. We define filtration on $\mathrm{CH}_l(X_K)$ by

$$\mathrm{Fil}^p \mathrm{CH}_l(X_K) = \{z \in \mathrm{CH}_l(X_K) \mid \Phi(z, \omega) = 0 \text{ for any } \omega \in \mathrm{Fil}^{m+1-p} H^l(\Omega^{l+m}_{X_K/\mathbf{Q}})\},$$

and define filtration on $\mathrm{CH}_l(X)$ by

$$\mathrm{Fil}^p \mathrm{CH}_l(X) = \bigcup_{X_K} \mathrm{Fil}^p \mathrm{CH}_l(X_K) \subset \mathrm{CH}_l(X),$$

where the sum runs for all models X_K with $\mathrm{Tr.deg}_{\mathbf{Q}} K < \infty$.

Remark 2.2. $\mathrm{Fil}^1 \mathrm{CH}_l(X) = \mathrm{CH}_l(X)_{\mathrm{hom}}$.

Remark 2.3. If we assume the existence of Beilinson's conjectural filtration $F_{\mathcal{MM}}$ on Chow group, which comes form the theory of mixed motives, we have $F^p_{\mathcal{MM}} \mathrm{CH}_l(X) \subset \mathrm{Fil}^p \mathrm{CH}_l(X)$, but these are not equal in general.

We define a subspace of $\mathrm{Fil}^p \mathrm{CH}_l(X)$ by

$$\mathrm{Fil}^p \mathrm{CH}_l(X)_{\mathrm{alg}} = \sum_{Y,\Gamma} \mathrm{Image}\,(\mathrm{Fil}^p \mathrm{CH}_0(Y) \xrightarrow{\Gamma_*} \mathrm{Fil}^p \mathrm{CH}_l(X)),$$

where the sum runs for all projective smooth varieties Y and $\Gamma \in \mathrm{CH}_{\dim Y + l}(Y \times X)$, and Γ_* is the algebraic correspondence; $\Gamma_*(z) = p_{X*}(\Gamma.\ p_Y^* z)$, where p_X and p_Y denote the projections from $Y \times X$ to each component.

Remark 2.4. $\mathrm{Fil}^1 \mathrm{CH}_l(X)_{\mathrm{alg}} = \mathrm{CH}_l(X)_{\mathrm{alg}}$.

2.2 Infinitesimal invariants

Let X be a projective smooth variety over $\mathbf{C}$. For $z \in \mathrm{Fil}^p \mathrm{CH}_l(X)$, there exists a subfield $K \subset \mathbf{C}$ of finite transcendental degree over $\mathbf{Q}$, and a projective smooth variety X_K over K such that $X \simeq X_K \times_{\mathrm{Spec}\, K} \mathrm{Spec}\, \mathbf{C}$ and $z \in \mathrm{Fil}^p \mathrm{CH}_l(X_K)$. By the definition of filtration, we have a K-linear map

$$\Phi^p_{K/\mathbf{Q}}(z) : I^p_l(X_K) = \mathrm{Gr}^{m-p} H^l(X_K, \Omega^{l+m}_{X_K/\mathbf{Q}}) \longrightarrow \Omega^m_{K/\mathbf{Q}};\ [\omega] \longmapsto \Phi(z, \omega),$$

that is called infinitesimal invariant for z. By Proposition 2.1, the K-vector space $I^p_l(X_K)$ is isomorphic to the homology of the complex

$$\Omega^{m-p-1}_{K/\mathbf{Q}} \otimes H^{l-1}(\Omega^{l+p+1}_{X_K/K}) \longrightarrow \Omega^{m-p}_{K/\mathbf{Q}} \otimes H^l(\Omega^{l+p}_{X_K/K}) \longrightarrow \Omega^{m-p+1}_{K/\mathbf{Q}} \otimes H^{l+1}(\Omega^{l+p-1}_{X_K/K}).$$

We set a subspace of $H^l(\Omega^{l+p}_{X_K/K})$ by the image of the differential;

$$H^l(\Omega^{l+p}_{X_K/K})_0 = \mathrm{Image}\,((\Omega^1_{K/\mathbf{Q}})^\vee \otimes H^{l-1}(\Omega^{l+p+1}_{X_K/K}) \xrightarrow{\delta} H^l(\Omega^{l+p}_{X_K/K})).$$

Then we have a complex

$$\Omega^{m-p-1}_{K/\mathbf{Q}} \otimes H^{l-1}(\Omega^{l+p+1}_{X_K/K}) \longrightarrow \Omega^{m-p}_{K/\mathbf{Q}} \otimes H^l(\Omega^{l+p}_{X_K/K})_0 \longrightarrow \Omega^{m-p+1}_{K/\mathbf{Q}} \otimes H^{l+1}(\Omega^{l+p-1}_{X_K/K}),$$

and we denote its homology by $I^p_l(X_K)_0$, which is a subspace of $I^p_l(X_K)$.

Proposition 2.5. *If $z \in \mathrm{Fil}^p\,\mathrm{CH}_l(X)_{\mathrm{alg}}$, then the infinitesimal invariant $\Phi^p_{K/\mathbf{Q}}(z)$ is trivial on $I^p_l(X_K)_0$.*

Proof. This is proved by the same way as Proposition 2.13. in [5]. □

3 Jacobian varieties

3.1 Computation for invariants

Let $K \subset \mathbf{C}$ be a subfield of finite transcendental degree over $\mathbf{Q}$, and let C be a projective smooth curve over K. We have an exact sequence

$$0 \longrightarrow \Omega^{p+1}_{K/\mathbf{Q}} \otimes \mathcal{O}_C \longrightarrow \Omega^{p+1}_{C/\mathbf{Q}} \xrightarrow{\epsilon} \Omega^p_{K/\mathbf{Q}} \otimes \Omega^1_{C/K} \longrightarrow 0.$$

We denote by $\alpha^p : \bigwedge^{p+1} H^0(\Omega^1_{C/\mathbf{Q}}) \to \Omega^p_{K/\mathbf{Q}} \otimes H^0(\Omega^1_{C/K})$ the composition of natural map $\bigwedge^{p+1} H^0(\Omega^1_{C/\mathbf{Q}}) \to H^0(\Omega^{p+1}_{C/\mathbf{Q}})$ and $\epsilon : H^0(\Omega^{p+1}_{C/\mathbf{Q}}) \to \Omega^p_{K/\mathbf{Q}} \otimes H^0(\Omega^1_{C/K})$. Let V be a subspace of

$$U(C) = \mathrm{Image}\,(\alpha^0) = \mathrm{Ker}\,(H^0(\Omega^1_{C/K}) \longrightarrow \Omega^1_{K/\mathbf{Q}} \otimes H^1(\mathcal{O}_C)).$$

We define a subspace of $H^0(\Omega^1_{C/K})$ by

$$V^p = \mathrm{Image}\,((\Omega^p_{K/\mathbf{Q}})^\vee \otimes \bigwedge^{p+1} \widetilde{V} \xrightarrow{\alpha^p} H^0(\Omega^1_{C/K})),$$

where $\widetilde{V} = (\alpha^0)^{-1}(V) \subset H^0(\Omega^1_{C/\mathbf{Q}})$.

Remark 3.1. $V = V^0 \subset V^1 \subset \cdots \subset V^m$, $(m = \dim_K \Omega^1_{K/\mathbf{Q}})$.

Then the K-linear map α^p induces a map β_V^p in the following commutative diagram;

$$\begin{array}{ccc} \bigwedge^{p+1}\tilde{V} & \xrightarrow{\alpha^p} & \Omega^p_{K/\mathbf{Q}}\otimes H^0(\Omega^1_{C/K}) \\ \downarrow & & \downarrow \\ \bigwedge^{p+1}V & \xrightarrow{\beta^p_V} & \Omega^p_{K/\mathbf{Q}}\otimes H^0(\Omega^1_{C/K})/V^{p-1}. \end{array}$$

The composition of β_V^p and the natural quotient map to $H^0(\Omega^1_{C/K})/(V^{p-1}+U(C))$ is denoted by

$$\bar{\beta}^p_V : \bigwedge^{p+1} V \longrightarrow \Omega^p_{K/\mathbf{Q}}\otimes H^0(\Omega^1_{C/K})/(V^{p-1}+U(C)).$$

Let J be the Jacobian variety of C, and let $w_l^p \in \mathrm{CH}_l(J)$ be the algebraic cycle defined in Section 1.

Remark 3.2. $w_l^p \in \mathrm{Fil}^p\,\mathrm{CH}_l(J)$.

Let ϕ_l^p be the infinitesimal invariant for w_l^p;

$$\phi_l^p = \Phi^p_{K/\mathbf{Q}}(w_l^p) : I_l^p(J) \longrightarrow \Omega^m_{K/\mathbf{Q}}.$$

By the identification $H^j(\Omega^i_{J/K}) \simeq \bigwedge^i H^0(\Omega^1_{C/K})\otimes \bigwedge^j H^1(\mathcal{O}_C)$, we can compute ϕ_l^p by using β_V^p. We denote by

$$\langle\ ,\ \rangle : H^0(\Omega^1_{C/K})\times H^1(\mathcal{O}_C) \longrightarrow H^1(\Omega^1_{C/K}) \simeq K$$

the natural pairing.

Theorem 3.3. *For $\xi\in\Omega^{m-p}_{K/\mathbf{Q}}$, $v_1,\dots,v_{l+p}\in V$ and $\sigma_1,\dots,\sigma_l\in H^1(\mathcal{O}_C)$, if $\sigma_1\in (V^{p-1})^\perp$, then*

$$\phi_l^p(\xi\otimes v_1\wedge\cdots\wedge v_{l+p}\otimes\sigma_1\wedge\cdots\wedge\sigma_l) = \sum_{\mathbf{j}}\langle v_{\hat{\mathbf{j}}},\sigma_{\hat{1}}\rangle(\xi\wedge\langle\beta_V^p(v_{\mathbf{j}}),\sigma_1\rangle)\in\Omega^m_{K/\mathbf{Q}},$$

where the sum runs for all subset $\mathbf{j}=\{j_1,\dots,j_{p+1}\}\subset\{1,\dots,l+p\}$, and

$$\langle v_{\hat{\mathbf{j}}},\sigma_{\hat{1}}\rangle = \mathrm{sgn}(j_1,\dots,j_{p+1},k_1,\cdots,k_{l-1})\cdot\det\begin{pmatrix}\langle v_{k_1},\sigma_2\rangle & \cdots & \langle v_{k_1},\sigma_l\rangle\\ & \cdots & \\ \langle v_{k_{l-1}},\sigma_2\rangle & \cdots & \langle v_{k_{l-1}},\sigma_l\rangle\end{pmatrix},$$
$$(\{j_1,\dots,j_{p+1}\}\amalg\{k_1,\cdots,k_{l-1}\}=\{1,\dots,l+p\}).$$

Proof. This is proved by the same way as Theorem 3.9. in [5]. □

Corollary 3.4. *If there exists a subspace $V \subset U(C)$ such that $\bar{\beta}^p_V \neq 0$ and $\dim_K V \geq l+p$, then $w^p_l \notin \mathrm{Fil}^p\,\mathrm{CH}_l(J \times_{\mathrm{Spec}\,K} \mathrm{Spec}\,\mathbf{C})_{\mathrm{alg}}$.*

Proof. By the assumption, there exist $\xi \in \Omega^{m-p}_{K/\mathbf{Q}}$, $v_1, \dots, v_{p+1} \in V$ and $\sigma_1 \in (V^{p-1} + U(C))^{\perp} \subset H^1(\mathcal{O}_C)$ such that $\xi \wedge \langle \beta^p_V(v_1 \wedge \cdots \wedge v_{p+1}), \sigma_1 \rangle \neq 0$. Since $\sigma_1 \in U(C)^{\perp}$, there exists $\gamma \in (\Omega^1_{K/\mathbf{Q}})^{\vee} \otimes H^0(\Omega^1_{X_K/K})$ such that $\sigma_1 = \delta(\gamma)$, where δ is the differential map $\delta : (\Omega^1_{K/\mathbf{Q}})^{\vee} \otimes H^0(\Omega^1_{X_K/K}) \to H^1(\mathcal{O}_C)$. We take $v_{p+2}, \dots, v_{l+p} \in V$ and $\sigma_2, \dots, \sigma_l \in (\sum_{i=1}^{p+1} \mathbf{Q} v_i)^{\perp} \subset H^1(\mathcal{O}_C)$ such that $v_1 \wedge \cdots \wedge v_{l+p} \neq 0$ and $\langle v_{p+1+i}, \sigma_{j+1} \rangle = \delta_{ij}$. Then

$$v_1 \wedge \cdots \wedge v_{l+p} \otimes \sigma_1 \wedge \cdots \wedge \sigma_l = \delta(\gamma \wedge v_1 \wedge \cdots \wedge v_{l+p} \otimes \sigma_2 \wedge \cdots \wedge \sigma_l)$$

is contained in $H^l(\Omega^{l+p}_{J/K})_0$, and by Theorem 3.3,

$$\phi^p_l(\xi \otimes v_1 \wedge \cdots \wedge v_{l+p} \otimes \sigma_1 \wedge \cdots \wedge \sigma_l) = \xi \wedge \langle \beta^p_V(v_1 \wedge \cdots \wedge v_{p+1}), \sigma_1 \rangle \neq 0.$$

By Proposition 2.5, w^p_l is not contained in $\mathrm{Fil}^p\,\mathrm{CH}_l(J)_{\mathrm{alg}}$. □

3.2 Example

Let $f(x) = a_0 x^{e_1} + a_1 x^{e_1 - 1} + \cdots + a_{e_1} \in \mathbf{C}[x]$ be a separable polynomial of degree e_1, and let C be the smooth compactification of the affine curve $\mathrm{Spec}\,\mathbf{C}[x,y]/(y^{e_2} - f(x))$. Then the genus of C is $g = \{(e_1 - 1)(e_2 - 1) - (e_0 - 1)\}/2$, where $e_0 = \gcd\{e_1, e_2\}$. We set $K = \mathbf{Q}(a_0, \dots, a_{e_1}) \subset \mathbf{C}$. We can consider C_K as a hypersurface in weighted projective space $\mathbf{P} = \mathbf{P}_K(1, e_2/e_0, e_1/e_0)$ over K defined by the weighted homogeneous polynomial

$$F(z_0, z_1, z_2) = a_0 z_1^{e_1} + a_1 z_0^{e_2/e_0} z_1^{e_1 - 1} + \cdots + a_{e_1} z_0^{e_1 e_2/e_0} - z_2^{e_2} \in K[z_0, z_1, z_2],$$

where $\deg z_0 = 1$, $\deg z_1 = e_2/e_0$, $\deg z_2 = e_1/e_0$. There is a natural identification

$$\begin{array}{ccc} H^0(\Omega^1_{C_K/K}) & \simeq & H^0(\mathbf{P}, \mathcal{O}_{\mathbf{P}}((e_1 e_2 - e_0 - e_1 - e_2)/e_0)); \\ \dfrac{x^i y^j dx}{e_2 y^{e_2 - 1}} & \leftrightarrow & z_0^{(e_1 e_2 - e_0 - (j+1)e_1 - (i+1)e_2)/e_0} z_1^i z_2^j. \end{array}$$

For $\omega_1, \dots, \omega_{p+1} \in V \subset U(C_K)$, we compute $\beta^p_V(\omega_1 \wedge \cdots \wedge \omega_{p+1})$, using this identification. Let $B_i \in H^0(\mathbf{P}, \mathcal{O}_{\mathbf{P}}((e_1 e_2 - e_0 - e_1 - e_2)/e_0))$ be the weighted homogeneous polynomial corresponding to ω_i. Since $\omega_i \in U(C_K)$, there exist weighted homogeneous polynomials $H_{i,j,k}$ such that

$$B_i \frac{\partial F}{\partial a_j} = z_0^{ie_2/e_0} z_1^{e_1-i} B_i \equiv H_{i,j,0} \frac{\partial F}{\partial z_0} + H_{i,j,1} \frac{\partial F}{\partial z_1} + H_{i,j,2} \frac{\partial F}{\partial z_2} \mod (F).$$

We set weighted homogeneous polynomials by

$$\begin{aligned} G_{i,j,0} &= \frac{e_2}{e_0} z_1 H_{i,j,2} - \frac{e_1}{e_0} z_2 H_{i,j,1} && \in H^0(\mathbf{P}, \mathcal{O}_{\mathbf{P}}((e_1e_2 - e_0)/e_0)), \\ G_{i,j,1} &= \frac{e_1}{e_0} z_2 H_{i,j,0} - z_0 H_{i,j,2} && \in H^0(\mathbf{P}, \mathcal{O}_{\mathbf{P}}((e_1e_2 - e_2)/e_0)), \\ G_{i,j,2} &= z_0 H_{i,j,1} - \frac{e_2}{e_0} z_1 H_{i,j,0} && \in H^0(\mathbf{P}, \mathcal{O}_{\mathbf{P}}((e_1e_2 - e_1)/e_0)). \end{aligned}$$

For $\mathbf{j} = \{j_1, \dots, j_p\} \subset \{0, \dots, e_1\}$, there is a weighted homogeneous polynomials $A_{\mathbf{j}} \in H^0(\mathbf{P}, \mathcal{O}_{\mathbf{P}}((e_1e_2 - e_0 - e_1 - e_2)/e_0))$ such that

$$A_{\mathbf{j}} \Big(\frac{\partial F}{\partial z_k} \Big)^p \equiv \det \begin{pmatrix} B_1 & \cdots & B_{p+1} \\ G_{1,1,k} & \cdots & G_{p+1,1,k} \\ & \cdots & \\ G_{1,p,k} & \cdots & G_{p+1,p,k} \end{pmatrix} \mod (F),$$

and $\eta_{\mathbf{j}}$ denotes the element in $H^0(\Omega^1_{C_K/K})$ corresponding to $A_{\mathbf{j}}$.

Theorem 3.5.

$$\beta_V^p(\omega_1 \wedge \cdots \wedge \omega_{p+1}) = \sum_{\mathbf{j}} da_{j_1} \wedge \cdots \wedge da_{j_p} \otimes [\eta_{\mathbf{j}}] \in \Omega^p_{K/\mathbf{Q}} \otimes H^0(\Omega^1_{C_K/K})/V^{p-1}.$$

Proof. This is proved by the same way as Theorem 4.1. in [5]. □

Theorem 3.6. *In the following cases, w_l^p is not contained in* $\mathrm{Fil}^p\,\mathrm{CH}_l(J)_{\mathrm{alg}}$*;*

1. $e_2 = e_1$, $1 \le p \le \mathrm{Tr.deg}_{\mathbf{Q}}\,\mathbf{Q}(a_2, \dots, a_{e_1-2})$, $l + p \le e_1 - 2$, *and $f(x)$ is general,*
2. $e_2 > e_1$, $1 \le p \le \max\{\mathrm{Tr.deg}_{\mathbf{Q}}\,\mathbf{Q}(a_2, \dots, a_{e_1-1}), \mathrm{Tr.deg}_{\mathbf{Q}}\,\mathbf{Q}(a_1, \dots, a_{e_1-2})\}$, $l + p \le e_1 - 1$, *and $f(x)$ is general.*

Proof. By Corollary 3.4, we find a subspace V such that $\bar{\beta}_V^p \neq 0$ and $\dim_K V \ge l + p$, We set

$$V = \bigoplus_{0 \le i \le (e_1e_2 - e_0 - e_1 - e_2)/e_2} K \cdot \frac{x^i dx}{y^{e_2-1}} \subset H^0(\Omega^1_{C_K/K}).$$

If $e_2 \ge e_1$, then V is contained in $U(C_K)$ for general $f(x)$. By using Theorem 3.5, we can show that

$$V^p \subset U^p = \bigoplus_{\substack{ie_2 + je_1 \leq e_1e_2 - e_0 - e_1 - e_2 \\ i \geq 0,\ 0 \leq j \leq p}} K \cdot \frac{x^i y^j dx}{y^{e_2 - 1}} \subset H^0(\Omega^1_{C_K/K}),$$

and $V^p \not\subset U^{p-1}$ for general $f(x)$. This means that $\bar{\beta}^p_V$ is not trivial. □

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