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THE GEGRAPHY OF A CERTAIN CLASS OF LEFSCHETZ FIBRATIONS FROM THE TOPOLOGICAL VIEWPOINT

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1. Introduction

If 4-dimensional manifolds admitt fibration structures, then we can understand their topology in detail. Elliptic surfaces, complex surfaces admitting elliptic fibrations whose generic fibers are smooth elliptic curves, were deeply studied by Kodaira, Kas, Moishezon and so on. Matsumoto considered smooth 4-manifolds admitting topological elliptic fibrations, that is, smooth fibrations whose generic fibers are homeomorphic to a real 2-dimensional torus. Matsumoto [11] and Ue [29] obtained the classification theorem of such topological elliptic surfaces, which asserts that diffeomorpism types of minimal, topological elliptic surfaces are determined by the Euler characteristic e, the genus of the base space and the multiplicities of multiple fibers. It is interesting that exotic differential structures were found in such topological elliptic surfaces. Using the gauge theory Donaldson showed that Dolgachev surfaces give exotic differentiable structures on the rational elliptic surface $E(1) \cong \mathbb{C}P^2 \sharp 9\overline{\mathbb{C}P^2}$, that is, Dolgachev surfaces are homeomorphic to $E(1)$ but not diffeomorphic. By using Donaldson's polynomial invariants, the author and Kametani [5] also showed that surfaces obtained from the elliptic surface $E(k)$ by logarithmic transformations give exotic differentiable structures on $E(k)$. They calculated the invariants by using the fact that the moduli space of irreducible Hermitian-Einstein connections on a $U(2)$-bundle P coincides with the moduli space of stable holomorphic structures on P. So, the study of moduli spaces of stable rank 2 vector bundles over smooth complex algebraic surfaces has contributed to the study of differentiable structures on their underlying smooth 4-manifolds.

After that, the center of 4-dimensional topology has shifted from the study of differentiable structures to one of symplectic structures. The study of *Lefschetz fibrations* in 4-dimensinal topology becomes popular from the latter half in the 1990's. A Lefschetz fibration is a smooth fibration of a smooth 4-manifold over a surface with finitely many critical points as complex analogs of Morse functions. The importance of Lefschetz fibrations from the viewpoint of topology was reverified by Matsumoto [12]. The study of Lefschetz pencils and Lefschetz fibrations has been the center of 4-dimensional symplectic topology by the support of the remarkable works of Donaldson [1] and Gompf [4], which implies that Lefschetz fibrations provide a topological characterization of symplectic 4-manifolds. Therefore, most of symplectic 4-manifolds correspond to most of 4-manifolds with Lefschetz fibrations.

The geography problem in complex surfaces is the characterization of pairs of integers which are realized as (c_1^2, c_2) of complex surfaces, and it is well studied in algebraic geometry. By the classification of complex surfaces due to Kodaira, a simply connected complex surface is rational, elliptic or of general type. We know completely the range which rational surfaces and elliptic surfaces cover in the (c_1^2, c_2)-plane. Minimal surfaces of general type must satisfy that c_1^2, $c_2 > 0$ and $(c_2 - 36)/5 \leq c_1^2 \leq 3c_2$ (the Noether inequality and Bogomolov-Miyaoka-Yau inequality).

A simply connected complex surface is Kähler and so symplectic. Therefore, the geography problem for symplectic 4-manifolds comes into our mind. This problem is raised by McCarthy and Wolfson [13]:

(1) Which pairs of integers are realized as (c_1^2, c_2) of a symplectic 4-manifold ?
(2) If there is a symplectic 4-manifold corresponding to a given lattice point (c_1^2, c_2), are there many distinct symplectic structures on it ?

The remarkable works of Donaldson and Gompf suggest that the geography of symplectic 4-manifolds is nearly the same as one of Lefschetz fibrations. Every lattice point (c_1^2, c_2) except finitely many lying in $(c_2 - 36)/5 \leq c_1^2 \leq 2c_2$ is realized as the total space of a Lefschetz fibration [18]. Fintushel and Stern showed that there exists a minimal Lefschetz fibration which does not satisfy the Noether inequality [3]. In [25], Stipsicz addressed the Bogomolov-Miyaoka-Yau inequality for Lefschetz fibrations.

Topologists often construct new manifolds by the cut-and-paste method. As one can make a new manifold from given manifolds by taking the connected sum, one can make a new Lefschetz fibration from given Lefschetz fibrations by taking the *fiber sum*. Instead of investigating all of the investigation objects, we restrict them to *prime* (or *irreducible*) things and often investigate these. Therefore, it is natural to investigate the geography of *irreducible* Lefschetz fibrations, that is, Lefschetz fibrations which cannot be decomposed as any nontrivial fiber sum. Lefschetz fibrations with spheres of square -1 have the following characteristics :

(1) Lefschetz fibrations come from Lefschetz pencils which Lefschetz introduced to the study of topology of algebraic varieties. The blow-ups of a Lefschetz pencil along the base locus yields a Lefschetz fibration with sections of square -1.
(2) Lefschetz fibrations with sections of square -1 cannot be decomposed as any nontrivial fiber sum [27].

Thus, we can judge that Lefschetz fibrations with spheres of square -1 are fundamental. In this note, we consider the geography problem of Lefschetz fibrations with spheres of square -1. We have only to understand the horizontal direction of a Lefschetz fibration to investigate the geography because the vertical direction is understood well. It is our fundamental idea to draw information in the horizontal direction by using spheres of square -1.

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2. Topological invariants of Lefschetz fibrations and the numbers of singular fibers

2.1. The definition of Lefschetz fibrations. A smooth map $f : X \to \Sigma$ from a closed, connected, oriented smooth 4-manifold X onto a closed, connected, oriented smooth 2-manifold Σ is said to be a *Lefschetz fibration*, if f admits finitely many critical points $C = \{p_1, p_2, \ldots, p_k\}$ on which f is injective and around which there are orientation-preserving complex coordinate neighborhoods such that locally f can be expressed as $f(z_1, z_2) = z_1^2 + z_2^2$. It is a consequence of this definition that $f|_{X \setminus C} : X \setminus C \to \Sigma \setminus f(C)$ is a smooth fiber bundle with fiber a closed oriented 2-manifold. If the genus of a generic fiber is g, we refer to f as a *genus-g Lefschetz fibration*. Moreover, we assume that f is relatively minimal, that is, there is no fiber containing a sphere of square -1.

A fiber containing a critical point is called a *singular fiber*, which is obtained by collapsing a simple closed curve, called a *vanishing cycle*, on a nearby generic fiber to a point. A singular fiber is called *reducible* or *irreducible* according to whether the corresponding vanishing cycle separates or dose not separate in the generic fiber. In particular, if a vanishing cycle α separates the closed surface Σ_g of genus g into two components with genera h and $g - h$ $(1 \le h \le [g/2])$, then the reducible singular fiber corresponding to α is said to be of type II_h.

The local monodromy around a singular fiber of a Lefschetz fibration $f : X \to S^2$ is a positive Dehn twist t_a along the corresponding vanishing cycle a.

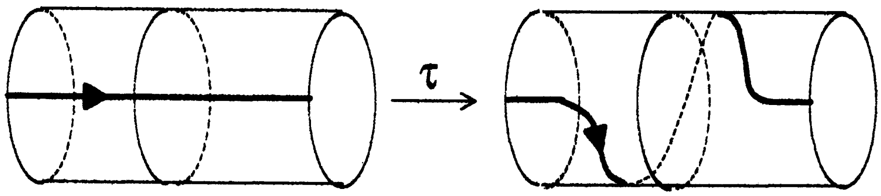

Figure 1 : positive Dehn twist

The product of all the local monodromies of f is trivial in the mapping class group Γ_g of genus g. Such a relation in Γ_g

$$t_{a_1} t_{a_2} \cdots t_{a_\mu} = 1$$

is called a *positive relation*, where $a_1, a_2, \ldots, a_\mu$ are vanishing cycles of f. The following theorem implies that genus-g Lefschetz fibrations correspond to positive relations in Γ_g.

Theorem 2.1 (Matsumoto [12]). *Suppose that $g \ge 2$. Then, there is a one-to-one correspondence :*

$$\left\{ \begin{array}{l} \textit{isomorphism classes of} \\ \textit{LF with } n \textit{ singular fibers} \end{array} \right\} \xleftrightarrow{\text{one-to-one}} \{\textit{conjugacy classes of } \rho\},$$

where $\rho : \pi_1(S^2 - \{f(p_i)\text{'s}\}, b_0) \to \Gamma_g$ is the monodromy representation.

It is the following theorems to become the clue that the study of Lefschetz fibrations became popular.

Theorem 2.2 (Donaldson [1]). *Every symplectic 4-manifold admits a Lefschetz pencil whose closed fibers are symplectic submanifolds.*

Thurston investigated the symplectic structures on surface-bundles over surfaces and gave the first example of a symplectic closed manifold which is not Kähler. Gompf proved the following theorem by generalizing the symplecticness of surface-bundles over surfaces.

Theorem 2.3 (Gompf [4]). *Let $f : X \to S^2$ be a Lefschetz fibration and let $[F]$ denote the homology class of the fiber. If $[F] \neq 0$ in $H_2(X;\mathbb{R})$, then X admits a symplectic structure such that fibers are symplectic submanifolds.*

If the fiber- genus g is greater than 1, then the homology class of a generic fiber of f is not torsion in $H_2(X;\mathbb{Z})$, and so this theorem states that such an X admits a symplectic structure with symplectic fibers.

From now on, we suppose that the fiber- genus g is greater than 1 and we can use the symplectic topology. Combining the remarkable theorems of Donaldson and Gompf gives the following topological characterization of symplectic 4-manifolds.

Corollary 2.1. *A 4-manifold X admits a symplectic structure if and only if it admits a Lefschetz pencil.*

2.2. **The signature of Lefschetz fibrations.** The Hirzebruch's signature theorem implies that the pair (c_1^2, c_2) of Chern numbers are determined by the signature and the Euler characteristic. So, it is important to calculate the signature and the Euler characteristic of a 4-manifold admitting a Lefschetz fibration. Every singular fiber of a genus-g Lefschetz fibration $f : X \to S^2$ contributes $+1$ to the Euler characteristic $e(X)$. If the fibration f has μ singular fibers, then we have $e(X) = 4(1-g) + \mu$.

Compared with the Euler characteristic, it is difficult to calculate the signature of X. Now we introduce some signature formulae. They are the formula for hyperelliptic Lefschetz fibrations and the formula for general (possibly non-hyperelliptic) Lefschetz fibrations. Let $F_1, F_2, \ldots, F_\mu$ be singular fibers of $f : X \to S^2$. Let $N(F_i)$ denote the tubular neighborhood of F_i $(i = 1, 2, \ldots, \mu)$. We set $X_0 = X - \bigcup_{i=1}^{\mu} N(F_i)$. Then the restriction $f_{|X_0} : X_0 \to f(X_0)$ is the associated Σ_g-bundle over the punctured sphere. Since an irreducible singular fiber and a reducible singular fiber contribute 0 and -1 to the signature $\sigma(X)$ respectively, it follows from the Novikov's additivity that we have

$$\sigma(X) = \sigma(X_0) - \sum_{h=1}^{[g/2]} s_h$$

, where s_h denotes the number of singular fibers of type II_h. The signature $\sigma(X_0)$ of the bundle part X_0 can be calculated from the signature cocycle τ_g. The signature cocycle τ_g is defined algebraically. Let $P = S^2 - \coprod_{i=1}^3 \mathrm{Int}\ D_i^2$ be a pants and $E(\alpha,\beta) \to P$ the Σ_g-bundle defined by monodromies $\alpha, \beta \in \Gamma_g$. Then, Meyer showed [15] that for the signature of $E(\alpha,\beta)$ we have

$$\sigma(E(\alpha,\beta)) = -\tau_g(\alpha,\beta).$$

If f has μ singular fibers, then we decompose the μ punctured sphere $f(X_0)$ into μ pairs $P_1, P_2, \ldots, P_\mu$ of pants. Then it follows from the Novikov's additivity and Meyer's theorem that we have

$$\sigma(X_0) = \sum_{i=1}^{\mu} \sigma(f^{-1}(P_i)) = -\sum_{i=1}^{\mu} \tau_g(t_{a_{i-1}} \cdots t_{a_2} t_{a_1},\ t_{a_i}).$$

Here $a_1, a_2, \ldots, a_\mu$ are vanishing cycles of f.

Matsumoto-Endo's signature formula: A *hyperelliptic* Lefschetz fibration is a Lefschetz fibration whose monodromy representation ρ is equivalent to one taking isotopy classes commuting with the hyperelliptic involution $\iota : \Sigma_g \to \Sigma_g$. Since the hyperelliptic mapping class group Γ_2^{hyp} of genus 2 agrees with Γ_2, every genus-2 Lefschetz fibration is hyperelliptic.

When we restrict the signature cocycle τ_g to the hyperelliptic mapping class group Γ_g^{hyp}, its cohomology class $[\tau_g^H] \in H^2(\Gamma_g^{\mathrm{hyp}}; \mathbb{Z})$ is of finite order. So we can calculate the terms of signature cocycles by the coboundary maps called Meyer's functions. Matsumoto [12] and Endo [2] calculated Meyer's functions and obtained the signature formula for hyperelliptic Lefschetz fibrations.

Theorem 2.4 (Matsumoto [12], Endo [2]). *Suppose that $f : X \to S^2$ is a genus-g hyperelliptic Lefschetz fibration with n_0 irreducible singular fibers and s_h singular fibers of type II_h $(h = 1, 2, \ldots, [g/2])$. Then, we have*

$$\sigma(X) = -\frac{g+1}{2g+1} n_0 + \sum_{h=1}^{[g/2]} \left(\frac{4h(g-h)}{2g+1} - 1 \right) s_h.$$

Smith's signature formula: Smith obtained the signature formula for general (possibly non-hyperelliptic) Lefschetz fibrations by using the geometry of the moduli space of stable curves. We denote the Deligne-Mumford compactified moduli space of stable curves of genus g by $\overline{\mathcal{M}_g}$. Let $f : X \to S^2$ be a genus-g Lefschetz fibration. Then we can define the moduli map $\phi_f : S^2 \to \overline{\mathcal{M}_g}$ of f by

$$\phi_f(x) := [f^{-1}(x)] \in \overline{\mathcal{M}_g} \quad (\forall x \in S^2).$$

In particular, if $f : X \to \mathbb{C}P^1$ is holomorphic, then the image $\phi_f(\mathbb{C}P^1)$ is a rational curve in $\overline{\mathcal{M}_g}$.

Theorem 2.5 (Smith [23]). *For any genus-g Lefschetz fibration $f : X \to S^2$ with μ singular fibers, namely $\mu = n_0 + \sum_{h=1}^{[g/2]} s_h$, the signature of X is given by*

$$\sigma(X) = 4\langle c_1(\lambda),\ [\phi_f(S^2)]\rangle - \mu$$

where $\lambda \to \overline{\mathcal{M}_g}$ denotes the Hodge bundle with fiber the determinant line $\wedge^g H^0(C; K_C)$ above $[C]$.

For a projective fibration $f : X \to \mathbb{C}P^1$, this theorem follows from Mumford's formula. Smith's formula is a generalization of the Atiyah's formula for smooth fibrations, and related work by Meyer.

2.3. **The numbers of singular fibers of Lefschetz fibrations.** It is conjectured that symplectic 4-manifolds which are not the blow-ups of ruled surfaces satisfy $e \geq 0$ for the Euler characteristic e. Since the Euler characteristic e of a genus-g Lefschetz fibration is given by $e = 4(1-g) + \mu$, it is a problem whether to satisfy $\mu \geq 4(g-1)$. Let $N(g)$ denote the minimal number of singular fibers in genus-g Lefschetz fibrations over S^2. Then, the following estimates on $N(g)$ is given.

Theorem 2.6 (Korkmaz-Ozbagci [7], Stipsicz [26]). *We have the estimates on $N(g)$ as follows :*

(1) $N(2) = 7$, *or* 8
(2) $N(g) \geq \frac{1}{5}(4g+2)$

Moreover, by considering the abelialization of the global monodromy of a Lefschetz fibration, we obtain the congruence on the number of singular fibers. Noting that the abelialization $H_1(\Gamma_2; \mathbb{Z})$ of Γ_2 is isomorphic to the cyclic group of order 10, the following proposition is proved.

Proposition 2.1 (Persson). *Suppose that a genus-2 Lefschetz fibration has n_0 irreducible singular fibers and s reducible singular fibers. Then, we have*

$$n_0 + 2s \equiv 0 \pmod{10}.$$

If $g \geq 3$, then $H_1(\Gamma_g; \mathbb{Z}) = 0$, and so we can get no information on the number of singular fibers. However, we can get information for hyperelliptic Lefschetz fibrations. Since the abelialization $H_1(\Gamma_g^{\mathrm{hyp}}; \mathbb{Z})$ of the hyperelliptic mapping class group Γ_g^{hyp} is isomorphic to $\mathbb{Z}/2(2g+1)$ if g is even and $\mathbb{Z}/4(2g+1)$ if g is odd, we obtain the congruence on the number of singular fibers of a hyperelliptic fibration.

Proposition 2.2 (Endo [2]). *Suppose that $f : X \to S^2$ is a genus-g hyperelliptic Lefschetz fibration with n_0 irreducible singular fibers and s_h singular fibers of type II_h $(h = 1, 2, \ldots, [g/2])$. Then, we have*

$$n + 4\sum_{h=1}^{[g/2]} h(2h+1)s_h \equiv 0 \left(\mathrm{mod} \begin{cases} 2(2g+1) & (\textit{if } g \textit{ is even}) \\ 4(2g+1) & (\textit{if } g \textit{ is odd}) \end{cases}\right).$$

3. Examples of Lefschetz fibrations

Let $\zeta_1, \zeta_2, \ldots, \zeta_{2g+1}$ be Lickorish generators of the mapping class group Γ_g of genus g. These are given by positive Dehn twists along the following illustrated curves on Σ_g.

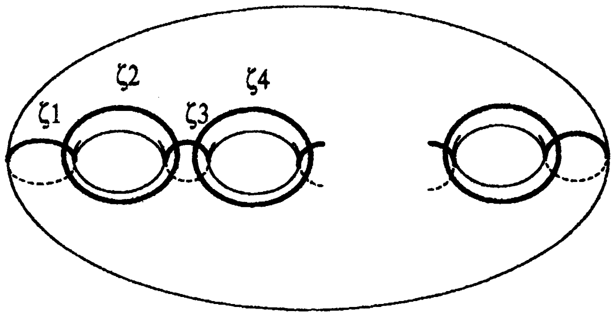

Figure 2 : Lickorish generators

Typical positive relations in Γ_g are the followings :

$$W_1 : (\zeta_1 \cdot \zeta_2 \cdots \zeta_{2g} \cdot \zeta_{2g+1} \cdot \zeta_{2g+1} \cdot \zeta_{2g} \cdots \zeta_2 \cdot \zeta_1)^2 = 1$$
$$W_2 : (\zeta_1 \cdot \zeta_2 \cdots \zeta_{2g} \cdot \zeta_{2g+1})^{2g+2} = 1$$
$$W_3 : (\zeta_1 \cdot \zeta_2 \cdots \zeta_{2g})^{4g+2} = 1$$

From these positive relations, we can construct hyperelliptic genus- g Lefschetz fibrations with only irreducible singular fibers and with sections of self-intersection -1. Furthermore, these Lefschetz fibrations are double branched covers of the Hirzebruch surfaces and so holomorphic.

3.1. **Examples of genus-2 Lefschetz fibrations.** The Hirzebruch surface $\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{C}P^1} \oplus \mathcal{O}_{\mathbb{C}P^1}(n))$ has two disjoint holomorphic sections Δ_n and Δ_{-n} of square $\pm n$.

(1) $M_1 = \mathbb{C}P^2 \sharp 13\overline{\mathbb{C}P^2}$

The positive relation $W_1 : (\zeta_1 \cdot \zeta_2 \cdot \zeta_3 \cdot \zeta_4 \cdot \zeta_5^2 \cdot \zeta_4 \cdot \zeta_3 \cdot \zeta_2 \cdot \zeta_1)^2 = 1$ describes the genus-2 Lefschetz fibration on the rational surface M_1 obtained as a double covering of $\mathbb{F}_0$ branched along a smooth algebraic curve in the linear system $|6\Delta + 2F|$. This fibration is obtained from the composition of the covering projection with the bundle projection $\mathbb{F}_0 \to S^2$ and has 20 irreducible singular fibers and sections of square -1.

(2) $M_2 = K3 \sharp 2\overline{\mathbb{C}P^2}$

The positive relation $W_2 : (\zeta_1 \cdot \zeta_2 \cdot \zeta_3 \cdot \zeta_4 \cdot \zeta_5)^6 = 1$ describes the genus-2 Lefschetz fibration on M_2 obtained as a double covering of $\mathbb{F}_1 = \mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$ branched along a smooth algebraic curve in the linear system $|6L|$, where L is a line in $\mathbb{C}P^2$ avoiding the blown-up point. This fibration has 30 irreducible singular fibers and sections of square -1.

(3) $M_3 = H'(1)$ (Horikawa surface)

The positive relation $W_3 : (\zeta_1 \cdot \zeta_2 \cdot \zeta_3 \cdot \zeta_4)^{10} = 1$ describes the genus-2 Lefschetz fibration on M_3 obtained as a double covering of $\mathbb{F}_2$ branched along the disjoint union of a smooth curve in the linear system $|5\Delta_2|$ and Δ_{-2}. This fibration has 40 irreducible singular fibers and a section of square -1. This section is a lift of the component of the branched set coming from Δ_{-2}. On the other hand, a fiber sum of two copies of the rational genus-2 Lefschetz fibration $\mathbb{C}P^2 \sharp 13\overline{\mathbb{C}P^2} \to S^2$ is a genus-2 Lefschetz fibration which has 40 irreducible singular fibers and the total space is homeomorphic to $H'(1)$ but not diffeomorphic. See §5.

(4) $S^2 \times T^2 \sharp 4\overline{\mathbb{C}P^2}$

Matsumoto showed that $S^2 \times T^2 \sharp 4\overline{\mathbb{C}P^2}$ has a genus-2 Lefschetz fibration with 6 irreducible singular fibers and 2 reducible singular fibers. This also has a section of square -1. The positive relation describing this fibration is $(\alpha_1 \cdot \alpha_2 \cdot \alpha_3 \cdot \alpha_4)^2 = 1$, where $\alpha_1, \dots, \alpha_4$ are given by positive Dehn twists along curves indicated below :

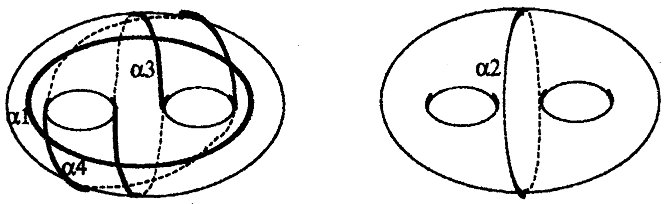

Figure 3.

3.2. **Examples of genus-3 Lefschetz fibrations.**

(1) M_1, M_2 and M_3 corresponding to positive relations W_1, W_2 and W_3 for $g = 3$ have hyperelliptic and holomorphic genus-3 Lefschetz fibrations.

(2) $S^2 \times T^2 \sharp 8\overline{\mathbb{C}P^2}$

This has a non-hyperelliptic genus-3 Lefschetz fibration with positive relation $(\alpha_1 \cdot \alpha_2 \cdot \alpha_3 \cdot \alpha_4 \cdot \beta_1^2 \cdot \beta_2^2)^2$ indicated below :

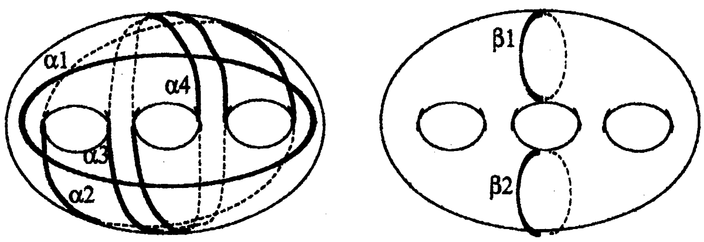

Figure 4.

This fibration also has a section of square -1.

(3) Fuller's example

Fuller constructed a non-hyperelliptic and non-holomorphic genus-3 Lefschetz fibration with positive relation $(\beta_1 \cdot \beta_2 \cdot \zeta_4 \cdot \zeta_3 \cdot \zeta_2 \cdot \zeta_1 \cdot \zeta_5 \cdot \zeta_4 \cdot \zeta_3 \cdot \zeta_2 \cdot \zeta_6 \cdot \zeta_5 \cdot \zeta_4 \cdot \zeta_3 \cdot (\zeta_1 \cdots \zeta_6)^{10}) = 1$, where β_1 and β_2 are given by positive Dehn twists along curves indecated in Figure 4. This fibration also has a section of square -1.

(4) $T^4 \sharp 4\overline{\mathbb{C}P^2}$

Smith [24] showed that $T^4 \sharp 4\overline{\mathbb{C}P^2}$ has a non-hyperelliptic and holomorphic genus-3 Lefschetz fibration with 12 irreducible singular fibers and 4 sections of square -1.

This fibration is obtained by using the inverse of the usual Kummer construction of a K3 surface which is elliptically fibered over S^2 with 16 disjoint (-2)-spheres containing 4 sections and 12 singular fibers.

4. Indecomposability of Lefschetz fibrations as fiber sum

As for spheres of square -1 in the total space of a Lefschetz fibration, the problems about the minimality or the fiber sum decomposability have been treated. Given two genus-g Lefschetz fibrations $f_i : X_i \to S^2$ $(i = 1, 2)$, we can construct a new genus-g Lefschetz fibration as follows.

Removing regular neighborhoods of generic fibers in each, we glue the boundaries of these remainders by using a fiber-preserving diffeomorphism of Σ_g which can be extended to f_i $(i = 1, 2)$. Thus we obtain a genus-g Lefschetz fibration, which is denoted by $X_1 \sharp_f X_2 \to B_1 \sharp B_2$ and is called the *fiber sum* of f_1 and f_2. Which Lefschetz fibration can be decomposed as a nontrivial fiber sum ? Stipsicz gave an answer for this problem.

Theorem 4.1 ([27], [21]). *If a Lefschetz fibration $f : X \to S^2$ admits a section of square -1, then X cannot be decomposed as any nontrivial fiber sum $X = X_1 \sharp_f X_2$.*

Stipsicz also proved the following theorem on the minimality of fiber sum.

Theorem 4.2 ([26]). *For a Lefschetz fibration $f : X \to S^2$, the fiber sum $X \sharp_f X$ is minimal.*

Here the minimality implies that a manifold is minimal in the DIFF category, that is, it cannot contain any smooth sphere of square -1. These theorems naturally lead us to the following conjecture :

Conjecture 4.1 (Stipsicz). *For nontrivial Lefschetz fibrations $f_i : X_i \to S^2$ $(i = 1, 2)$, the fiber sum $X_1 \sharp_f X_2$ is minimal.*

In the term of the mapping class group Γ_g, this conjecture implies that if the word w in Γ_g describing a positive relation is given by a positive Dehn twist along a closed curve parallel to a boundary component of $\Sigma_{g,k}$ with boundary, then the word w cannot be written as a nontrivial product $w_1 w_2$ where the word w_i describes a positive relation in Γ_g. As far as the author can know, this conjecture is an open problem.

5. 2-spheres of square -1 in Lefschetz fibrations

5.1. pseudo-holomorphic curves in symplectic 4-manifolds and the Gromov invariant. Since we suppose that the fiber genus g of a Lefschetz fibration $X \to S^2$ is greater than 1, then X has a symplectic structure. When X has a smooth sphere of square -1, we would like to make use of it to draw information in the horizontal direction. Since the one except for the differentiable structure is not supposed on a embedded sphere at present, we know the intersection of it with a generic fiber only algebraically. However, due to the development of 4-dimensional topology in recent years we found that a smooth sphere of square -1 can be exchanged for a pseudo-holomorphic rational curve of square -1.

We recall some definitions and some results about pseudo-holomorphic curves in symplectic 4-manifolds. Let (M, ω) be a connected, closed symplectic 4-manifold with J an ω-compatible almost complex structure. A smooth map $\varphi : \Sigma \to M$ from a possibly disconnected compact Riemann surface (Σ, j) to (M, J) is said to be *J-holomorphic* if the differential $d\varphi$ satisfies

$$d\varphi \circ j = J \circ d\varphi.$$

We call the image $\varphi(\Sigma)$ a *J-holomorphic curve* or *pseudo-holomorphic curve.* Then, McDuff [14] showed that pseudo-holomorphic curves have locally positive intersections. Hence, the number of algebraic intersections of pseudo-holomorphic curves give information about geometric intersections. This is one of the advantages which are brought by using pseudo-holomorphic curves.

In [28], Taubes defined the *Gromov invariant* Gr_T by counting with signs the number of pseudo-holomorphic curves in a given homology class and showed that Gromov invariants can be calculated in terms of Seiberg-Witten invariants. Given a cohomology class $\alpha \in H^2(M;\mathbb{Z})$, we define $d(\alpha)$ by

$$d(\alpha) = -K_M \cdot \alpha + \alpha \cdot \alpha,$$

where K_M denotes the symplectic canonical class of (M, ω) and $\cdot$ is the cup product pairing. If $d(\alpha) > 0$, let $\Omega \subset M$ be a set of $d(\alpha)/2$ distinct points. If $d(\alpha) \leq 0$, we set $\Omega = \emptyset$. Then, we consider the space $\mathcal{H}(\alpha, J)$ of J-holomorphic curves representing $PD(\alpha)$ and going through Ω. Here we denote $PD(\alpha)$ the Poincaré dual of α. Then, $\mathcal{H}(\alpha, J)$ is an oriented 0-manifold for a generic choice of J and Ω. Taubes proved the regularity theorem for $\mathcal{H}(\alpha, J)$ and introduced the Gromov invariant Gr_T as follows: For a nonzero class α, $\mathrm{Gr}_T(\alpha)$ is defined to be the algebraic number of pseudo-holomorphic curves in $\mathcal{H}(\alpha, J)$, that is,

$$\mathrm{Gr}_T(\alpha) = \sum_{C \in \mathcal{H}(\alpha, J)} \varepsilon(C),$$

where $\varepsilon(C) = \pm 1$. We set $\mathrm{Gr}_T(0) = 1$. The Gromov invariants Gr_T are independent of a generic choice of J and Ω. Furthermore, he obtained the following structure theorem.

Theorem 5.1 (Taubes' structure theorem [28]). *Let (M, ω) be a closed symplectic 4-manifold with $b_2^+(M) > 1$. Then the followings hold :*

(1) $\mathrm{Gr}_T(K_M) = \pm 1$. *In particular, the homology class $PD(K_M)$ has a (possibely disconnected) smooth pseudo-holomorphic representative. Hence, if M is minimal, then $c_1^2(M) = K_M^2 \geq 0$.*

(2) *(The duality formula) For any cohomology class $\alpha \in H^2(M;\mathbb{Z})$,* $\mathrm{Gr}_T(\alpha) = \pm \mathrm{Gr}_T(K_M - \alpha)$.

(3) *If* $\mathrm{Gr}_T(\alpha) \neq 0$, *then* α *satisfies* $d(\alpha) = 0$ *and can be represented by a pseudo-holomorphic curve* C *such that each component* C_i *of* C *is of genus* $g(C_i) = 1 + [C_i]^2$.

5.2. **2-spheres of square -1 in Lefschetz fibrations.** Now we can state the following theorem on smoothly embedded spheres in a symplectic 4-manifold with self-intersection number -1.

Theorem 5.2 ((−1)-curve theorem, Taubes [28], Li-Liu [9]). *Let* (M, ω) *be a closed symplectic 4-manifold. Suppose that* M *is neither the blow-up of a rational surface nor the blow-up of a ruled surface. Then, any smoothly embedded sphere of square* -1 *is* $\mathbb{Z}$*-homologous to a pseudo-holomorphic rational curve of square* -1 *after the appropriate choice of an orientation of the sphere.*

Taubes showed this theorem for $b_2^+(M) > 1$, and Li and Liu showed this theorem for $b_2^+(M) = 1$.

Next we consider spheres of square -1 in Lefschetz fibrations. Let $f : X \to S^2$ be a genus-g Lefschetz fibration. Since we suppose that $g \geq 2$, X has a symplectic structure ω with an ω-compatible almost complex structure J for which the fibers are pseudo-holomorphic (Theorem 2.3). Let $E \in H^2(X; \mathbb{Z})$ be the Poincaré dual of the homology class which is represented by a smoothly embedded sphere of square -1 in X. By changing the orientation of this sphere if necessary, we may assume that $E \cdot [\omega] > 0$. We denote by $\mathcal{E}_X$ the set of all the Poincaré dual of the homology classes E which can be represented by smoothly embedded spheres of square -1 and satisfy $E \cdot [\omega] > 0$. Moreover, let F denote the Poincaré dual of the homology class represented by a generic fiber. Then, we have the following theorem:

Theorem 5.3. *Suppose that* X *is neither the blow-up of a rational surface nor the blow-up of a ruled surface. Moreover, we suppose that* $\mathcal{E}_X$ *is not empty, and set* $\mathcal{E}_X = \{E_1, E_2, \ldots, E_m\}$. *Then, we have*

$$m \leq \left(\sum_{i=1}^{m} E_i \right) \cdot F \leq 2g - 2.$$

We need the following lemma to prove this theorem.

Lemma 5.1. *If* X *is neither the blow-up of a rational surface nor the blow-up of a ruled surface, then* $E \cdot F \geq 1$ *for any* $E \in \mathcal{E}_X$.

Proof. There exists an ω-compatible almost complex structure J on X such that fibers are J-holomorphic curves. By the (-1)-curve theorem, E can be represented by a J-holomorphic (-1)-curve C.

Suppose that $E \cdot F \leq 0$. Since pseudo-holomorphic curves have locally positive intersections [14], C must be contained in a certain singular fiber. However, this contradicts that $f : X \to S^2$ is relatively minimal. Therefore, $E \cdot F \geq 1$. □

Outline of the proof of Theorem 5.3. Equip X with an almost complex structure J such that fibers are J-holomorphic curves. Then there are symplectic representatives $C_1, \ldots, C_m$ of $E_1, \ldots, E_m$. Since $E_i \cdot F \geq 1$, we have $m \leq (\sum_{i=1}^m E_i) \cdot F$ (Lemma 5.1).

Case of $b_2^+(X) > 1$: There is a minimal symplectic 4-manifld Y of X and we let $\pi : X \to Y$ be the symplectic blow-down map. Since X is obtained from Y by blowing up with symplectic exceptional curves $C_1, \ldots, C_m$ representing $E_1, \ldots, E_m$, we have

$$K_X = \pi^* K_Y + \sum_{i=1}^m E_i.$$

By the adjunction formula, we have

$$2g - 2 = \pi^* K_Y \cdot F + \left(\sum_{i=1}^m E_i \right) \cdot F.$$

On the other hand, it follows from the blow-up formula of the Gromov invariant that there is a pseudo-holomorphic curve S representing $\pi^* K_Y$. If we let S_j be any irreducible component of S, then $S_j \cdot F \geq 0$ because of the positivity of intersections of pseudo-holomorphic curves and $F^2 = 0$, and so $\pi^* K_Y \cdot F \geq 0$. Hence, $2g - 2 \geq (\sum_{i=1}^m E_i) \cdot F$.

Case of $b_2^+(X) = 1$: Since X is not the blow-up of a ruled surface, there is a symplectic minimal model of X. In this case the author does not know whether K_X can have a pseudo-holomorphic representative or not, that is to say, whether $\mathrm{Gr}_T(K_X) \neq 0$ or not. Since X is neither rational nor ruled, it follows from a result of Li and Liu [10] that $\mathrm{Gr}_T(nK_Y) \neq 0$ for $n \geq 2$. By the blow-up formula of Gromov invariants, we have $\mathrm{Gr}_T(n\pi^* K_Y) \neq 0$ for $n \geq 2$. Hence, $n\pi^* K_Y$ can be represented by a pseudo-holomorphic curve, and so we can show that $(n\pi^* K_Y) \cdot F \geq 0$. Thus $\pi^* K_Y \cdot F \geq 0$. Therefore, we can prove the required inequality in the same manner as the case of $b_2^+(X) > 1$. □

6. The geography of non-minimal Lefschetz fibrations

6.1. **The geography of Lefschetz fibrations.** The geography problem for complex surfaces is the characterization of pairs of integers which are realized as $(c_1^2,\ c_2)$ of complex surfaces. We would like to consider the geography problem for symplectic 4-manifolds, in particular Lefschetz fibrations. By the Hirzebruch's signature theorem, we have

$$K_X^2 = 3\sigma(X) + 2e(X).$$

Hence, the pair $(c_1^2,\ c_2)$ of Chern numbers is equivalent to the pair $(\sigma,\ e)$. Matsumoto-Endo's signature formula implies that the signature of the total space of a hyperelliptic Lefschetz fibration is calculated from the number of singular fibers. The Euler characteristic of the total space of a Lefschetz fibration is also calculated from the number of singular fibers. Hence, we regard *the geography of Lefschetz fibrations* as characterizing the pair of the numbers of singular fibers $(n_0,\ s_1,\ \dots,\ s_{[g/2]})$.

6.2. **The case** $g = 2$. : We consider a genus-2 Lefschetz fibration $f : X \to S^2$ with spheres of square -1. If X is neither rational nor ruled, then Theorem 5.3 states that $\mathcal{E}_X$ is one of the following three :

Type (1, 1) : $\mathcal{E}_X = \{E_1, E_2\}$, $E_1 \cdot F = E_2 \cdot F = 1$.
Type (1) : $\mathcal{E}_X = \{E\}$, $E \cdot F = 1$.
Type (2) : $\mathcal{E}_X = \{E\}$, $E \cdot F = 2$.

In the first and the second cases, spheres representing $\mathcal{E}_X$ are sections of $f : X \to S^2$. Note that $E_1 \cdot E_2 = 0$ for E_1 and E_2 in Type (1, 1), which follows from the proof of Corollary 3 in [8].

Theorem 6.1. *Only finitely many pairs* (c_1^2, c_2) *can be realized as genus-2 Lefschetz fibrations* $X \to S^2$ *with 2-spheres of square* -1. *Here,* $c_1^2 = c_1^2(X)([X])$ *and* $c_2 = c_2(X)([X])$.

Outline of the proof of Theorem 6.1. We suppose that $f : X \to S^2$ has n_0 irreducible singular fibers and s reducible singular fibers. By the Hirzebruch signature theorem, the Matsumoto's local signature formula [12] (Theorem 2.4) and Proposition 2.1, we have

$$\begin{cases} K_X^2 = 3\sigma(X) + 2e(X), \\ \sigma(X) = -\dfrac{3}{5}n_0 - \dfrac{1}{5}s, \text{ and} \\ e(X) = n_0 + s - 4, \\ n_0 + 2s \equiv 0 \pmod{10}. \end{cases}$$

We suppose that $b_2^+(X) > 1$ and $\mathcal{E}_X$ is of type (2). Set $A = K_X - E$. By the adjunction formula, we have $K_X \cdot F = 2$, $K_X \cdot E = -1$ and so $A \cdot F = A \cdot E = 0$. Since $\mathrm{Gr}_T(A) \neq 0$, A can be represented by a pseudo-holomorphic curve $C = \bigcup_{i=1}^m C_i$. Noting that $A \cdot F = 0$, each component C_i of C is contained in a fiber, and so $[C_i]^2 = 0$ or -1. Because of the relative minimality of f, fibers contain no sphere-component and each component C_i is not a sphere. Since $E \cdot F = 2$ and $A \cdot E = 0$, each component C_i is neither a generic fiber nor an irreducible singular fiber, and so it is a component of a reducible singular fiber. Hence C consists of components $C_1, C_2, \dots, C_m$ with $[C_i]^2 = -1$, $[C_i] \cdot [C_j] = 0$ $(i \neq j)$ and $genus(C_i) = 1$ $(i, j = 1, 2, \dots, m)$. Then, by the adjunction formula, we have $K_X \cdot [C_i] = 1$ $(i = 1, 2, \dots, m)$, and so

$$\begin{aligned} 2A^2 &= A^2 + A^2 = K_X \cdot A + A^2 \\ &= K_X \cdot \left(\sum_{i=1}^m [C_i]\right) + \left(\sum_{i=1}^m [C_i]\right) \cdot \left(\sum_{i=1}^m [C_i]\right) = \sum_{i=1}^m K_X \cdot [C_i] + \sum_{i=1}^m [C_i] \cdot [C_i] \\ &= m - m = 0. \end{aligned}$$

Hence, we get $K_X^2 = A^2 + 2A \cdot E + E^2 = -1$.

Therefore, the pair (n_0, s) satisfies

$$\begin{cases} n_0 + 7s = 35, \\ n_0 + 2s \equiv 0 \quad (\text{mod } 10). \end{cases}$$

Since f is not trivial [26], the required pairs (n_0, s) are $(14, 3)$ and $(28, 1)$.

We can also show the other cases in the same manner. Thus only finitely many pairs (n_0, s) arise, and equivalently only finitely many pairs (c_1^2, c_2) do so.

TABLE 1. Possible pairs (n_0, s) as geography

b_2^+	Possible pairs (n, s)	$\mathcal{E}_X$
$b_2^+ > 1$	$(16, 2)$, $(30, 0)$	Type(1, 1)
	$(14, 3)$, $(28, 1)$ $(12, 4)$, $(26, 2)$, $(40, 0)$	Type (1)
	$(14, 3)$, $(28, 1)$	Type (2)
$b_2^+ = 1$	$n_0 + 2s = 20$, $n_0 > 0$, $s \geq 0$	
	$n_0 + 2s = 10$, $n_0 > 0$, $s \geq 0$	

Remark 6.1. The pairs $(n_0, s) = (20, 0)$, $(30, 0)$ and $(40, 0)$ are realized as $\mathbb{C}P^2 \sharp 13\overline{\mathbb{C}P^2}$, $K3\sharp 2\overline{\mathbb{C}P^2}$ and the Horikawa surface $H'(1)$, respectively. Moreover, the pair $(n_0, s) = (6, 2)$ is realized as $S^2 \times T^2 \sharp 4\overline{\mathbb{C}P^2}$.

We can prove the following corollary from Table 1.

Corollary 6.1. *We have the followings :*

(1) *The fiber sum $X \sharp_f X$ of a genus-2 Lefschetz fibration $X \to S^2$ is minimal.*

(2) *The fiber sum $(\mathbb{C}P^2 \sharp 13\overline{\mathbb{C}P^2}) \sharp_f (\mathbb{C}P^2 \sharp 13\overline{\mathbb{C}P^2})$ has a topological 2-sphere of square -1 but cannot contain any smooth 2-sphere of square -1. In particular, this fiber sum is homeomorphic to the Horikawa surface $H'(1)$ but not diffeomorphic.*

(3) *If a holomorphic genus-2 Lefschetz fibration $f : X \to \mathbb{C}P^1$ without reducible singular fibers is not minimal, then f is isomorphic to $\mathbb{C}P^2 \sharp 13\overline{\mathbb{C}P^2}$, $K3\sharp 2\overline{\mathbb{C}P^2}$ or $H'(1)$.*

6.3. **The case $g = 3$.** : We consider a genus-3 Lefschetz fibration $f : X \to S^2$ with spheres of square -1. If X is neither rational and ruled, Theorem 5.3 states that the set EX of spheres of square -1 consists of 11 types containing the followings:

Type (1, 1, 1, 1) : $\mathcal{E}_X = \{E_1, \ldots, E_4\}$, $E_1 \cdot F = \cdots = E_4 \cdot F = 1$
Type (2, 1, 1) : $\mathcal{E}_X = \{E, E_1, E_2\}$, $E \cdot F = 2$, $E_1 \cdot F = E_2 \cdot F = 1$
Type (3, 1) : $\mathcal{E}_X = \{E, E_1\}$, $E \cdot F = 3$, $E_1 \cdot F = 1$
Type (2, 2) : $\mathcal{E}_X = \{E_1, E_2\}$, $E_1 \cdot F = E_2 \cdot F = 2$

Type (4) $\quad : \mathcal{E}_X = \{E\},\ E \cdot F = 4$
etc.

We can prove the following lemma in the same manner as the case of $g = 2$.

Lemma 6.1. *If $\left(\sum_{E\in\mathcal{E}_X} E\right) \cdot F = 4$, then $K_X^2 \leq s - k$ for some $k \in \{1, 2, 3, 4\}$. Moreover, in case of type (1, 1, 1), $K_X^2 \leq s - k$ for some $k \in \{2, 3, 4\}$.*

Hyperelliptic case: Let $f : X \to S^2$ be a non-minimal hyperelliptic genus-3 Lefschetz fibration with n_0 irreducible singular fibers and s reducible singular fibers. Then, by Endo's signature formula (Theorem 2.4) we have $\sigma(X) = -4n_0/7 + s/7$, and so the pair $(c_1^2,\ c_2)$ of Chern numbers is equivalent to the pair $(n_0,\ s)$ of singular fibers for hyperelliptic fibrations. Thus it follows from the Hirzebruch's signature theorem and Proposition 2.2 that

$$\begin{cases} \dfrac{2}{7}n_0 + \dfrac{17}{7}s = K_X^2 + 16, \\ \\ n_0 + 12s \equiv 0 \pmod{28} \end{cases}$$

Hence, from Lemma 6.1, we have

Theorem 6.2. *Only finitely many $(n_0,\ s)$ can be realized as hyperelliptic genus-3 Lefschetz fibrations with $\left(\sum_{E\in\mathcal{E}_X} E\right) \cdot F = 4$. Hence, only finitely many $(c_1^2,\ c_2)$ can be realized as such hyperelliptic genus-3 Lefschetz fibrations.*

Non-hyperelliptic case: Let $f : X \to S^2$ be a non-minimal genus-3 Lefschetz fibration with n_0 irreducible singular fibers and s reducible singular fibers. Then, by Smith's signature formula we have $\sigma(X) = 4\langle c_1(\lambda),\ [\phi_f(S^2)]\rangle - (n_0 + s)$.

Let Δ_0 and Δ_1 be the divisor of irreducible and reducible nodal curves, respectively. Then the Deligne-Mumford moduli space $\overline{\mathcal{M}_3}$ of stable curves of genus 3 is given by adjoining Δ_0, Δ_1 to $\mathcal{M}_3$. Let $\overline{\mathcal{H}_3}$ denote the divisor of hyperelliptic curves of genus 3 in $\overline{\mathcal{M}_3}$. A theorem of Harer states that the Hodge class $c_1(\lambda)$ and $[\Delta_0], [\Delta_1]$ generate $H^2(\overline{\mathcal{M}_3}; \mathbb{Z})$, and the cohomology class $[\overline{\mathcal{H}_3}]$ is given, up to a positive rational multiple, by

$$[\overline{\mathcal{H}_3}] = 9c_1(\lambda) - [\Delta_0] - 3[\Delta_1].$$

Suppose that $f : X \to \mathbb{C}P^1$ is non-hyperelliptic and holomorphic. Since a holomorphic fibration f gives rise to a rational curve $[\phi_f(\mathbb{C}P^1)]$ in $\overline{\mathcal{M}_3}$ and has positive intersection with all effective divisors in which they are not contained. Hence, we have

$$\langle [\overline{\mathcal{H}_3}],\ [\phi_f(\mathbb{C}P^1)]\rangle \geq 0.$$

Since $\langle [\overline{\mathcal{H}_3}],\ [\phi_f(\mathbb{C}P^1)] \rangle$ is given, up to a positive rational multiple, by

$$\begin{aligned}\langle [\overline{\mathcal{H}_3}],\ [\phi_f(\mathbb{C}P^1)] \rangle &= \langle 9c_1(\lambda) - [\Delta_0] - 3[\Delta_1],\ [\phi_f(\mathbb{C}P^1)] \rangle \\ &= \frac{9}{4}(\sigma(X) + n_0 + s) - n_0 - 3s \\ &= \frac{9}{4}\sigma(X) + \frac{5}{4}n_0 - \frac{3}{4}s,\end{aligned}$$

we can obtain the following inequality :

$$\sigma(X) \geq -\frac{5}{9}n_0 + \frac{1}{3}s.$$

Thus we get the relations

$$\begin{cases} K_X^2 = 3\sigma(X) + 2e(X), \\ K_X^2 \leq s - k, \\ \sigma(X) \geq -\frac{5}{9}n_0 - \frac{1}{3}s, \\ e(X) = n_0 + s - 8, \\ 5n_0 \geq s \end{cases} \quad \text{hence,} \quad \begin{cases} n_0 \leq -6s + 3(16-k), \\ n_0 \geq \frac{1}{5}s, \\ n_0 > 0, \quad s \geq 0 \end{cases}$$

for some $k \in \{1, 2, 3, 4\}$. From these inequalities, we can estimate the numbers of singular fibers.

Theorem 6.3. *Only finitely many $(n_0,\ s)$ can be realized as pairs of the numbers of singular fibers of non-hyperelliptic and holomorphic genus-3 Lefschetz fibrations with $\left(\sum_{E \in \mathcal{E}_X} E\right) \cdot F = 4$.*

For example, $T^4 \sharp 4\overline{\mathbb{C}P^2}$ has a non-hyperelliptic and holomorphic genus-3 Lefschetz fibration with 12 irreducible singular fibers and 4 sections of square -1. See §3.

7. Concluding remark

We close this note by some remarks about Smith's signature formula for Lefschetz fibrations.

The Hodge class $c_1(\lambda)$ is ample on the moduli space $\mathcal{M}_g$ but not on the Deligne-Mumford compactified moduli space $\overline{\mathcal{M}_g}$. The moduli space $\mathcal{M}_2$ of genus 2 is affine [16], and so an ample divisor on $\mathcal{M}_2$ is empty. Hence, $c_1(\lambda)$ is a combination of $[\Delta_0]$ and $[\Delta_1]$. In fact, Mumford [16] showed that

$$10c_1(\lambda) = [\Delta_0] + 2[\Delta_1].$$

Therefore, it follows from Smith's formula that the signature of a genus-2 Lefschetz fibration $X \to S^2$ with n_0 irreducible singular fibers and s reducible singular fibers is given by

$$\sigma(X) = -\frac{3}{5}n_0 - \frac{1}{5}s.$$

This is the formula proved by Matsumoto [12]. Hence, the following problem comes to mind :

Problem. Prove Endo's signature formula for hyperelliptic genus-$g(\geq 3)$ Lefschetz fibrations from Smith's signature formula.

For Lefschetz fibrations, we will show the slope inequality due to Konno [6] from Smith's formula. We recall the slope λ_f of a holomorphic fibration $f : X \to B$. Relative numerical invariants K_f^2, χ_f are defined by

$$K_f^2 = c_1(X)^2 - 8\chi(\mathcal{O}_F)\chi(\mathcal{O}_B),$$
$$\chi_f = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_F)\chi(\mathcal{O}_B).$$

Then the *slope* of f is defined as the following ratio:

$$\lambda_f = \frac{K_f^2}{\chi_f}.$$

In particular, the slope λ_f of a genus-g fibration $f : X \to \mathbb{C}P^1$ is given topologically as follows:

$$\lambda_f = \frac{K_f^2}{\chi_f} = \frac{4(3\sigma(X) + 2e(X) + 8g - 8)}{\sigma(X) + e(X) + 4g - 4}.$$

Konno obtained the following inequality:

Theorem 7.1. (Konno [6]). *For a non-hyperelliptic genus-3 holomorphic fibration $f : X \to B$, we have the inequality $\lambda_f \geq 3$.*

We can prove this inequality for non-hyperelliptic, holomorphic genus-3 Lefschetz fibrations over $\mathbb{C}P^1$ from Smith's formula. Let $f : X \to \mathbb{C}P^1$ be such a Lefschetz fibration with n_0 irreducible singular fibers and s reducible singular fibers. Since f is holomorphic, we have $\langle [\overline{\mathcal{H}_3}],\ [\phi_f(\mathbb{C}P^1)]\rangle \geq 0$. On the other hand, we have

$$\begin{aligned}
\langle [\overline{\mathcal{H}_3}],\ [\phi_f(\mathbb{C}P^1)]\rangle &= \langle 9c_1(\lambda) - [\Delta_0] - 3[\Delta_1],\ [\phi_f(\mathbb{C}P^1)]\rangle \\
&= \frac{9}{4}\langle 4c_1(\lambda),\ [\phi_f(\mathbb{C}P^1)]\rangle - \langle [\Delta_0],\ [\phi_f(\mathbb{C}P^1)]\rangle \\
&\qquad\qquad - 3\langle [\Delta_1],\ [\phi_f(\mathbb{C}P^1)]\rangle \\
&= \frac{9}{4}(\sigma(X) + n_0 + s) - n_0 - 3s \\
&= \frac{9}{4}\sigma(X) + \frac{5}{4}n_0 - \frac{3}{4}s \\
&= \frac{9\sigma(X) + 5(n_0 + s) - 8s}{4} \\
&= \frac{9\sigma(X) + 5(e(X) + 8) - 8s}{4} \\
&= \frac{9\sigma(X) + 5e(X) - 8s + 40}{4}.
\end{aligned}$$

Hence, we obtain $9\sigma(X) + 5e(X) + 40 \geq 9\sigma(X) + 5e(X) - 8s + 40 \geq 0$. Moreover, because of $e(X) + 8 = n_0 + s > 0$, we have $9(\sigma(X) + e(X) + 8) = \{9(\sigma(X) + e(X) +$

$8)-4(e(X)+8)\}+4(e(X)+8) = (9\sigma(X)+5e(X)+40)+4(e(X)+8) > 0$. Therefore, it follows from these inequalities that we have

$$\lambda_f - 3 = \frac{9\sigma(X)+5e(X)+40}{\sigma(X)+e(X)+8} \geq 0.$$

Proposition 7.1 (Smith [22]). *Let $f : X \to S^2$ be a genus-3 Lefschetz fibration with only irreducible singular fibers. Suppose that*

(1) $e(X)+1 \not\equiv 0 \pmod 7$ *and*
(2) $9\sigma(X)+5e(X)+40 < 0$.

Then f is not isomorphic to a holomorphic fibration.

The assumption (1) is equivalent to the fact that the number of irreducible singular fibers is not divisible by 7, and so it follows from Endo's signature formula that f is not hyperelliptic. The assumption (2) is equivalent to $\langle[\overline{\mathcal{H}_3}],\ [\phi_f(\mathbb{C}P^1)]\rangle < 0$, or $\lambda_f < 3$. Therefore, f is not isomorphic to a holomorphic fibration. For example, since the genus-3 Lefschetz fibration given by Fuller satisfies the assumptions (1) and (2), this fibration is not isomorphic to a holomorphic fibration.

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