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Kyoto University
ADIABATIC LIMITS OF $\eta$-INVARIANTS AND THE MEYER FUNCTION FOR SMOOTH THETA DIVISORS

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1. Introduction

Let $\Sigma_g$ be a closed oriented surface of genus $g$ and let $M_g$ be the mapping class group of genus $g$, namely the group of all isotopy classes of orientation-preserving diffeomorphisms of $\Sigma_g$. Meyer introduced a cocycle $\tau_g : M_g \times M_g \rightarrow \mathbb{Z}$, called the signature cocycle or the Meyer cocycle, and he gave a signature formula for the signature of surface bundles over surfaces ([21]). Let $[\tau_g] \in H^2(\mathcal{M}_g, \mathbb{Z})$ denotes the cohomology class of $\tau_g$. When $g = 1$, since $M_1 = SL_2(\mathbb{Z})$, $H^1(SL_2(\mathbb{Z}), \mathbb{Z}) = 0$ and $3[\tau_1] = 0$, there exists a unique 1-cocycle $\phi_1 : SL_2\mathbb{Z} \rightarrow \frac{1}{3}\mathbb{Z}$ such that cobounds $\tau_1$. The function $\phi_1$ is called the Meyer function of genus one, which has the following property: Let $\pi : Z \rightarrow X$ be a $\Sigma_1$-bundle over a compact oriented surface with boundary $\partial Z = c_1 \cup \cdots \cup c_k$. Let $A_1, \cdots, A_k$ be the monodromies around each component of the boundary. Since the Picard-Lefschetz transformation along $c_i$ is an automorphism of $H^1(\Sigma_1, \mathbb{Z})$ preserving the intersection form, one has $A_i \in SL_2(\mathbb{Z})$ by fixing a symplectic basis of $H^1(\Sigma_1, \mathbb{Z})$. Then the signature of $Z$, which is defined as the signature of the cup-product pairing on $H^2(Z, \partial Z, \mathbb{R})$, satisfies

$$\text{Sign}(Z) = \sum_{i=1}^{k} \phi_1(A_i).$$

The explicite formula of $\phi_1$ was obtained by Meyer ([21]).

When $g = 2$, since $5[\tau_2] = 0 \in H^2(M_2, \mathbb{Z}) \cong \mathbb{Z}/10\mathbb{Z}$ and $H^1(M_2, \mathbb{Z}) = 0$, there exists a unique 1-cocycle $\phi_2 : M_2 \rightarrow \frac{1}{3}\mathbb{Z}$ satisfying (1), for every $\Sigma_2$-bundles over compact oriented surfaces. The function $\phi_2$ is called the Meyer function of genus two.

In [1], Atiyah investigated the Meyer function $\phi_1$ from the several view points. For an odd dimensional closed oriented Riemannian manifold $M$, let $\eta(M)$ be the $\eta$-invariant of $M$ with respect to the signature operator of $M$ [2]. For $\sigma \in SL_2\mathbb{Z}$, let $\pi : M_2 \rightarrow S^1$ be the mapping
torus associated with \( \sigma \), i.e., \( \Sigma_1 \)-bundle over \( S^1 \) with monodromy \( \sigma \). Then Atiyah showed the following identity, when \( M_\sigma \) is equipped with a certain metric:

\[
\phi_1(\sigma) = \eta(M_\sigma)
\]

Moreover, he gave several interpretation of \( \phi_1 \) interms of the following quantities: (1)Hirzebruch’s signature defect; (2)the transformation lows of the logarithm of the Dedekind \( \eta \)-function; (3)the logarithm of the monodromy of Quillen’s line bundle; (4)the special value of the Shimizu \( L \)-function at the origin.

In this note, we study an extension of the result of Atiyah to the case \( g = 2 \) and higher dimensional manifold. We shall construct a higher dimensional analogue of the Meyer function for smooth theta divisors of odd dimension.

**Notation**: For a complex manifold \( M \), \( T^{1,0}M \) (resp. \( T^{0,1}M \)) denotes the holomorphic (resp. anti-holomorphic) tangent bundle and \( TM \) denotes the real tangent bundle. We set \( d^c := \frac{1}{4\pi\sqrt{-1}}(\partial - \bar{\partial}) \). Hence \( dd^c = \frac{1}{2\pi} d\bar{\partial} \).

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2. Preliminaries from Riemannian geometry

In this section, we recall some results of Riemannian geometry which will be used in the proof of the main theorem. Following [10], we define connections of fiber bundles and the connection of relative tangent bundles. Let \( M \) be a manifold and let \( \pi : Z \to B \) be a fiber bundle with typical fiber \( M \).

The relative tangent bundle \( T(Z/B) \) is the subbundle of \( TZ \) defined by

\[
T(Z/B) := \text{Ker}\{\pi_* : TZ \to \pi^*TB\}.
\]

A vector of \( T(Z/B) \) is said to be **vertical**.

**Definition 2.1.** A subbundle \( T_SZ \subset TZ \) with \( TZ = T(Z/B) \oplus T_SZ \) is called a **connection** of the fiber bundle \( \pi : Z \to B \).

For a connection, one has \( T_SZ \cong \pi^*TB \) via the projection \( \pi_* : TZ \to \pi^*TB \). A vector of \( T_SZ \) is said to be **horizontal**.

When \( Z \) is trivial, i.e., \( Z = M \times B \), \( TZ \) is naturally isomorphic to the direct sum \( \pi^*TM \oplus (\pi^*TM)^\sigma \). This connection is called the **trivial connection** of the trivial fiber bundle.

Given a connection, one can define the projection \( P_2 : TZ \to T(Z/B) \) with kernel \( T_HZ \). We often identify \( P_2 \) with the corresponding connection \( T_HZ := \text{Ker}(P_2) \). In the rest of Section 2, we fix a connection \( T_HZ \), or equivalently \( P_2 \). One can define the pull-back of a connection, as follows: Let \( B' \) be a manifold and let \( h : B' \to B \) be a \( C^\infty \) map. The fiber product \( Z' := \pi'^{-1} B' \subset \pi^{-1} B \) satisfies the following commutative diagram:

\[
\begin{array}{ccc}
Z' & \xrightarrow{\hat{h}} & Z \\
\pi' \downarrow & & \downarrow \pi \\
B' & \xrightarrow{h} & B
\end{array}
\]

Since the map \( P_2 \circ \hat{h}_* : TZ' \to h^*T(Z/B) \) is surjective, \( \text{Ker}(P_2 \circ \hat{h}_*) \) is a subbundle of \( TZ' \). Since \( T'(Z'/B') \) is canonically isomorphic to \( h^*T(Z/B) \), the map \( P_2 \circ \hat{h}_* \) is identified with a projection from \( TZ' \) to \( T(Z'/B') \).
Definition 2.2. The connection of $\pi': Z'\to B'$ induced from $T_H Z$ by $h$ is defined by

$$T_H Z' := \text{Ker}(P_Z \circ \tilde{h}_* : T(Z')\to T(Z/B)),$$

under the identification between $T(Z'/B')$ and $h^*T(Z/B)$. The projection corresponding to $T_H Z'$ is denoted by $h^*P_Z$.

We fix a metric $g^{Z/B}$ on the relative tangent bundle, a Riemannian metric $g^B$ on $B$, and the connection $T_H Z$ and the corresponding projection $P_Z$. We define the Riemannian metric $g^Z$ on the total space $Z$ by

$$g^Z := g^{Z/B} \oplus \pi^*g^B,$$

under the isomorphism $TZ \cong T(Z/B) \oplus T_{H}Z \cong T(Z/B) \oplus \pi^*TB$. Let $\nabla^Z$ be the Levi-Civita connection of $(Z,g^Z)$. We define the connection $\nabla^Z/B$ on $T(Z/B)$ by

$$\nabla^Z/B := P_Z \circ \nabla^Z.$$

Then $\nabla^Z/B$ preserves the metric $g^{Z/B}$.

Lemma 2.3. The connection $\nabla^Z/B$ is independent of a choice of $g^B$

Proof. See [10, Proposition 10.2] \hfill \Box

Lemma 2.4. Let $B'$ be a manifold and let $h : B'\to B$ be a $C^\infty$-map, and set $Z' := Z \times_B B'$. Let $g^{Z'/B'} = h^*g^{Z/B}$ be the metric on $T(Z'/B')$ induced from $g^{Z/B}$, and let $P_{Z'} = h^*P_Z$ be the connection of $Z'$ induced from $P_Z$. Then $\nabla^{Z'/B'} = h^*\nabla^{Z/B}$.

Proof. See [15] \hfill \Box

With respect to the decomposition $TZ = T(Z/B) \oplus T_{H}Z$, We put for $\epsilon \in \mathbb{R}^+$

$$g^{Z,\epsilon} := g^{Z/B} \oplus \pi^*g^B,$$

The Levi-Civita connections of $(Z,g^{Z,\epsilon})$ and $(B,g^B)$ are denoted by $\nabla^{Z,\epsilon}$ and $\nabla^B$, respectively. Let $R^{Z,\epsilon}$ and $R^B$ be the curvature of $\nabla^{Z,\epsilon}$ and $\nabla^B$, respectively. Then $g^Z := g^{Z,1}$ and $\nabla := \nabla^{Z,1}$. We define another connection $\nabla$ on $Z$ by

$$\nabla := \nabla^{Z/B} \oplus \pi^*\nabla^B,$$

and we put

$$S^{(\epsilon)} := \nabla^{Z,\epsilon} - \nabla \in \Lambda^1(\text{End}(TZ)), \quad S := S^{(1)}.$$

Then $\nabla$ preserves the Riemannian metric $g^{Z,\epsilon}$, and $P_Z$ is parallel with respect to $\nabla$, i.e. $\nabla \circ P_Z - P_Z \circ \nabla = 0$.

Let $\{e_1, \cdots, e_k\}$ be a local orthogonal framing for $(T(Z/B),g^{Z/B})$, and let $\{f_1, \cdots, f_l\}$ be a local orthogonal framing for $(T_{H}Z, \pi^*g^B)$.

Proposition 2.5. With respect to the splitting $TZ = T(Z/B) \oplus T_{H}B$, the following identity holds:

$$\lim_{\epsilon \to 0} R^{Z,\epsilon} = \begin{pmatrix} R^{Z/B} & P_Z(\nabla S) \\ 0 & \pi^*R^B \end{pmatrix}.$$

Proof. See [7] (3.195). \hfill \Box
3. \(\eta\)-invariants

In this section, we recall the definition and some properties of \(\eta\)-invariants. Let \((M, g^M)\) be a coed oriented Riemannian manifold of dimension \((2l - 1)\). Denote the space of \(C^\infty\) \(k\)-forms on \(M\) by \(\mathcal{A}^k(M)\). Let \(* : \mathcal{A}^k(M) \to \mathcal{A}^{2l-k-1}(M)\) be the Hodge star operation with respect to \(g^M\). The signature operator \(D : \oplus_{p\geq 0} \mathcal{A}^{2p}(M) \to \oplus_{p\geq 0} \mathcal{A}^{2p}(M)\) of \(M\) is defined by

\[
D : \omega \mapsto (\sqrt{-1})^{l-1}(-1)^{p+1}(d \ast d \ast)\omega, \quad \omega \in \mathcal{A}^{2p}(M).
\]

Then \(D\) is an elliptic self-adjoint differential operator of first order acting on \(\oplus_{p\geq 0} \mathcal{A}^{2p}(M)\). Let \(\sigma(D)\) be the spectrum of \(D\). The \(\eta\)-function of \(M\) is defined by

\[
\eta(s) := \sum_{\lambda \in \sigma(D) \setminus \{0\}} \frac{\text{sign}\lambda}{\lambda^s},
\]

for \(s \in \mathbb{C}\) with \(\text{Re}(s) \gg 0\). Then \(\eta(s)\) extends meromorphically to \(\mathbb{C}\) and is holomorphic at \(s = 0\) by [2], [7].

**Definition 3.1.** The real number \(\eta(0)\) is called the \(\eta\)-invariant of \((M, g^M)\) and is denoted by \(\eta(M, g^M)\).

Let \((X, g^X)\) be a \(4k\)-dimensional, oriented, compact, Riemannian manifold with boundary \(Y\). Put \(g^Y := g^X|_Y\), and fix a color neighborhood \(U \supset Y\) such that \(U \cong Y \times [0, 1]\). Assume that \(g^X|_U = g^Y \oplus dt^2\) under the above isomorphism. Let \(\nabla^L\) be the Levi-Civita connection of \((X, g^X)\).

**Theorem 3.2 (Atiyah-Patodi-Singer [2]).** The following equation holds:

\[
\text{Sign}(X) = \int_X L(TX, \nabla^L) - \eta(Y, g^Y)
\]

Here \(L\) denotes the Hirzebruch \(L\)-polynomial, which is a multiplicative genus associated with the power series: \(L(x) := x / \tanh(x)\).

Let \(X, B\) and \(M\) be closed oriented manifolds. Let \(\pi : X \to B\) be a \(C^\infty\)-submersion, whose fibers are isomorphic to \(M\). Assume that \(\dim M = 4k\). Let \(g^{X/B}\) be a metric on \(T(X/B)\) and \(g^B\) be a metric on \(TB\). Let \(T_H X \subset TX\) be a connection. We identify \(T_H X\) with \(\pi^* TB\) via \(\pi\). With respect to the decomposition \(TX = T(X/B) \oplus \pi^* TB\), we define the metric on \(X\) by \(g^X := g^{X/B} \oplus \pi^* g^B\) and we consider the one parameter family of metrics on \(X\) defined by

\[
g^{X,\epsilon} := g^{X/B} \oplus \epsilon^{-1} \pi^* g^B, \quad \epsilon \in \mathbb{R}^+.
\]

**Theorem 3.3 (Bismut-Cheeger, [6]).** The limit \(\lim_{\epsilon \to 0} \eta(X, g^{X,\epsilon})\) exists.

The limit \(\lim_{\epsilon \to 0} \eta(X, g^{X,\epsilon})\) is called the adiabatic limit of the \(\eta\)-invariants and is denoted by \(\eta^0(X)\). By definition, \(\eta^0(X, g^X)\) depends on the three data: \(g^{X/B}, g^B\) and \(T_H X\).

4. Family of smooth theta divisors

We fix the following notation. Let \(\mathfrak{S}_g\) be the Siegel upper-half space of degree \(g\) and let \(\Gamma_g\) be the integral symplectic group, i.e.,

\[
\mathfrak{S}_g := \{ \tau \in M(g, \mathbb{C}) \mid \tau^t = \tau, \ \text{Im} \tau > 0 \}
\]

\[
\Gamma_g := \{ \gamma \in GL(2g, \mathbb{Z}) \mid \gamma J_g \gamma^t = J_g \},
\]

where \(J_g = \left( \begin{smallmatrix} 0 & I_g \\ -I_g & 0 \end{smallmatrix} \right)\) and \(I_g\) denotes the \(g \times g\) identity matrix. \(\Gamma_g\) acts on \(\mathfrak{S}_g\) by

\[
\gamma \cdot \tau := (A \tau + B)(C \tau + D)^{-1}, \quad \gamma = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \in \Gamma_g, \ \tau \in \mathfrak{S}_g.
\]
For \( \tau \in \mathcal{S}_g \), write \( \tau = \tau_1, \cdots, \tau_g \) and set
\[
\Lambda_\tau := \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_g \oplus \mathbb{Z} \tau_1 \oplus \cdots \oplus \mathbb{Z} \tau_g \subset \mathbb{C}^g
\]
where \( 1_g = \tau_1, \cdots, \tau_g \) and \( \tau = \tau_1, \cdots, \tau_g \in \mathcal{S}_g \). Define the \( \mathbb{Z}^{2g} \)-action on \( \mathbb{C}^g \times \mathcal{S}_g \) by
\[
(m, n) \cdot (z, \tau) := (z + m\tau + n, \tau), \quad (z, \tau) \in \mathbb{C}^g \times \mathcal{S}_g, \; m, n \in \mathbb{Z}^{2g}.
\]
Then
\[
f : A_g := (\mathbb{C}^g \times \mathcal{S}_g)/\mathbb{Z}^{2g} \rightarrow \mathcal{S}_g
\]
is the universal family of principally polarized Abelian varieties over \( \mathcal{S}_g \), whose fiber at \( \tau \) is \( A_\tau := \mathbb{C}^g/\Lambda_\tau \). For \( (a, b) \in \mathbb{R}^{2g}, z \in \mathbb{C}^g \) and \( \tau \in \mathcal{S}_g \) we define the theta function with characteristic by
\[
\vartheta_{a,b}(z, \tau) := \sum_{n \in \mathbb{Z}^g} e\left(\frac{1}{2}(n + a)^t \tau (n + a) + (n + a)^t (z + b)\right),
\]
where \( e(t) = \exp(2\pi i t) \). Let
\[
f : \Theta_{a,b} := \{(z, \tau) \in A_g \mid \vartheta_{a,b}(z, \tau) = 0\} \rightarrow \mathcal{S}_g.
\]
be the universal family of theta divisors. For simplicity we write \( \vartheta \) for \( \vartheta_{0,0} \) and set \( \Theta = \Theta_{0,0} \).

On \( A_g, \Gamma_g \) acts by
\[
\gamma \cdot (z, \tau) := (z(C\tau + D)^{-1}, (A\tau + B)(C\tau + D)^{-1}), \quad \gamma \in \Gamma_g, \; z \in \mathbb{C}^g, \; \tau \in \mathcal{S}_g.
\]

Then \( \gamma \) acts on \( \Theta \) by \( \gamma \cdot \Theta = \Theta \). For any \( (m, n) \in \mathbb{R}^{2g} \), we define an automorphism \( t_{m,n} : A_g \rightarrow A_g \) by
\[
(t_{m,n}) \cdot (z, \tau) := (z + m\tau + n, \tau).
\]
Then \( t_{m,n} \) has no fixed points when \( (m, n) \in \mathbb{R}^{2g} \setminus \mathbb{Z}^{2g} \) and the subgroup \( \mathbb{Z}^{2g} \subset \mathbb{R}^{2g} \) acts trivially on \( A_g \). For \( \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \), we define
\[
\tilde{\gamma} := t_{m,n} \circ \gamma \in \text{Aut}(A_g), \quad (m, n) := \frac{1}{2}((C^t D)_0, (A^t B)_0).
\]
Then \( \tilde{\gamma} \) preserves the family \( f : \Theta \rightarrow \mathcal{S}_g \).

**Proposition 4.1.** For any \( \gamma_1, \gamma_2 \in \Gamma_g \),
\[
\tilde{\gamma}_1 \circ \tilde{\gamma}_2 = \tilde{\gamma}_1 \tilde{\gamma}_2.
\]
**Proof.** See [15] \( \square \)

We set
\[
g^{A_g/\mathcal{S}_g} := dz \cdot (\text{Im}\tau)^{-1} \cdot d\bar{z}.
\]
Then \( g^{A_g/\mathcal{S}_g} \) is a \( \Gamma_g \)-invariant Hermitian metric on the relative tangent bundle \( T(A_g/\mathcal{S}_g) \). The next purpose of this section is to construct a \( \Gamma_g \)-invariant Kähler metric on \( TA_g \) such that \( g^{A_g/\mathcal{S}_g}|_{A_\tau} = dz \cdot (\text{Im}\tau)^{-1} \cdot d\bar{z} \) for all \( \tau \in \mathcal{S}_g \).

Put \( T^{2g} := \mathbb{R}^{2g} \setminus \mathbb{Z}^{2g} \). Define a \( \mathbb{Z}^{2g} \)-action on \( \mathbb{R}^{2g} \times \mathcal{S}_g \) by \( (m, n) \cdot (x, y, \tau) := (x + m, y + n, \tau) \) for \( (m, n) \in \mathbb{Z}^{2g}, (x, y) \in \mathbb{R}^{2g}, \tau \in \mathcal{S}_g \). Then \( (\mathbb{R}^{2g} \times \mathcal{S}_g)/\mathbb{Z}^{2g} \) is the trivial \( T^{2g} \)-bundle \( T^{2g} \times \mathcal{S}_g \). We define a \( C^\infty \)-map \( \tilde{\rho} : \mathbb{R}^{2g} \times \mathcal{S}_g \rightarrow \mathbb{C}^g \times \mathcal{S}_g \) by
\[
\tilde{\rho}(x, y, \tau) := (x\tau + y, \tau), \quad x, y \in \mathbb{R}^g, \; \tau \in \mathcal{S}_g.
\]
Since \( \tilde{\rho} \) is a \( \mathbb{Z}^{2g} \)-equivariant map, \( \tilde{\rho} \) induces a \( C^\infty \)-isomorphism \( \rho : T^{2g} \times \mathcal{S}_g \rightarrow A_g \) as \( T^{2g} \)-bundles over \( \mathcal{S}_g \). Define a \( \Gamma_g \)-action on \( T^{2g} \times \mathcal{S}_g \) by
\[
\gamma \cdot ((x, y), \tau) := ((x, y)\gamma^{-1}, \gamma \cdot \tau), \quad \gamma \in \Gamma_g.
\]
Then for any $\gamma \in \Gamma_g$, the following diagram is commutative.

\[
\begin{array}{ccc}
T^{2g} \times \mathcal{S}_g & \longrightarrow & A_g \\
\downarrow & & \downarrow \\
T^{2g} \times \mathcal{S}_g & \longrightarrow & A_g
\end{array}
\]

Since the trivial connection on $T^{2g} \times \mathcal{S}_g$ is $\Gamma_g$-invariant, $A_g$ has the induced $\Gamma_g$-invariant connection $T_H A_g \subset T A_g$ via the $\Gamma_g$-equivariant isomorphism $\rho$. We denote the $\Gamma_g$-equivariant projection corresponding to $T_H A_g$ by $P_{\rho}$. Let $P_{\rho}^C : T A_g \otimes \mathbb{C} \rightarrow T(A_g/\mathcal{S}_g) \otimes \mathbb{C}$ be the complexification of $P_{\rho}$. Then $P_{\rho}^C$ is also $\Gamma_g$-equivariant.

Under the projection, the horizontal lift of a $(1,0)$ (resp. $(1,0)$) tangent vector is a $(1,0)$ (resp. $(1,0)$) tangent vector. Therefore the extension $P_{\rho}^C : T A_g \otimes \mathbb{C} \rightarrow T(A_g/\mathcal{S}_g) \otimes \mathbb{C}$ decomposes

\[
P_{\rho}^C = P_{\rho}^{1,0} \oplus P_{\rho}^{0,1},
\]

under the isomorphism $T A_g \otimes \mathbb{C} = T^{1,0} A_g \oplus T^{0,1} A_g$ and $T(A_g/\mathcal{S}_g) \otimes \mathbb{C} = T^{1,0}(A_g/\mathcal{S}_g) \oplus T^{0,1}(A_g/\mathcal{S}_g)$. Hence $P_{\rho}$ induces a $\Gamma_g$-equivariant $C^\infty$-isomorphism

\[
T^{1,0} A_g \cong T^{1,0}(A_g/\mathcal{S}_g) \oplus f^* T^{1,0} \mathcal{S}_g.
\]

Let $g^{\mathcal{S}_g}$ be the Bergman metric on $\mathcal{S}_g$ with Kähler form

\[
\omega_{\mathcal{S}_g} = -2\sqrt{-1} \partial \overline{\partial} \log \det \text{Im} \tau.
\]

Then $g^{\mathcal{S}_g}$ is $\Gamma_g$-invariant. Using the $\Gamma_g$-equivariant isomorphism (3), we define the $\Gamma_g$-invariant Hermitian metric $g^{A_g}$ on $T A_g$ by

\[
g^{A_g} := g^{A_g/\mathcal{S}_g} \oplus f^* g^{\mathcal{S}_g}.
\]

**Theorem 4.2.** The Hermitian metric $g^{A_g}$ is Kähler.

**Proof.** See [15] \square

We put

\[
A_k(\Gamma_g, \chi) = \{ f \in \mathcal{O}(\mathcal{S}_g) \mid f(\gamma \cdot \tau) = j(\tau, \gamma)^k \chi(\gamma) f(\tau), \gamma \in \Gamma_g \}
\]

where $\chi$ is a character of $\Gamma_g$ and $j(\tau, \gamma) = \text{det}(C \tau + D)$ for $\gamma \in \left( \begin{array}{cc} A & B \\ C & D \end{array} \right)$. An element of $A_k(\Gamma_g, \chi)$ is called a Siegel modular form of weight $k$ with character $\chi$. In particular, an element of $A_k(\Gamma_g, 1)$ is called a Siegel modular form. Let $\mathcal{F}_g^k := \mathcal{S}_g \times \mathbb{C}^g$ be the trivial holomorphic line bundle over $\mathcal{S}_g$ with the $\Gamma_g$-action

\[
\gamma \cdot (\tau, \xi) := (\gamma \cdot \tau, j(\gamma, \tau)^k \xi).
\]

A Siegel modular form of weight $k$ is regarded as a $\Gamma_g$-invariant holomorphic section of $\mathcal{F}_g^k$. Define the Peterson metric on $\mathcal{F}_g^k$ by

\[
\| \xi \|_{\mathcal{F}_g^k}^2 := (\text{det Im} \tau)^k \| \xi \|^2, \quad (\tau, \xi) \in \mathcal{F}_g^k.
\]

By the automorphic property of $\text{det Im}(\gamma \cdot \tau) = |j(\tau, \gamma)|^{-2} \text{det Im} \tau$, we see that $\| \cdot \|_{\mathcal{F}_g^k}$ is $\Gamma_g$-invariant.

Let $\mathcal{N}_g := \{ \tau \in \mathcal{S}_g \mid \text{Sing} \Theta_\tau \neq \emptyset \}$ be the Andreotti-Mayer locus, which is the locus of Abelian varieties whose theta divisors is singular. The followings are known for the locus $\mathcal{N}_g$. 


Theorem 4.3 ([12]). \( N_g \) is a divisor of \( \mathfrak{S}_g \), consisting of two irreducible components as a divisor of the modular variety \( \Gamma_g \backslash \mathfrak{S}_g \):
\[
N_g = \theta_{null,g} + 2N_g'.
\]
Here \( \theta_{null,g} \) is the zero divisor of Igusa’s modular form \( \chi_g(\tau) \) which is the Siegel modular form of weight \( 2g^2 - 2(2g + 1) \) defined as the product of all even theta constants and \( N_g' = 0 \) for \( g = 2, 3 \).

For a generic point \( \tau \in \theta_{null,g} \), \( \text{Sing}(\Theta_{\tau}) \) consists of one ordinary double point.

Theorem 4.4 ([25]). There is a Siegel cusp form \( \Delta_g(\tau) \) of weight \( \frac{(g + 3) \cdot g!}{2} \) with zero divisor \( N_g \).

By the Proposition 4.3, this implies that there exists \( J_g(\tau) \) which is a Siegel modular form of weight \( \frac{(g + 3) \cdot g!}{4} - 2^{g - 3}(2^g + 1) \) with zero divisor \( N_g' \) such that
\[
\Delta_g := \chi_g(\tau) J_g(\tau)^2.
\]

We put \( \mathfrak{S}_g' := \mathfrak{S}_g - N_g' \), \( \Theta_g' := \Theta|_{\mathfrak{S}_g'} \). Then \( f : \Theta' \rightarrow \mathfrak{S}_g' \) is a family of smooth theta divisors.

Endow \( T^{1,0}(\Theta'/\mathfrak{S}_g') \) the Hermitian metric \( g^{\Theta'/\mathfrak{S}_g'} := g^{\Theta|_{\mathfrak{S}_g'}} \). Let \( g^{\Theta'} := g^{\Theta|_{\mathfrak{S}_g'}} \) be the restriction of the Kähler metric \( g^{\Theta|_{\mathfrak{S}_g}} \). Consider \( g^{\Theta'/\mathfrak{S}_g'} \) and \( g^{\Theta'} \) as Riemannian metric on \( T(\Theta'/\mathfrak{S}_g') \) and \( T\Theta' \).

Let
\[
T_H \Theta' := (T(\Theta'/\mathfrak{S}_g'))^\perp
\]
be the orthogonal complement of \( T(\Theta'/\mathfrak{S}_g') \) in \( T\Theta' \), which induces a connection \( P_{\Theta'} : T\Theta' \rightarrow T\Theta'/\mathfrak{S}_g' \).

Hence we obtain the connection \( \nabla^{\Theta'/\mathfrak{S}_g'} \) on \( T(\Theta'/\mathfrak{S}_g') \) by using \( g^{\Theta'/\mathfrak{S}_g'} \) and \( P_{\Theta'} \) as in Section 2.2. Let \( \nabla^h \) be the holomorphic Hermitian connection on \( T^{1,0}(\Theta'/\mathfrak{S}_g') \) with respect to the Hermitian metric \( g^{\Theta'/\mathfrak{S}_g'} \).

Lemma 4.5. Under the \( C^\infty \)-isomorphism \( T(\Theta'/\mathfrak{S}_g') \otimes \mathbb{C} \cong T^{1,0}(\Theta'/\mathfrak{S}_g') \otimes T^{0,1}(\Theta'/\mathfrak{S}_g') \), the following equality of connections holds.
\[
\nabla^{\Theta'/\mathfrak{S}_g'} \otimes \mathbb{C} = \nabla^h \otimes \overline{\nabla}^h
\]

Proof. Let \( \nabla^L \) be the Levi-Civita connection on \( TA_g \) and let \( \nabla^H \) be the holomorphic Hermitian connection on \( T^{1,0}A_g \). Since \( g^{A_g} \) is Kähler, the following equality holds ([18])
\[
\nabla^L \otimes \mathbb{C} = \nabla^H \otimes \overline{\nabla}^H
\]
under the isomorphism \( TA_g \otimes \mathbb{C} = T^{1,0}A_g \otimes T^{0,1}A_g \). By (2), we get
\[
\nabla^{\Theta'/\mathfrak{S}_g'} \otimes \mathbb{C} = (P_{\rho} \nabla^L P_{\rho}) \otimes \mathbb{C} = P_{\rho}^C (\nabla^L \otimes \mathbb{C}) P_{\rho}^C = P_{\rho}^{1,0} \nabla^H P_{\rho}^{1,0} \oplus P_{\rho}^{0,1} \overline{\nabla}^H P_{\rho}^{0,1}.
\]

Since \( P_{\rho}^{1,0} \nabla^H P_{\rho}^{1,0} = \nabla^h \) (see [18] Capier I, Section 6), we get the result. \( \square \)

Let \( g_{1_g} \) be the restriction of the Hermitian metric \( |dz|^2 \) on \( TA_g / \mathfrak{S}_g \) to the relative tangent bundle \( T\Theta'/\mathfrak{S}_g' \). Let \( F(T\Theta'/\mathfrak{S}_g', g_{1_g}) \) be the corresponding Chern-Weil form for \( F(x) \) and the holomorphic Hermitian connection of \( (T\Theta'/\mathfrak{S}_g', g_{1_g}) \).

Proposition 4.6 ([24], Proposition 2.1). The following equality holds:
\[
[F(T\Theta'/\mathfrak{S}_g', g_{1_g})]^{(g,g)} = 0.
\]

In particular one has
\[
[f_* F(T\Theta'/\mathfrak{S}_g', g_{1_g})]^{(1,1)} = 0.
\]
Let $\|\Delta_{2g}(\tau)\|^2 := (\det \im \tau)^{(2g+3)(2g)!} |\Delta_{2g}(\tau)|^2$ denote the Peterson norm of the Siegel modular form $\Delta_{2g}(\tau)$ and let $B_k$ be the $k$-th Bernoulli number, i.e.,

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \sum_{k=1}^{\infty} (-1)^{k+1} B_k \frac{x^{2k}}{(2k)!}.$$  

**Theorem 4.7.** The following equality holds:

$$\left[ f_{*}L(T(\mathcal{O}/\mathcal{O}_g'), \nabla\mathcal{O}/\mathcal{E}_g') \right]^{(2)} = \frac{(-1)^{g} 2^{g+1} (2^{2g+2} - 1)}{(2g+1)(g+1)} B_{g+1} dd^{c} \log \det \im \tau$$

$$= \frac{(-1)^{g} 2^{g+3} (2^{2g+2} - 1)}{(2g+3)!} B_{g+1} dd^{c} \log \|\Delta_{2g}(\tau)\|^2.$$  

By Lemma 4.5 and the fact that $(\nabla^h)^2$ is a $(1,1)$-form, we see that the left-hand side is equal to $\left[ f_{*}L(T_{1,0}(\mathcal{O}/\mathcal{O}_g'), \nabla^h) \right]^{(1,1)}$. By Proposition 4.6 we obtain

$$\left[ L(T_{1,0}(\mathcal{O}/\mathcal{O}_g'), \nabla^h) \right]^{(1,1)} = -dd^{c} f_{*} \left[ L(T_{1,0}(\mathcal{O}/\mathcal{O}_g'), g_{1_{g}}, g_{\mathcal{O}/\mathcal{E}_g'}) \right]^{(2g-1,2g-1)}.$$  

Hence we deduced the proof to the computation of the Bott-Chern form and we can compute it by using the same idea in [25]. Since this is rather complicated, we omit the proof.

**Remark 4.8.** In Section 5, it will be crucial that $dd^{c} \log \|\Delta_{2}(\tau)\|^2$ is $\Gamma_{g}$-invariant and that $dd^{c} \log \|\Delta_{2}(\tau)\|^2$ is an exact form as a 2-form on $\Gamma_{g}\backslash \mathcal{O}_g$.

5. The signature cocycle for smooth theta divisors

Since $\Gamma_{g}$ acts on $\mathcal{O}_g$ properly discontinuously the space $\Gamma_{g}\backslash \mathcal{O}_g$ has naturally orbifold structure and can be regarded as the moduli space of smooth theta divisors. We shall consider the orbifold fundamental group of $\Gamma_{g}\backslash \mathcal{O}_g$ and construct a 2-cocycle of this group.

In the rest of this section we fix a generic base point $\ast \in \mathcal{O}_g$, i.e., $\ast$ satisfies $\{ \gamma \in \Gamma_{g} | \gamma \ast = \ast \} = \{ \pm 1_{2g} \}$. Let $(B, b)$ be a topological space with a base point and let $\pi : \tilde{B} \rightarrow B$ be the universal covering. Then the fundamental group $\pi_{1}(B, b)$ acts on $\tilde{B}$ as the deck transformation. Fix a lift $\tilde{b} \in \tilde{B}$ of $b \in B$. We set

$$[B, \Gamma_{g}\backslash \mathcal{O}_g]^{orb} := \{(p, \beta) \mid p : \tilde{B} \rightarrow \mathcal{O}_g, \beta : \pi_{1}(B, b) \rightarrow \Gamma_{g}, \text{s.t. } p(\tilde{b}) = \ast, p(\gamma x) = \beta(\gamma) p(x) \}/\sim.$$  

Here the relation $(p_{0}, \beta_{0}) ~ (p_{1}, \beta_{1})$ holds if and only if $\beta_{0} = \beta_{1}$ and there is a map $\tilde{p} : \tilde{B} \times [0,1] \rightarrow \mathcal{O}_g$ such that $\tilde{p}(x, 0) = p_{0}, \tilde{p}(x, 1) = p_{1}$ and $p(\gamma x, t) = \beta(\gamma) p(x, t)$ for any $\gamma \in \Gamma_{g}, x \in \tilde{B}, t \in [0,1]$.

**Definition 5.1.** We define the orbifold fundamental group of $\Gamma_{g}\backslash \mathcal{O}_g$ by

$$S_{g} := \{ [S_{1}, \Gamma_{g}\backslash \mathcal{O}_g]^{orb} \}$$

$$= \{ (\alpha, \gamma) \mid \gamma \in \Gamma_{g}, \alpha : [0,1] \rightarrow \mathcal{O}_g, \text{s.t. } \alpha(0) = \ast, \alpha(t) = \gamma \cdot \alpha(t+1), \text{ } t \in [0,1] \}/\sim.$$  

Then

$$S_{g} = \{ (\alpha, \gamma) | \gamma \in \Gamma_{g}, \alpha : [0,1] \rightarrow \mathcal{O}_g, \text{s.t. } \alpha(0) = \gamma \cdot \alpha(1) = \ast \}/\sim.$$  

Here $(\alpha_{0}, \gamma_{0}) ~ (\alpha_{1}, \gamma_{1})$ if and only if $\gamma_{0} = \gamma_{1}$ and there exists a homotopy $\alpha(s, t) : [0,1] \times [0,1] \rightarrow \mathcal{O}_g$ connecting $\alpha_{0}$ and $\alpha_{1}$, such that $\alpha(s, 0) = \gamma_{0} \cdot \alpha(s, 1) = \ast$ for $s \in [0,1]$.

The group law of $S_{g}$ is defined as follows. Let $[(\alpha_{1}, \gamma_{1})], [(\alpha_{2}, \gamma_{2})] \in S_{g}$. Then $\gamma_{2}^{-1} \cdot \alpha_{1}$ is a path path from $\gamma_{2}^{-1} \ast$ to $(\gamma_{2} \gamma_{1})^{-1} \ast$. We define the new path $\alpha : [0,1] \rightarrow \mathcal{O}_g$ by $\alpha(t) := \alpha_{2}(2t)$ for $0 \leq t \leq \frac{1}{2}, \alpha(t) := \gamma_{2}^{-1} \cdot \alpha_{1}(2t-1)$ for $\frac{1}{2} \leq t \leq 1$. Then we define $[(\alpha_{1}, \gamma_{1})] \cdot [(\alpha_{2}, \gamma_{2})] := [(\alpha, \gamma_{1} \gamma_{2})] \in S_{g}$.
Let \( p : S_g \to \Gamma_g \) be the projection to the second factor. Since the kernel of \( p \) is isomorphic to \( \pi_1(\mathcal{G}_g', *) \), we have an exact sequence

\[(5) \quad 1 \to \pi_1(\mathcal{G}_g', *) \to S_g \to \Gamma_g \to 1.\]

**Remark 5.2.** When \( g = 1 \), \( \Gamma_1 \setminus \mathcal{G}_1 = SL_2\mathbb{Z} \setminus \mathcal{G}_1 \) is the moduli space of curves of genus 1 and \( S_1 = \mathcal{M}_1 \). When \( g = 2 \), \( \Gamma_2 \setminus \mathcal{G}_2 \) is the moduli space of curves of genus 2 by the Torelli theorem and \( S_2 = \mathcal{M}_2 \).

Recall that a \( \pi_1(B, b) \)-equivariant map \( f : (\tilde{B}, \tilde{b}) \to (\mathcal{G}_g^*, *) \) induces the homomorphism of groups \( f_* : \pi_1(B, b) \to S_g \) such that \( f_*([c]) = [f \circ c] \) for \( [c] \in \pi_1(B, b) \).

**Proposition 5.3.** Let \((B, b)\) be a compact orientated surface with base point and with non empty boundary. Then the map

\[ [B, \Gamma_g \setminus \mathcal{G}_g']^{orb} \ni [f] \mapsto f_* \in \text{Hom}(\pi_1(B, b), S_g). \]

is a bijection.

**Proof.** It is known that \( B \) is homotopy equivalent to an \( n \)-bouquet \( \bigvee_{k=1}^{n} S^1_k \) for some \( n \) and the fundamental group \( \pi_1(B, b) \cong \pi_1(\bigvee_{k=1}^{n} S^1_k, o) \) is isomorphic to the free group of rank \( n \). Hence we get

\[ [B, \Gamma_g \setminus \mathcal{G}_g']^{orb} \cong [\bigvee_{k=1}^{n} S^1_k, \Gamma_g \setminus \mathcal{G}_g']^{orb} \cong \text{Hom}(\pi_1(\bigvee_{k=1}^{n} S^1_k, o), S_g) \cong \text{Hom}(\pi_1(B, b), S_g). \]

which completes the proof.

In the rest of this section we assume that \( B = S^2 - \bigcap_{k=1}^{3} D_k \), where \( D_1, D_2, D_3 \) are mutually disjoint open discs. Since \( B \) is homotopy equivalent to a 2-bouquet \( \pi_1(B, b) \) is the free group of rank 2. Let \( g_1, g_2 \) be generators of \( \pi_1(B, b) \) represented by the loops which are mutually homotopy equivalent to \( \partial D_1, \partial D_2 \). By Proposition 5.3 we have a bijection

\[(6) \quad [B, \Gamma_g \setminus \mathcal{G}_g']^{orb} \cong S_g \times S_g, \]

which is given by \([f] \mapsto (f_*(g_1), f_*(g_2)) \in S_g \times S_g \) for \([f] \in [B, \Gamma_g \setminus \mathcal{G}_g']^{orb} \).

For \([f] \in [B, \Gamma_g \setminus \mathcal{G}_g']^{orb} \) the fiber product \( \pi : \tilde{B} \times_f \Theta \to \tilde{B} \) is a \( \pi_1(B, b) \)-equivariant fiber bundle because \( f : \tilde{B} \to \mathcal{G}_g \) is a \( \pi_1(B, b) \)-equivariant map. We get the fiber bundle \( \pi : (\tilde{B} \times_f \Theta) / \pi_1(B, b) \to B \), which is uniquely determined by \([f] \in [B, \Gamma_g \setminus \mathcal{G}_g']^{orb} \) up to an isomorphism and which is 2g-dimensional compact oriented manifold with boundary. For \((\sigma_1, \sigma_2) \in S_g \times S_g \), Let \( \pi : X(\sigma_1, \sigma_2) \to B \) denote the corresponding fiber bundle under the isomorphism (6).

**Definition 5.4.** Define the map \( c_{2g} : S_{2g} \times S_{2g} \to \mathbb{Z} \) by

\[ c_{2g}(\sigma_1, \sigma_2) := \text{Sign}(X(\sigma_1, \sigma_2)). \]

We call \( c_{2g} \) the signature cocycle for smooth theta divisors.

**Remark 5.5.** We only consider the case of an even genus because in the case of an odd genus \( \text{Sign}(X(\sigma_1, \sigma_2)) \) always vanishes.

**Lemma 5.6.** The following relation holds:

\[ c_{2g}(\sigma_1, \sigma_2) + c_{2g}(\sigma_1 \sigma_2, \sigma_3) = c_{2g}(\sigma_2, \sigma_3) + c_{2g}(\sigma_2 \sigma_3, \sigma_1), \]

for any \( \sigma_1, \sigma_2, \sigma_3 \in S_{2g} \). In particular, \( c_{2g} \) is a 2-cocycle of the group \( S_{2g} \) ([11]).

**Proof.** By the same argument in [1], we obtain the assertion.

Let \([c_{2g}] \in H^2(S_{2g}, \mathbb{Z})\) be the cohomology class of \( c_{2g} \). When \( g = 1 \), \( c_2 \) is the Meyer cocycle.
6. Construction of the Meyer function

Let $\sigma = [(\alpha, \gamma)]$ be an element of $S_{2g}$, where $\alpha : \mathbb{R} \to G'_{2g}$ and $\gamma \in \Gamma_{2g}$. Let $\mathbb{R} \times \alpha G'$ be the fiber product, which has a natural $\pi_1(S^1)$-action. We define the mapping torus $M_\sigma$ for $\sigma$ by

$$\pi : M_\sigma := (\mathbb{R} \times \alpha G')/\pi_1(S^1) \to S^1.$$  

Since the metric $g^{G'}/G'$ on $T(G'/G')$ and the connection $P_{G'}$ on $G'$ are $\Gamma_{2g}$-invariant and the map $p : S^1 = \mathbb{R} \to G'_{2g}$ is $\pi_1(S^1)$-equivariant, the metric $g^M_{\sigma}/S^1$ on $T(M_{\sigma}/S^1)$ and the connection on $P_{\sigma}$ on $M_{\sigma}$ are naturally induced via the map $p$. Using the connection $P_{\sigma}$ we define the $1$-parameter family of Riemannian metrics $\{g^M_{\sigma, \epsilon}\}_{\epsilon > 0}$ on $M_{\sigma}$ by

$$g^M_{\sigma, \epsilon} := g^M_{\sigma}/S^1 \oplus \epsilon^{-1} \pi^* dt^2, \quad \epsilon \in \mathbb{R}_{>0}.$$  

Here we regard $S^1$ as $\mathbb{R}/\mathbb{Z}$ and $t \in \mathbb{R}$ as a coordinate of $S^1$. By the theorem 3.3, the adiabatic limit

$$\eta^0(M_{\sigma}, g^{M_{\epsilon}}_{\sigma}) := \lim_{\epsilon \to 0} \eta(M_{\sigma}, g^{M_{\epsilon}}_{\sigma})$$

exists. Recall that the Siegel modular form $\Delta_{2g}(\tau)$ with zero divisors $N_{2g}$ (see Section 3.3). Since the $1$-form $d\log||\Delta_{2g}(\tau)||^2$ is $\Gamma_{2g}$-invariant the pull-back $p^* d\log||\Delta_{2g}(\tau)||^2$ can be regarded as a $1$-form on $S^1$.

Definition 6.1. For $\sigma \in S_{2g}$ we fix $(p, \gamma)$ which represents $\sigma = [(p, \gamma)]$, where $\gamma \in \Gamma_{2g}$ and $p : \mathbb{R} \to G'_{2g}$. We set

$$\Phi_{2g}(p, \gamma) := \eta^0(M_{\sigma}, g^{M_{\epsilon}}_{\sigma}) + \frac{(-1)^g 2^{2g+3} (2^{2g+2} - 1) B_{9+1}}{(2g+3)!} \int_{S^1} p^* d\log||\Delta_{2g}(\tau)||^2.$$  

The following theorem is the main result of this paper.

Theorem 6.2. (a) The value $\Phi_{2g}(p, \gamma)$ is independent of a choice of $(p, \gamma)$ which represents $\sigma \in S_{2g}$. In particular $\Phi_{2g}$ is a function on $S_{2g}$.

(b) The cocycle $-c_{2g}$ is the coboundary of the function $\Phi_{2g}$. In particular $[c_{2g}] \otimes \mathbb{Q} = 0 \in H^2(S_{2g}, \mathbb{Z})$.

As a corollary of the Theorem 6.2, it follows that $\phi_2 = \Phi_2$ by the uniqueness of Meyer's function of genus $2$. On the other hand, $\Delta_2(\tau)$ coincides with the Igusa's modular form $\chi_2(\tau)$ ((25)), which is the product of all even theta constants. Then we can derive the following formula:

Corollary 6.3 ((15)). Let $\sigma = [(p, \gamma)]$ be an element of $S_2 = M_2$ as before. Then we have

$$\phi_2(\sigma) = \eta^0(M_{\sigma}, g^{M_{\epsilon}}_{\sigma}) - \frac{2}{15} \int_{S^1} p^* d\log||\chi_2(\tau)||^2.$$  

Proof of Theorem 6.2. (a) Assume that $(p_0, \gamma)$ and $(p_1, \gamma)$ represents the same element $\sigma \in S_{2g}$. Put $I := [0, 1]$. There is a map

$$\bar{p} : I \times \mathbb{R} \to G'_{2g}$$

which satisfies $\bar{p}(s, 0) = *$ for $s \in I$ and $\bar{p}(s, t) = \gamma \cdot \bar{p}(s, t + 1)$ for $(s, t) \in I \times \mathbb{R}$ and the following condition

$$(7) \quad \bar{p}(s, t) = p_0(t), \quad s \in [0, \frac{1}{3}] \quad \text{and} \quad \bar{p}(s, t) = p_1(t), \quad s \in (\frac{2}{3}, 1].$$

Since $\bar{p}$ is $\pi_1(I \times \mathbb{R})$-equivariant, the fiber product $(I \times \mathbb{R}) \times_{\mathbb{R}} G'$ has the $\pi_1(I \times S^1)$-action and the quotient space

$$\pi : \bar{M}_{\sigma} := (I \times \mathbb{R}) \times_{\mathbb{R}} G'/\pi_1(I \times S^1) \to I \times S^1$$

has the induced metric $g^{M_{\sigma}/I\times S^{1}}$ on $T(\tilde{M}_{\sigma}/I\times S^{1})$ from the metric $g^{\Theta'/\mathfrak{S}_{2}}$ and the connection $P_{\sigma}$ on $M_{\sigma}$ from the connection $P_{\Theta'}$ mutually via the map $p$. Using the connection $\tilde{P}_{\sigma}$ we set

$$g^{\tilde{M}_{\sigma},\epsilon} := g^{M_{\sigma}/I\times S^{1}} \oplus \epsilon^{-1} \pi^{*}(ds^{2} \oplus dt^{2}), \quad \epsilon \in \mathbb{R}_{>0}.$$  

Let $g^{M_{\sigma},\epsilon}$ be the metrics on $M_{\sigma}$, induced from the map $p_{i}$ for $i = 0, 1$ as above. The condition (7) implies that

$$g^{\tilde{M}_{\sigma},\epsilon}|_{[0,\frac{1}{a}) \times S^{1}} = g^{M_{\sigma},\epsilon} \oplus \epsilon^{-1} dt^{2}, \quad g^{\tilde{M}_{\sigma},\epsilon}|_{[\frac{1}{a},1[\times S^{1}} = g^{M_{\sigma},\epsilon} \oplus \epsilon^{-1} dt^{2}.$$  

Then we can apply the Atiyah-Patodi Singer's index theorem to $(\tilde{M}_{\sigma}, g^{\tilde{M}_{\sigma},\epsilon})$:

$$(8) \quad \text{Sign}(\tilde{M}_{\sigma}) = \int_{I \times S^{1}} \tilde{\pi}_{*}L(T(\tilde{M}_{\sigma}, g^{\tilde{M}_{\sigma},\epsilon})) = (\eta(M_{\sigma}, g^{M_{\sigma},\epsilon}) - \eta(M_{\sigma}, g^{M_{\sigma},\epsilon})).$$

Since $\tilde{M}_{\sigma}$ is isomorphic to the product $M_{\sigma} \times I$, we have (see [3]),

$$\text{Sign}(\tilde{M}_{\sigma}) = \text{Sign}(M_{\sigma}) \times \text{Sign}(I) = 0.$$

By Proposition 2.4 and the Proposition 2.5, we get

$$\lim_{\epsilon \to 0} \int_{I \times S^{1}} \tilde{\pi}_{*}L(T(\tilde{M}_{\sigma}, g^{\tilde{M}_{\sigma},\epsilon})) = \int_{I \times S^{1}} \tilde{\pi}_{*}L(T(\tilde{M}_{\sigma}/(I \times S^{1})))$$

$$= \int_{I \times S^{1}} (\tilde{\pi}_{*}L(T(\tilde{M}_{\sigma}/(I \times S^{1})), \nabla^{\tilde{M}_{\sigma}/(I \times S^{1}))})^{(2)}$$

$$= \int_{I \times S^{1}} [\tilde{\pi}_{*}p^{*}L(T(\Theta'/\mathfrak{S}_{2}'), \nabla^{\Theta'/\mathfrak{S}_{2}'})]^{(2)}$$

$$= \int_{I \times S^{1}} \tilde{\pi}_{*}L(T(\overline{M}_{\sigma}/(S^{1} \times I)), \nabla^{\overline{M}_{\sigma}/(S^{1} \times I)})^{(2)}$$

where $\nabla^{\tilde{M}_{\sigma}/(S^{1} \times I)}$ is the connection on the relative tangent bundle $T(\tilde{M}_{\sigma}/(S^{1} \times I))$ associated with $g^{\tilde{M}_{\sigma}/(S^{1} \times I)}$ and $\tilde{P}_{\sigma}$ and we used the commutativity of fiber integrals and base changes in the last equality. By the Proposition 4.7, we have

$$(11) \quad \int_{I \times S^{1}} \tilde{\pi}_{*}L(T(\Theta'/\mathfrak{S}_{2}'), \nabla^{\Theta'/\mathfrak{S}_{2}'})^{(2)}$$

$$= -\frac{2^{2g+3}(2^{2g+2} - 1)B_{2g+2}}{(2g + 3)!} \int_{I \times S^{1}} \tilde{p}^{*}d\delta \log||\Delta_{2g}(\tau)||^{2}$$

$$= -\frac{2^{2g+3}(2^{2g+2} - 1)B_{2g+2}}{(2g + 3)!} \left( \int_{[0,1] \times S^{1}} \tilde{p}^{*}d\delta \log||\Delta_{2g}(\tau)||^{2} \right) - \int_{[0,1] \times S^{1}} \tilde{p}^{*}d\delta \log||\Delta_{2g}(\tau)||^{2},$$

where we used the $\Gamma_{2g}$-invariance of the 1-form $d\delta \log||\Delta_{2g}(\tau)||^{2}$ in the last equality. By (25) ~ (12) and the Definition 6.1, we obtain

$$0 = \Phi_{2g}(p_{1}, \gamma) - \Phi_{2g}(p_{0}, \gamma),$$

which completes the proof of (a).

(b) Let $\sigma_{1} = [(p_{1}, \gamma_{1})], \sigma_{2} = [(p_{2}, \gamma_{2})], \sigma_{3} := (\sigma_{1}\sigma_{2})^{-1} = (p_{3}, (\gamma_{1}\gamma_{2})^{-1})\in S_{2g}$. Set $B := S^{2} - \bigcup_{k=1}^{3} D_{k}$. Recall that the fiber bundle $\pi : X(\sigma_{1}, \sigma_{2}) \to B$ for $\sigma_{1}, \sigma_{2}$ defined at the Section 3.2. By the definition of $\Phi_{2g}$, we have $\Phi_{2g}(\sigma^{-1}) = -\Phi_{2g}(\sigma)$ for any $\sigma \in S_{2g}$. Therefore to show that $-c_{2g}$ is the coboundary of $\Phi$, we have to show that

$$(12) \quad \text{Sign}(X(\sigma_{1}, \sigma_{2})) = -\sum_{i=1}^{3} \Phi_{2g}(\sigma_{i})$$
Let $U_i$ be the neighborhood of $\partial D_i$ in $B$ such that $U_i \cong [0,1) \times \partial D_i$. Let $\beta_i: \widetilde{U}_i \cong [0,1) \times \mathbb{R} \rightarrow \widetilde{B}$ be the lift of the map $U_i \rightarrow B$. Let $g_1, g_2 \in \pi_1(B, b)$ be the generators represented by the loops $\partial D_1, \partial D_2$. Let $[(p, \alpha)] \in [B, \Gamma_{2g}] \setminus \mathcal{S}_{2g}^{\text{torb}}$ be the corresponding element for $(\sigma_1, \sigma_2) \in S_{2g} \times S_{2g}$ under the isomorphism (6) where $\alpha: \pi_1(B, b) \rightarrow \Gamma_{2g}$ is a group homomorphism and $p: \widetilde{B} \rightarrow \mathcal{S}_{2g}$ is a $\pi_1(B, b)$-equivariant homomorphism preserving the basepoint. Since $\partial D_1, \partial D_2$ and $\partial D_3$ are homotopy equivalent to the loops which represent $g_1, g_2$ and $(g_1 g_2)^{-1} \in \pi_1(B, b)$ we can assume that

\[(13) \quad p \circ \beta_i|_{\widetilde{U}_i}(s, t) = p_i(t), \quad (s, t) \in \widetilde{U}_i \cong [0,1) \times \mathbb{R}, \quad i = 1 \sim 3.\]

Let $g^{X(\sigma_1, \sigma_2)/B}$ and $P_{X(\sigma_1, \sigma_2)}$ be the metric on $TX(\sigma_1, \sigma_2)$ and the connection on $X(\sigma_1, \sigma_2)$ induced from the metric $g^{S_i/\mathcal{E}_{S_i}}$ and the connection $P_{\mathcal{E}_i}$ via the map $p$. Let $g^B$ be the metric on $TB$ such that $g^B|_{U_i} = ds_i^2 \oplus dt^2$. Using the connection $P_{X(\sigma_1, \sigma_2)}$ we define the metric on $TX(\sigma_1, \sigma_2)$ by

\[g^{X(\sigma_1, \sigma_2), \epsilon} := g^{X(\sigma_1, \sigma_2)/B} \oplus \epsilon^{-1} \pi^* g^B, \quad \epsilon \in \mathbb{R}_{>0}.\]

Let $g^{M_{\sigma_i}, \epsilon}$ be the metric on $M_{\sigma_i}$ induced from $p_i$ for $i = 1 \sim 3$ as above. Let $\nabla^{X(\sigma_1, \sigma_2)/B}$ be the connection on $T(X(\sigma_1, \sigma_2))$ defined by the metric $g^{X(\sigma_1, \sigma_2)/B}$ and the connection $P_{X(\sigma_1, \sigma_2)}$. Since the condition (13) implies that the metric $g^{X(\sigma_1, \sigma_2), \epsilon}$ is a product metric near the boundary of $X(\sigma_1, \sigma_2)$ we can apply the Atiyah-Patodi-Singer's index theorem to $(X(\sigma_1, \sigma_2), g^{X(\sigma_1, \sigma_2), \epsilon})$:

\[\text{Sign}(X(\sigma_1, \sigma_2)) = \int_{X(\sigma_1, \sigma_2)} L(TX(\sigma_1, \sigma_2), g^{X(\sigma_1, \sigma_2), \epsilon}) - \sum_{i=1}^{3} \eta(M_{\sigma_i}, g^{M_{\sigma_i}, \epsilon})\]

\[= \int_{B} \pi_* L(T(X(\sigma_1, \sigma_2)/B), \nabla^{X(\sigma_1, \sigma_2)/B}) - \sum_{i=1}^{3} \eta^0(M_{\sigma_i}, g^{M_{\sigma_i}, \epsilon})\]

\[= \int_{B} p^* \left[ f_* L(T(\mathcal{E}'_{S_{2g}}/\mathcal{E}_{S_{2g}}'), \nabla^{\mathcal{E}'_{S_{2g}}}) \right]^{(2)} - \sum_{i=1}^{3} \eta^0(M_{\sigma_i}, g^{M_{\sigma_i}, \epsilon})\]

\[= - \sum_{i=1}^{3} \int_{\partial D_i} - \frac{2^{2g+3}(2^{2g+2} - 1)B_{2g+2}d\log||\Delta_{2g}(\tau)||^2}{(2g + 3)!}\]

\[- \sum_{i=1}^{3} \eta^0(M_{\sigma_i}, g^{M_{\sigma_i}, \epsilon})\]

\[= - \sum_{i=1}^{3} \Phi_{2g}(\sigma_i)\]

which completes the proof of (b).

\(\square\)

7. The first cohomology of $S_g$

The uniqueness of a 1-cocycle that cobounds the 2-cocycle $c_{2g}$ is equivalent to the vanishing of $H^1(S_{2g}, \mathbb{Z})$. In deed, if there is another 1-cocycle $\Phi_{2g}': S_{2g} \rightarrow \mathbb{R}$ that cobounds $c_{2g}$, the difference $\Phi_{2g} - \Phi_{2g}'$ is an element of $\text{Hom}(S_{2g}, \mathbb{R}) \cong H^1(S_{2g}, \mathbb{R})$. While $H^1(S_1, \mathbb{Z}) = H^1(S_2, \mathbb{Z}) = 0$, the uniqueness no longer valid for higher genus.

**Theorem 7.1.** The following holds:

\[H^1(S_g, \mathbb{Z}) = \begin{cases} 0 & \text{if } 1 \leq g \leq 3, \\ \mathbb{Z} & \text{if } g \geq 4. \end{cases}\]
In particular, the cochain cobounding the signature cocycle $c_{2g}$ is not unique when $g \geq 2$.

By (5) and [11], we have the 5-term exact sequence

\[
1 \to H^1(\Gamma, \mathbb{Z}) \to H^1(S, \mathbb{Z}) \to H^1(\pi_1(\mathfrak{S}_g, *), \mathbb{Z})^\Gamma_\tau \overset{\delta}{\to} H^2(\Gamma, \mathbb{Z}) \to H^2(S, \mathbb{Z}).
\]

We have $H^1(\Gamma, \mathbb{Z}) = 0$ for $g \geq 1$ and $H^2(\Gamma, \mathbb{Z}) = \mathbb{Z}$ for $g \geq 3$. By the Hurwitz theorem we see that

\[
H^1(\pi_1(\mathfrak{S}_g, *), \mathbb{Z}) \cong H^1(\mathfrak{S}_g, \mathbb{Z}).
\]

**Lemma 7.2.** Let $X$ be a connected complex manifold of $\dim_C X \geq 2$. Assume that

\[
H^1(X, \mathbb{Z}) = H^2(X, \mathbb{Z}) = 0.
\]

Let $D = \sum_{\lambda \in \Lambda} n_{\lambda} D_{\lambda}$ be a divisor on $X$ such that $n_{\lambda} \neq 0$ and $D_{\lambda}$ is irreducible for all $\lambda \in \Lambda$. Then

\[
H^1(X - D, \mathbb{Z}) \cong \mathbb{Z}^\Lambda.
\]

The generator of the cohomology $H^1(X - D, \mathbb{Z})$ corresponding to $\lambda \in \Lambda$ is represented by the map $l_{\lambda} \mapsto 1$ and $l_{\mu} \mapsto 0$ for $\mu \neq \lambda \in \Lambda$, where $l_{\mu}$ denotes the loop around a small disk and intersecting $D_{\mu}$ transversally.

**Proof.** Since the real codimension of $\text{Sing} D$ in $X$ is greater than or equal to 4, we have $\pi_k(X, X - \text{Sing} D, *) = 0$ for $1 \leq k \leq 3$. The relative Hurwitz theorem asserts that $\pi_k(X, X - \text{Sing} D, \mathbb{Z}) = 0$ for $k \leq 3$. Hence $H^k(X, X - \text{Sing} D, \mathbb{Z}) = 0$ for $k \leq 3$, which together with the cohomology exact sequence for the triple $(X, X - \text{Sing} D, X - D)$, yields that

\[
H^2(X, X - D, \mathbb{Z}) \cong H^2(X - \text{Sing} D, X - D, \mathbb{Z}).
\]

By the cohomology exact sequence for the pair $(X, X - D)$ and (16), we obtain

\[
H^1(X - D, \mathbb{Z}) \cong H^2(X, X - D, \mathbb{Z}).
\]

Since $D - \text{Sing} D$ is a closed submanifold in $X - \text{Sing} D$ and $X - D = (X - \text{Sing} D) - (D - \text{Sing} D)$, the Thom isomorphism asserts that

\[
H^2(X - \text{Sing} D, X - D, \mathbb{Z}) \cong H^0(D - \text{Sing} D, \mathbb{Z}).
\]

By the irreducibility of $D_{\lambda}$, $D_{\lambda} - \text{Sing} D_{\lambda}$ is path connected so that

\[
H^0(D - \text{Sing} D, \mathbb{Z}) \cong \mathbb{Z}^\Lambda.
\]

The result follows from (17)~(20). \hfill \Box

**Lemma 7.3.** The following holds:

\[
H^1(\pi_1(\mathfrak{S}_g, *), \mathbb{Z})^\Gamma_\tau = \begin{cases} 
\mathbb{Z} & 1 \leq g \leq 3 \\
\mathbb{Z}^2 & g \geq 4.
\end{cases}
\]

By regarding $H^1(\mathfrak{S}_g, \mathbb{C})$ as the de Rham cohomology group, the image of the generators under the natural map $H^1(\mathfrak{S}_g, \mathbb{Z}) \to H^1(\mathfrak{S}_g, \mathbb{C})$ are represented by the 1-forms $\frac{1}{2\pi\sqrt{-1}} \log \chi_g(\tau)$ and $\frac{1}{2\pi\sqrt{-1}} \log J_g(\tau)$. Here $J_g(\tau) \equiv 1$ and hence $\log J_g(\tau) = 0$ for $1 \leq g \leq 3$.

**Proof.** By Proposition 4.3, Proposition 4.4, the isomorphism (15) and Lemma 7.2, we get the assertion. \hfill \Box
Recall that the automorphic factor \( j(\tau, \gamma) \) is a nowhere vanishing holomorphic function on \( \mathfrak{S}_g \). Since \( \mathfrak{S}_g \) is simply connected, the logarithm of \( j(\tau, \gamma) \) makes sense. Choose a branch of the logarithm of \( j(\tau, \gamma) \) and denote it by \( \log_\sigma j(\tau, \gamma) \) for \( \gamma \in \Gamma_g \). Define the function \( \lambda_\sigma : \Gamma_g \times \Gamma_g \to \mathbb{Z} \) by

\[
(21) \quad \lambda_\sigma(A, B) := \frac{1}{2\pi\sqrt{-1}} \{ \log_\sigma j(\tau, AB) - \log_\sigma j(\tau, B) + \log_\sigma j(\tau, A) - \log_\sigma j(\tau, B) \}, \quad (A, B) \in \Gamma_g \times \Gamma_g.
\]

**Lemma 7.4.** The function \( \lambda_\sigma \) is a 2-cocycle of \( \Gamma_g \), whose cohomology class generates \( H^2(\Gamma_g, \mathbb{Z}) \).

**Proof.** For \( g = 1 \) see [4]. When \( g \geq 1 \), we follow [4]. Let \( G := \text{Sp}(2g, \mathbb{R}) \) be the symplectic group and let \( G^\delta \) be the same group endowed with the discrete topology. Let \( u \in H^2(G^\delta, \mathbb{Z}) \) be the cohomology class corresponding to the universal covering

\[ 0 \to \mathbb{Z} \to \tilde{G} \to G \to 1. \]

We choose the branch \( \log_\sigma j(\tau, \gamma) \) satisfying

\[
(22) \quad \text{Im} \log_\sigma j(\sqrt{-1} \cdot 1_{2g}, \gamma) \in [0, 2\pi).
\]

Since the function \( \lambda_\sigma \) is measurable, the cohomology class \( [\lambda_\sigma] \) is a constant multiple of \( u \) by [20]. Therefore it suffices to determine the restriction of the cohomology class \( [\lambda_\sigma] \) to the maximal compact subgroup of \( G \). We shall identify the unitary group \( U(g) \) with the maximal compact subgroup of \( G \) by the inclusion map defined as

\[ \iota : U(g) \ni Z \mapsto \begin{pmatrix} \text{Re} Z & \text{Im} Z \\ -\text{Im} Z & \text{Re} Z \end{pmatrix} \in G, \quad Z \in U(g). \]

Since \( j(\sqrt{-1} \cdot 1_{2g}, \iota(Z)) = \det(Z)^{-1} \) for \( Z \in U(g) \) and the isotropy subgroup at \( \sqrt{-1} \cdot 1_{2g} \in \mathfrak{S}_g \) is just \( U(g) \), we have

\[
(23) \quad 2\pi\sqrt{-1} \overline{\lambda}_\sigma(Z_1, Z_2) = -\log_\sigma \det(Z_1 Z_2) + \log_\sigma \det(Z_1) + \log_\sigma \det(Z_2)
\]

for \( (Z_1, Z_2) \in U(g) \times U(g) \). By (23), the restriction of the cohomology class \( [\overline{\lambda}_\sigma] \) to \( U(g) \) is the pullback of the cohomology class \( [\lambda_\sigma] \) to \( U(g) \) is the pullback of the cohomology class \( [\lambda_\sigma] \) corresponding to the universal covering \( 0 \to \mathbb{Z} \to \tilde{U}(1) \cong \mathbb{R} \to U(1) \to 1 \), via the map \( \det : U(1) \to U(1) \). Since the induced map \( (\det)_* : \pi_1(U(1)) \to \pi_1(U(1)) \) is an isomorphism, we obtain \( [\overline{\lambda}_\sigma] = u \). Since the cohomology class \( [\overline{\lambda}_\sigma] \) is independent of the choice of the branch of \( \log_\sigma j(\tau, \gamma) \) and since the restriction of \( u \) to \( \Gamma_g \) is the generator of the cohomology \( H^2(\Gamma_g, \mathbb{Z}) \) we obtain the assertion.

**Lemma 7.5.** Let \( g \geq 2 \). The map \( \delta : H^1(\pi_1(\mathfrak{S}_g', *), \mathbb{Z}) \to H^2(\Gamma_g, \mathbb{Z}) \) is given by

\[
(m, n) \mapsto \langle k_1(g)m + k_2(g)n \rangle \in H^2(\Gamma_g, \mathbb{Z}) \cong \mathbb{Z}
\]

for \( (m, n) \in H^1(\pi_1(\mathfrak{S}_g', *), \mathbb{Z}) \cong \mathbb{Z}^{2g} \). Here,

\[
k_1(g) = 2^{g-2}(2^g + 1), \quad k_2(g) = \frac{(g + 3)g!}{4} - 2^{g-3}(2^g + 1)
\]

are the weights of Siegel modular forms \( \chi_g(\tau), J_g(\tau) \), respectively.

**Proof.** Let \( \sigma : \Gamma_g \to S_g \) be a section, and write \( \sigma(\gamma) = [(l_1, \gamma)] \in S_g \) for \( \gamma \in \Gamma_g \). We can assume that \( l_{-1} = -\gamma \cdot l_1 \) where \( l(t) := l(1 - t), \ t \in [0, 1] \) for a path \( l(t) \). Hence \( \sigma(\gamma^{-1}) = \sigma(\gamma)^{-1} \). Let \( \alpha \) be an element of \( H^1(\pi_1(\mathfrak{S}_g', *), \mathbb{Z})^{\Gamma_\tau} \cong \text{Hom}(\pi_1(\mathfrak{S}_g', *), \mathbb{Z})^{\Gamma_\tau} \). Then \( \delta(\alpha) : \Gamma_g \times \Gamma_g \to \mathbb{Z} \) is given by

\[
(A, B) \mapsto \alpha(\sigma(A)\sigma(B)\sigma(AB)^{-1}) \in \mathbb{Z}, \quad (A, B) \in \Gamma_g \times \Gamma_g,
\]

where we identify \( \sigma(\sigma(A)\sigma(B)\sigma(AB)^{-1}) \in \text{Im} \{ \pi_1(\mathfrak{S}_g', *) \to S_g \} \) with the corresponding preimage of \( \pi_1(\mathfrak{S}_g', *) \) under the inclusion \( \pi_1(\mathfrak{S}_g', *) \to S_g \). Write \( \sigma(\sigma(A)\sigma(B)\sigma(AB)^{-1}) = [(l_{(A, B)}), 1)] \in \pi_1(\mathfrak{S}_g', *) \).
Here $l_{(A,B)}$ is a loop on $\mathfrak{S}_{g}'$, which is the composition of the paths $l_{B}, B^{-1}l_{A}$ and $-l_{AB}$. Under the identification $H^{1}(\pi_{1}(\mathfrak{S}_{g}', *), Z)\Gamma_{g} \cong \mathbb{Z}^{2}$ given in Lemma 7.3, the cochain $\delta(m, n)$ is given by

$$
\delta(m, n)(A, B) = \frac{1}{2\pi i} \int_{l_{(A,B)}} d\log \chi_{g}(\tau)^{m} J_{\mathfrak{S}}^{n} \in \mathbb{Z}, \quad (A, B) \in \Gamma_{g} \times \Gamma_{g},
$$

for $(m, n) \in H^{1}(\pi_{1}(\mathfrak{S}_{g}', *), Z)\Gamma_{g} \cong \mathbb{Z}^{2}$. Using $\sigma$, we choose the branch $\log_{\sigma} j(\tau, \gamma)$ for $\gamma \in \Gamma_{g}$ such that

$$
\log_{\sigma} j(*, \gamma) := \frac{1}{k_{1}(g)} \int_{l_{\gamma^{-1}}} d\log \chi_{g}(\tau).
$$

Then we get

$$
2\pi i \delta(1, 0)(A, B) = \int_{l_{(A,B)}} d\log \chi_{g}(\tau) - \int_{l_{AB}^{-1}(A,B)} d\log \chi_{g}(AB\cdot \tau) - \int_{l_{B}^{-1}A} d\log \chi_{g}(\tau) + \int_{l_{A}^{-1}B} d\log \chi_{g}(\tau) - \int_{AB\cdot l_{(A,B)}} d\log \chi_{g}(\tau)
$$

By Lemma 7.4 we get $\delta(1, 0) = k_{1}(g) \in H^{2}(\Gamma_{g}, Z) \cong \mathbb{Z}$. Similarly, $\delta(0, 1) = k_{2}(g) \in H^{2}(\Gamma_{g}, Z) \cong \mathbb{Z}$. This completes the proof.

**Proof of Theorem 7.1.** Since $H^{1}(\Gamma_{g}, Z)$ in the exact sequence (5), we get $H^{1}(\mathfrak{S}_{g}, Z) = \ker \delta$. By Lemma 7.5, we get $\ker \delta = 0$ for $1 \leq g \leq 3$ and $\ker \delta \cong \mathbb{Z}$ for $g \geq 4$. This completes the proof of Theorem 7.1.

**8. The value for the Dehn twist**

In this section, we shall compute the value of $\Phi_{2g}$ for the *Dehn twist*, which is defined as follows (cf. [16]). Let $\Delta \subset \mathbb{C}$ be the unit disk. Recall that the Andreotti-Mayer locus $N_{2g}$ has two irreducible components $\theta_{null, 2g}$ and $N_{g}^{c}$ by Theorem 4.3. Let $\rho: \Delta \to \mathfrak{S}_{2g}$ be a $C^{\infty}$-map such that $\rho(0) \in \theta_{null, 2g}$ is a generic point, $\rho(z) \in N_{2g}$ for $z \in \Delta \setminus \{0\}$ and $\rho(\Delta)$ intersects with $\theta_{null, 2g}$ at $\rho(0)$ transversally. For simplicity we assume that the base point $\ast$ lies in $\rho(\partial \Delta)$ and we denote the monodromy corresponding to the loop $\rho|_{\partial \Delta} : \partial \Delta \to \mathfrak{S}_{2g}$ by $\sigma_{2g} \in S_{2g}$. The element $\sigma_{2g}$ is called the Dehn twist. We put

$$
\omega : X_{2g} := \Delta \times \rho \Theta \to \Delta,
$$

which is smooth family of theta divisors over $\Delta$ induced from the universal family $\pi : \Theta \to \mathfrak{S}_{2g}$ by $\rho$. Let $\hat{\rho} : X_{2g} \to \Theta$ be the lift of the map $\rho$ defined as the projection to the second factor. By
the assumption of $\rho$ and the Theorem 4.3, $\text{Sing}(\omega^{-1}(0))$ consists of one ordinary double point and $\omega^{-1}(z)$ is a smooth theta divisor for $z \in \Delta \setminus \{0\}$. Notice that $\partial X_{2g}$ endowed with the topology induced from $X_{2g}$ is diffeomorphic to the mapping torus $M_{\sigma_{2g}}^{-1}$ endowed with the natural orientation, i.e., $\partial X_{2g} = -M_{\sigma_{2g}}$.

**Theorem 8.1.** The following equality holds:

$$\Phi_{2g}(\sigma_{2g}) = \begin{cases} -\frac{g}{2} & \text{if } g = 1, \\ -\frac{1}{(2g+1)(2g+3)} & \text{if } g > 1. \end{cases}$$

**Proof.** Put $\Delta_r := \{z \in \Delta \mid |z| < r\} \subset \Delta$ for $0 < r < 1$. We choose $\rho$ such that the restriction $\rho|_{\Delta_{1/3}} : \Delta_{1/3} \rightarrow \rho(\Delta_{1/3}) \subset \mathfrak{S}_{2g}$ is a holomorphic embedding that

$$\rho(re^{\sqrt{-1}\theta}) = \rho\left(\frac{2}{3}e^{\sqrt{-1}\theta}\right), \quad 2 < r \leq 1, \quad 0 \leq \theta < 2\pi.$$

Let $g^\Delta$ be the metric on $T\Delta$ which is a product metric near the boundary $\partial \Delta$ and coincides with the metric $\rho^*g^\Theta$ on $\Delta_{1/3}$. Let $p \in X_{2g}$ be the unique singular point on the singular fiber $X_0$. Let $g^{X_{2g}/\Delta}$ be the metric on $T(X_{2g}/\Delta)|_{X_{2g} - \{p\}}$ induced from the metric $g^\Theta/\Theta$ via the map $\rho$. Let $g^{X_{2g}}$ be the metric on $TX_{2g}$ which coincides with $g^{X_{2g}/\Delta} \oplus \omega^*g^\Delta$, where we used the connection induced from the connection $P_{g'}$ on $\Theta'$ via the map $\rho$, on $X_{2g} - \{p\}$ and coincides with the metric induced from the metric $g^{\Theta'}$ via the map $\tilde{\rho}$ on a neighbourhood of $p$. Set

$$g^{X_{2g},\varepsilon} := g^{X_{2g}} \oplus \omega^{-1}\varepsilon, \quad \varepsilon \in \mathbb{R}_{>0}.$$

By the assumption of $g^\Delta$ and the condition (24), $g^{X_{2g},\varepsilon}$ is the product metric near the boundary $\partial X_{2g}$ for $\varepsilon \in \mathbb{R}_{>0}$. By the Atiyah-Patodi-Singer index theorem,

$$\text{Sign}(X_{2g}) = \int_{X_{2g}} L(TX_{2g}, g^{X_{2g},\varepsilon}) + \eta(M_{\sigma_{2g}}, g^{M_{\sigma_{2g}},\varepsilon}).$$

Here $\partial X_{2g}$ is identified with $-M_{\sigma_{2g}}$, and $g^{M_{\sigma_{2g}},\varepsilon}$ is the restriction of $g^{X_{2g},\varepsilon}$ to the boundary $\partial X_{2g} \cong -M_{\sigma_{2g}}$. By the formula in [26], the first term of the right-hand side of (25):

$$\lim_{\varepsilon \downarrow 0} L(TX_{2g}, g^{X_{2g},\varepsilon}) = L(T(X_{2g}/\Delta), \nabla^{X_{2g}/\Delta}) + P(-t, \cdots, (-t)^{2g})|_{t^{2g}} \cdot \mu(p)_{\varepsilon}$$

Here $L(T(X_{2g}/\Delta), \nabla^{X_{2g}/\Delta})$ is only defined on $X_{2g} - \{p\}$ but has the natural smooth extension on whole $X_{2g}$. The constant $\mu(p)$ is the Milnor number of the singular point $p$, $\delta_p$ is the Dirac delta current supported at $p$ and $P(x_1, \cdots, x_{2g}) \in \mathbb{C}[x_1, \cdots, x_{2g}]$ is defined by

$$\prod_{k=1}^{2g} L(x_k) = P(\sigma_1, \cdots, \sigma_{2g}),$$

where $L(x) = x/\tanh(x)$ and $\sigma_1 = \sum_k x_k, \sigma_2 = \sum_{i>j} x_i x_j, \cdots, \sigma_{2g} = \prod_k x_k$ are the fundamental symmetric polynomials. Notice that

$$P(-t, \cdots, (-t)^{2g})|_{t^{2g}} = L^{-1}(t)|_{t^{2g}}.$$

Since $p$ is a non-degenerate critical point of $\pi : X \rightarrow \Delta$, we get $\mu(p) = 1$, which together with (25), (26) and Theorem 4.7, yields that

$$\text{Sign}(X_{2g}) = \frac{(-1)^{g}2^{g+1}(2^{2g+1} - 1)}{(g+1)(2g+1)} B_{g+1} + \frac{(-1)^{g}2^{g+2}(2^{2g+2} - 1)}{(2g+2)!} B_{g+1} + \eta^0(M_{\sigma_{2g}}, g^{M_{\sigma_{2g}},\varepsilon}).$$
By (27) and Definition 6.1, we get
\[
\Phi_{2g}(\sigma_{2g}) = \eta^{0}(M_{\sigma_{2g}}, g^{M_{\sigma_{2g}}}) + \frac{(-1)^{2g+3}}{(2g + 3)!} \int_{\Delta} p^{*}d^{c}\left(\log|\Delta_{2g}(\tau)|^{2}(\det Im\tau)^{2g+2}\right)
\]
\[
= -\frac{(-1)^{2g+3}}{(2g + 2)!} \int_{\Delta} p^{*}d^{c}\log|\Delta_{2g}(\tau)|^{2}
\]
\[
= \frac{(-1)^{g+1}(2g+1)2^{2g+2}(2^{2g+2}-1)}{(2g+3)!} \int_{\Delta} p^{*}d^{c}\log|\Delta_{2g}(\tau)|^{2} + \text{Sign}(X_{2g})
\]
where we used the Poincaré-Lelong formula and Theorem 4.4 to get the last equality. When \( g = 1 \), since the singular fiber has two irreducible components and \( \text{Sign}(X_{2}) = -1 \), we obtain the proof for the case \( g = 1 \). We complete the proof by the following Lemma in the case \( g > 1 \).

**Lemma 8.2.** Let \( \pi : \mathcal{X} \rightarrow \Delta \) be a Lefschetz degeneration of relative dimension \( 2n - 1 \), i.e., \( \pi \) is a proper holomorphic surjective map from a \( 2n \)-dimensional complex manifold \( \mathcal{X} \) to the unit disk \( \Delta \) and there is a point \( p \in \mathcal{X} \) and an open neighbourhood \( p \in U \cong \{(z_{1}, \cdots , z_{2n})\in \mathbb{C}^{2n} | \sum_{k=1}^{2n}|z_{k}|^{2} < 1\} \) such that
\[
\pi(z_{1}, \cdots , z_{2n}) = \sum_{k=1}^{2n} z_{k}^{2}, \quad (z_{1}, \cdots , z_{2n}) \in U
\]
and \( \pi_{*} \) has maximal rank on \( \mathcal{X} \setminus p \). Assume that \( n > 1 \). Then \( \text{Sign}(\mathcal{X}) = 0 \).

**Proof.** For \( \in \Delta \), we set \( U_{t} := \mathcal{X}_{t} \cap U \). Then a sequence of inclusions
\[
\mathcal{X}_{0} \setminus U_{0} \subset \mathcal{X}_{0} \setminus \{p\} \subset \mathcal{X}_{0} \subset \mathcal{X}
\]
induces a sequence of isomorphisms:
\[
H_{2n}(\mathcal{X}_{0} \setminus U_{0}, Z) \cong H_{2n}(\mathcal{X}_{0} \setminus \{p\}, Z) \cong H_{2n}(\mathcal{X}_{0}, Z) \cong H_{2n}(\mathcal{X}, Z).
\]
Here the first isomorphism follows from the homotopy equivalence of \( \mathcal{X}_{0} \setminus U_{0} \) and \( \mathcal{X}_{0} \setminus \{p\} \), the second isomorphism follows from the fact \( \text{codim}_{\mathcal{X}}(p)/\mathcal{X}_{0} = 4n - 2 > 2n + 1 \), and the third isomorphism follows from the fact that the inclusion \( \mathcal{X}_{0} \rightarrow \mathcal{X} \) is a deformation retraction. By Ehresman’s Theorem, \( \mathcal{X} \setminus U \) is diffeomorphic to \( (\mathcal{X}_{0} \setminus U_{0}) \times \Delta \) as a fiber bundle over \( \Delta \). Since \( \Delta \) is contractible, the inclusion \( \mathcal{X}_{t} \setminus U_{t} \hookrightarrow \mathcal{X} \setminus U \) induces an isomorphism \( H_{2n}(\mathcal{X}_{t} \setminus U_{t}, Z) \cong H_{2n}(\mathcal{X} \setminus U, Z) \). By (28), the inclusion \( \mathcal{X}_{t} \setminus U_{t} \hookrightarrow \mathcal{X} \) induces an isomorphism \( H_{2n}(\mathcal{X}_{t} \setminus U_{t}, Z) \hookrightarrow H_{2n}(\mathcal{X}, Z) \). Hence, for any \( t \in \Delta \), any element of \( H_{2n}(\mathcal{X}, Z) \) can be represented by a cycle contained in \( \mathcal{X}_{t} \). Therefore the intersection matrix of \( H_{2n}(\mathcal{X}, Z) \) is trivial and \( \text{Sign}(\mathcal{X}) = 0 \). This completes the proof.

**Remark 8.3.** When \( g = 1 \), \( \sigma_{2} \in \Lambda_{2} \) is the Dehn twist along a separating simple closed curve on a Riemann surface of genus two. Since \( \text{Sign}(X_{2}) = -1 \) and \( B_{2} = \frac{1}{8} \), we obtain \( \Phi_{2}(\sigma_{2}) = \Phi_{2}(\sigma_{2}) = -\frac{4}{8} \), which confirms a result of Matsumoto ([19]).

**References**


