ADIABATIC LIMITS OF $\eta$-INVARIANTS AND THE MEYER FUNCTION FOR SMOOTH THETA DIVISORS

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1. Introduction

Let $\Sigma_g$ be a closed oriented surface of genus $g$ and let $\mathcal{M}_{g}$ be the mapping class group of genus $g$, namely the group of all isotopy classes of orientation-preserving diffeomorphisms of $\Sigma_g$. Meyer introduced a cocycle $\tau_g: \mathcal{M}_{g} \times \mathcal{M}_{g} \to \mathbb{Z}$, called the signature cocycle or the Meyer cocycle, and he gave a signature formula for the signature of surface bundles over surfaces ([21]). Let $[\tau_g] \in H^2(\mathcal{M}_g, \mathbb{Z})$ denotes the cohomology class of $\tau_g$. When $g = 1$, since $\mathcal{M}_1 = SL_2(\mathbb{Z})$, $H^1(SL_2(\mathbb{Z}), \mathbb{Z}) = 0$ and $3[\tau_1] = 0$, there exists a unique 1-cocycle $\phi_1: SL_2\mathbb{Z} \to \frac{1}{3}\mathbb{Z}$ such that cobounds $\tau_1$. The function $\phi_1$ is called the Meyer function of genus one, which has the following property: Let $\pi: Z \to X$ be a $\Sigma_1$-bundle over a compact oriented surface with boundary $\partial Z = c_1 \cup \cdots \cup c_k$. Let $A_1, \cdots, A_k$ be the monodromies around each component of the boundary. Since the Picard-Lefschetz transformation along $c_i$ is an automorphism of $H^1(\Sigma_1, \mathbb{Z})$ preserving the intersection form, one has $A_i \in SL_2(\mathbb{Z})$ by fixing a symplectic basis of $H^1(\Sigma_1, \mathbb{Z})$. Then the signature of $Z$, which is defined as the signature of the cup-product pairing on $H^2(Z, \partial Z, \mathbb{R})$, satisfies

\begin{equation}
\text{Sign}(Z) = \sum_{i=1}^{k} \phi_1(A_i).
\end{equation}

The explicit formula of $\phi_1$ was obtained by Meyer ([21]).

When $g = 2$, since $5[\tau_2] = 0 \in H^2(\mathcal{M}_2, \mathbb{Z}) \cong \mathbb{Z}/10\mathbb{Z}$ and $H^1(\mathcal{M}_2, \mathbb{Z}) = 0$, there exists a unique 1-cocycle $\phi_2: \mathcal{M}_2 \to \frac{1}{3}\mathbb{Z}$ satisfying (1), for every $\Sigma_2$-bundles over compact oriented surfaces. The function $\phi_2$ is called the Meyer function of genus two.

In [1], Atiyah investigated the Meyer function $\phi_1$ from the several view points. For an odd dimensional closed oriented Riemannian manifold $M$, let $\eta(M)$ be the $\eta$-invariant of $M$ with respect to the signature operator of $M$ [2]. For $\sigma \in SL_2\mathbb{Z}$, let $\pi: M_\sigma \to S^1$ be the mapping
torus associated with $\sigma$, i.e., $\Sigma_1$-bundle over $S^1$ with monodromy $\sigma$. Then Atiyah showed the following identity, when $M_\sigma$ is equipped with a certain metric:

\[ \phi_1(\sigma) = \eta(M_\sigma) \]

Moreover, he gave several interpretation of $\phi_1$ interms of the following quantities: (1) Hirzebruch's signature defect; (2) the transformation lows of the logarithm of the Dedekind $\eta$-function; (3) the logarithm of the monodromy of Quillen's line bundle; (4) the special value of the Shimizu $L$-function at the origin.

In this note, we study an extension of the result of Atiyah to the case $g = 2$ and higher dimensional manifold. We shall construct a higher dimensional analogue of the Meyer function for smooth theta divasors of odd dimension.

Notation: For a complex manifold $M$, $T^{1,0}M$ (resp. $T^{0,1}M$) denotes the holomorphic (resp. anti-holomorphic) tangent bundle and $TM$ denotes the real tangent bundle. We set $\omega := \frac{1}{4\pi \sqrt{-1}}(\partial - \overline{\partial})$. Hence $d\omega = \frac{1}{2\pi} \partial \overline{\partial}$. \n
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2. Preliminaries from Riemannian geometry

In this section, we recall some results of Riemannian geometry which will be used in the proof of the main theorem. Following [10], we define connections of fiber bundles and the connection of relative tangent bundles. Let $M$ be a manifold and let $\pi : Z \rightarrow B$ be a fiber bundle with typical fiber $M$.

The relative tangent bundle $T(Z/B)$ is the subbundle of $TZ$ defined by

\[ T(Z/B) := \text{Ker}(\pi_* : TZ \rightarrow \pi^* TB). \]

A vector of $T(Z/B)$ is said to be vertical.

Definition 2.1. A subbundle $T_H Z \subset TZ$ with $TZ = T(Z/B) \oplus T_H Z$ is called a connection of the fiber bundle $\pi : Z \rightarrow B$.

For a connection, one has $T_H Z \cong \pi^* TB$ via the projection $\pi_* : TZ \rightarrow \pi^* TB$. A vector of $T_H Z$ is said to be horizontal.

When $Z$ is trivial, i.e., $Z = M \times B$, $TZ$ is naturally isomorphic to the direct sum $(pr_1)^* TM \oplus (pr_2)^* TB$. This connection is called the trivial connection of the trivial fiber bundle.

Given a connection, one can define the projection $P_Z : TZ \rightarrow T(Z/B)$ with kernel $T_H Z$. We often identify $P_Z$ with the corresponding connection $T_H Z := \text{Ker}(P_Z)$. In the rest of Section 2, we fix a connection $T_H Z$, or equivalently $P_Z$. One can define the pull-back of a connection, as follows: Let $B'$ be a manifold and let $h : B' \rightarrow B$ be a $C^\infty$ map. The fiber product $Z' := Z \times_B B'$ is given by $\{ (x, b) \in Z \times B' \mid \pi(x) = h(b) \}$ satisfies the following commutative diagram:

\[
\begin{array}{ccc}
Z' & \stackrel{\tilde{h}}{\longrightarrow} & Z \\
\downarrow{\pi'} & & \downarrow{\pi} \\
B' & \stackrel{h}{\longrightarrow} & B
\end{array}
\]

Since the map $P_Z \circ \tilde{h}_* : TZ' \rightarrow h^* T(Z/B)$ is surjective, $\text{Ker}(P_Z \circ \tilde{h}_*)$ is a subbundle of $TZ'$. Since $T(Z'/B')$ is canonically isomorphic to $h^* T(Z/B)$, the map $P_Z \circ \tilde{h}_*$ is identified with a projection from $TZ'$ to $T(Z'/B')$.\n
Definition 2.2. The connection of \( \pi' : Z' \rightarrow B' \) induced from \( T_HZ \) by \( h \) is defined by

\[
T_HZ' := \ker(P_Z \circ \tilde{h}_* : T(Z'/B) \rightarrow T(Z/B)),
\]
under the identification between \( T(Z'/B') \) and \( h^*T(Z/B) \). The projection corresponding to \( T_HZ' \) is denoted by \( h^*P_Z \).

We fix a metric \( g^{Z/B} \) on the relative tangent bundle, a Riemannian metric \( g^B \) on \( B \), and the connection \( T_HZ \) and the corresponding projection \( P_Z \). We define the Riemannian metric \( g' \) on the total space \( Z \) by

\[
g' := g^{Z/B} \oplus \pi^*g^B,
\]
under the isomorphism \( TZ \cong T(Z/B) \oplus T_{H}Z \cong T(Z/B) \oplus \pi^*TB \). Let \( \nabla^Z \) be the Levi-Civita connection of \((Z, g^Z)\). We define the connection \( \nabla' \) on \( T(Z/B) \) by

\[
\nabla' := \pi^*\nabla^B.
\]
Then \( \nabla' \) preserves the metric \( g^{Z/B} \).

Lemma 2.3. The connection \( \nabla' \) is independent of a choice of \( g^B \)

Proof. See [10, Proposition 10.2]

Lemma 2.4. Let \( B' \) be a manifold and let \( h : B' \rightarrow B \) be a \( C^\infty \)-map, and set \( Z' := Z \times_B B' \). Let \( g^{Z/B} \) be the metric on \( T(Z'/B') \) induced from \( g^{Z/B} \), and let \( P_Z' := h^*P_Z \) be the connection of \( Z' \) induced from \( P_Z \). Then \( \nabla' = h^*\nabla' \).

Proof. See [15]

With respect to the decomposition \( TZ = T(Z/B) \oplus T_HZ \), We put for \( \epsilon \in \mathbb{R}^+ \)

\[
g^{Z,\epsilon} := g^{Z/B} \oplus \epsilon^{-1}\pi^*g^B.
\]
The Levi-Civita connections of \((Z, g^{Z,\epsilon})\) and \((B, g^B)\) are denoted by \( \nabla^{Z,\epsilon} \) and \( \nabla^B \) respectively. Let \( R^{Z,\epsilon} \) and \( R^B \) be the curvature of \( \nabla^{Z,\epsilon} \) and \( \nabla^B \), respectively. Then \( g := g^{Z,1} \) and \( \nabla := \nabla^{Z,1} \). We define another connection \( \nabla \) on \( Z \) by

\[
\nabla := \nabla^{Z/B} \oplus \pi^*\nabla^B,
\]
and we put

\[
S^{(\epsilon)} := \nabla^{Z,\epsilon} - \nabla \in A^1(\text{End}(TZ)), \quad S := S^{(1)}.
\]
Then \( \nabla \) preserves the Riemannian metric \( g^{Z,\epsilon} \), and \( P_Z \) is parallel with respect to \( \nabla \), i.e. \( \nabla \circ P_Z - P_Z \circ \nabla = 0 \).

Let \( \{e_1, \cdots, e_k\} \) be a local orthogonal framing for \((T(Z/B), g^{Z/B})\), and let \( \{f_1, \cdots, f_l\} \) be a local orthogonal framing for \((T_HZ, \pi^*g^B)\).

Proposition 2.5. With respect to the splitting \( TZ = T(Z/B) \oplus T_HB \), the following identity holds:

\[
\lim_{\epsilon \to 0} R^{Z,\epsilon} = \begin{pmatrix} R^{Z/B} & P_Z(\nabla S) \\ 0 & \pi^*R^B \end{pmatrix}.
\]

Proof. See [7] (3.195)
3. $\eta$-invariants

In this section, we recall the definition and some properties of $\eta$-invariants. Let $(M, g^M)$ be a coed oriented Riemannian manifold of dimension $(2l - 1)$. Denote the space of $C^\infty$ $k$-forms on $M$ by $\mathcal{A}^k(M)$. Let $*: \mathcal{A}^k(M) \to \mathcal{A}^{2l-k-1}(M)$ be the Hodge star operation with respect to $g^M$. The signature operator $D: \oplus_{p \geq 0} \mathcal{A}^{2p}(M) \to \oplus_{p \geq 0} \mathcal{A}^{2p}(M)$ of $M$ is defined by

$$D: \omega \mapsto (-1)^{l}(1)^{p+1}(d^*d^*)\omega, \quad \omega \in \mathcal{A}^{2p}(M).$$

Then $D$ is an elliptic self-adjoint differential operator of first order acting on $\oplus_{p \geq 0} \mathcal{A}^{2p}(M)$. Let $\sigma(D)$ be the spectrum of $D$. The $\eta$-function of $M$ is defined by

$$\eta(s) := \sum_{\lambda \in \sigma(D) \setminus \{0\}} \frac{\text{sign} \lambda}{\lambda^s},$$

for $s \in \mathbb{C}$ with $\text{Re}(s) > 0$. Then $\eta(s)$ extends meromorphically to $\mathbb{C}$ and is holomorphic at $s = 0$ by [2], [7].

**Definition 3.1.** The real number $\eta(0)$ is called the $\eta$-invariant of $(M, g^M)$ and is denoted by $\eta(M, g^M)$.

Let $(X, g^X)$ be a $4k$-dimensional, oriented, compact, Riemannian manifold with boundary $Y$. Put $g^X := g^X|_Y$ and fix a color neighborhood $U \supset Y$ such that $U \cong Y \times [0, 1)$. Assume that $g^X|_U = g^Y \oplus dt^2$ under the above isomorphism. Let $\nabla^L$ be the Levi-Civita connection of $(X, g^X)$.

**Theorem 3.2** (Atiyah-Patodi-Singer [2]). The following equation holds:

$$\text{Sign}(X) = \int_X L(TX, \nabla^L) - \eta(Y, g^Y)$$

Here $L$ denotes the Hirzebruch $L$-polynomial, which is a multiplicative genus associated with the power series: $L(x) := x/\tanh(x)$.

Let $X$, $B$ and $M$ be closed oriented manifolds. Let $\pi: X \to B$ be a $C^\infty$-submersion, whose fibers are isomorphic to $M$. Assume that $\dim X = 4k$. Let $g^{X/B}$ be a metric on $TX/B$ and let $g^B$ be a metric on $TB$. Let $T_H X \subset TX$ be a connection. We identify $T_H X$ with $\pi^*TB$ via $\pi$. With respect to the decomposition $TX = T(X/B) \oplus \pi^*TB$, we define the metric on $X$ by $g^X := g^{X/B} \oplus \pi^*g^B$ and we consider the one parameter family of metrics on $X$ defined by

$$g^{X, \varepsilon} := g^{X/B} \oplus \varepsilon^{-1} \pi^*g^B, \quad \varepsilon \in \mathbb{R}^+.$$

**Theorem 3.3** (Bismut-Cheeger, [6]). The limit $\lim_{\varepsilon \to 0} \eta(X, g^{X, \varepsilon})$ exists.

The limit $\lim_{\varepsilon \to 0} \eta(X, g^{X, \varepsilon})$ is called the adiabatic limit of the $\eta$-invariants and is denoted by $\eta^0(X)$. By definition, $\eta^0(X, g^X)$ depends on the three data: $g^{X/B}$, $g^B$ and $T_H X$.

4. Family of smooth theta divisors

We fix the following notation. Let $\mathfrak{S}_g$ be the Siegel upper-half space of degree $g$ and let $\Gamma_g$ be the integral symplectic group, i.e.,

$$\mathfrak{S}_g := \{ \tau \in \text{M}(g, \mathbb{C}) \mid \text{Im} \tau > 0 \},$$

$$\Gamma_g := \{ \gamma \in \text{GL}(2g, \mathbb{Z}) \mid \gamma J_g \tau J_g = J_g \},$$

where $J_g = \left( \begin{smallmatrix} 0 & 1_g \\ -1_g & 0 \end{smallmatrix} \right)$ and $1_g$ denotes the $g \times g$ identity matrix. $\Gamma_g$ acts on $\mathfrak{S}_g$ by

$$\gamma \cdot \tau := (A \tau + B)(C \tau + D)^{-1}, \quad \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g, \quad \tau \in \mathfrak{S}_g.$$
For \( \tau \in \mathcal{G}_g \), write \( \tau = t(\tau_1, \cdots, \tau_g) \) and set
\[
\Lambda_{\tau} := \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_g + \mathbb{Z}\tau_1 + \cdots + \mathbb{Z}\tau_g \subset \mathbb{C}^g
\]
where \( 1_g = t(e_1, \cdots, e_g) \) and \( \tau = t(\tau_1, \cdots, \tau_g) \in \mathcal{G}_g \). Define the \( \mathbb{Z}^{2g} \)-action on \( \mathbb{C}^g \times \mathcal{G}_g \) by
\[
(m, n) \cdot (z, \tau) := (z + m\tau + n, \tau), \quad (z, \tau) \in \mathbb{C}^g \times \mathcal{G}_g, \quad m, n \in \mathbb{Z}^{2g}.
\]
Then
\[
f : A_g := (\mathbb{C}^g \times \mathcal{G}_g)/\mathbb{Z}^{2g} \to \mathcal{G}_g
\]
is the universal family of principally polarized Abelian varieties over \( \mathcal{G}_g \), whose fiber at \( \tau \) is \( A_{\tau} := \mathbb{C}^g/\Lambda_{\tau} \). For \( (a, b) \in \mathbb{R}^{2g} \), \( z \in \mathbb{C}^g \) and \( \tau \in \mathcal{G}_g \) we define the theta function with characteristic by
\[
\vartheta_{a,b}(z, \tau) := \sum_{n \in \mathbb{Z}^g} e\left(\frac{1}{2}(n + a)^t(n + a) + (n + a)^t(z + b)\right),
\]
where \( e(t) = \exp(2\pi \sqrt{-}1t) \). Let
\[
f : \Theta_{a,b} := \{(z, \tau) \in A_g | \vartheta_{a,b}(z, \tau) = 0\} \to \mathcal{G}_g.
\]
be the universal family of theta divisors. For simplicity we write \( \vartheta \) for \( \vartheta_{0,0} \) and set \( \Theta = \Theta_{0,0} \).

On \( A_g \), \( \Gamma_g \) acts by
\[
\gamma : (z, \tau) := (z + m\tau + n, \tau), \quad (m, n) \in \mathbb{Z}^{2g} \subset \mathbb{R}^{2g}.
\]
Then \( t_{(m,n)} \) has no fixed points when \( (m, n) \in \mathbb{R}^{2g} \setminus \mathbb{Z}^{2g} \) and the subgroup \( \mathbb{Z}^{2g} \subset \mathbb{R}^{2g} \) acts trivially on \( A_g \). For \( \gamma = (A B) \), we define
\[
\tilde{\gamma} := t_{(m,n)} \circ \gamma \in \text{Aut}(A_g), \quad (m, n) := \frac{1}{2}((C^tD)_0, (A^tB)_0).
\]
Then \( \tilde{\gamma} \) preserves the family \( f : \Theta \to \mathcal{G}_g \).

**Proposition 4.1.** For any \( \gamma_1, \gamma_2 \in \Gamma_g \),
\[
\tilde{\gamma_1} \circ \tilde{\gamma_2} = \tilde{\gamma_1 \gamma_2}
\]

**Proof.** See [15] \[\square\]

We set
\[
g^{A_2/\mathcal{G}_g} := dx \cdot (\text{Im} \tau)^{-1} \cdot t dz.
\]
Then \( g^{A_2/\mathcal{G}_g} \) is a \( \Gamma_g \)-invariant Hermitian metric on the relative tangent bundle \( T(A_2/\mathcal{G}_g) \). The next purpose of this section is to construct a \( \Gamma_g \)-invariant Kähler metric on \( TA_2 \) such that \( g^{A_2/\mathcal{G}_g}|_{A_\tau} = dx \cdot (\text{Im} \tau)^{-1} \cdot t dz \) for all \( \tau \in \mathcal{G}_g \).

Put \( T^{2g} := \mathbb{R}^{2g} \setminus \mathbb{Z}^{2g} \). Define a \( \mathbb{Z}^{2g} \)-action on \( \mathbb{R}^{2g} \times \mathcal{G}_g \) by \( (m, n) \cdot (x, y, \tau) := (x + m, y + n, \tau) \) for \( (m, n) \in \mathbb{Z}^{2g} \), \( (x, y) \in \mathbb{R}^{2g} \), \( \tau \in \mathcal{G}_g \). Then \( (\mathbb{R}^{2g} \times \mathcal{G}_g)/\mathbb{Z}^{2g} \) is the trivial \( T^{2g} \)-bundle \( T^{2g} \times \mathcal{G}_g \).

We define a \( C^\infty \)-map \( \tilde{\rho} : \mathbb{R}^{2g} \times \mathcal{G}_g \to \mathbb{C}^g \times \mathcal{G}_g \) by
\[
\tilde{\rho}(x, y, \tau) := (x \tau + y, \tau), \quad x, y \in \mathbb{R}^g, \quad \tau \in \mathcal{G}_g.
\]
Since \( \tilde{\rho} \) is a \( \mathbb{Z}^{2g} \)-equivariant map, \( \tilde{\rho} \) induces a \( C^\infty \)-isomorphism \( \rho : T^{2g} \times \mathcal{G}_g \to A_g \) as \( T^{2g} \)-bundles over \( \mathcal{G}_g \). Define a \( \Gamma_g \)-action on \( T^{2g} \times \mathcal{G}_g \) by
\[
\gamma \cdot ((x, y), \tau) := ((x, y)\gamma^{-1}, \gamma \cdot \tau), \quad \gamma \in \Gamma_g.
\]
Then for any $\gamma \in \Gamma_g$, the following diagram is commutative.

$$
\begin{array}{ccc}
T^{2g} \times \mathfrak{S}_g & \xrightarrow{\rho} & \mathrm{A}_g \\
\gamma & \downarrow & \\
T^{2g} \times \mathfrak{S}_g & \xrightarrow{\rho} & \mathrm{A}_g
\end{array}
$$

Since the trivial connection on $T^{2g} \times \mathfrak{S}_g$ is $\Gamma_g$-invariant, $A_g$ has the induced $\Gamma_g$-invariant connection $T_H \mathrm{A}_g \subset T \mathrm{A}_g$ via the $\Gamma_g$-equivariant isomorphism $\rho$. We denote the $\Gamma_g$-equivariant projection corresponding to $T_H \mathrm{A}_g$ by $P_\rho$. Let $P^C_\rho : T \mathfrak{S}_g \otimes \mathbb{C} \to T(\mathrm{A}_g/\mathfrak{S}_g) \otimes \mathbb{C}$ be the complexification of $P_\rho$. Then $P^C_\rho$ is also $\Gamma_g$-equivariant.

Under the projection, the horizontal lift of a $(1,0)$ (resp. $(1,0)$) tangent vector is a $(1,0)$ (resp. $(1,0)$) tangent vector. Therefore the extension $P_\rho^C : T \mathfrak{S}_g \otimes \mathbb{C} \to T(\mathrm{A}_g/\mathfrak{S}_g) \otimes \mathbb{C}$ decomposes

$$
P^C_\rho = P^{1,0}_\rho \oplus P^{0,1}_\rho,$$

under the isomorphism $T \mathfrak{S}_g \otimes \mathbb{C} = T^{1,0} \mathfrak{S}_g \otimes T^{0,1} \mathfrak{S}_g$ and $T(\mathrm{A}_g/\mathfrak{S}_g) \otimes \mathbb{C} = T^{1,0}(\mathrm{A}_g/\mathfrak{S}_g) \otimes T^{0,1}(\mathrm{A}_g/\mathfrak{S}_g)$.

Hence $P_\rho$ induces a $\Gamma_g$-equivariant $C^\infty$-isomorphism

$$
T^{1,0} \mathrm{A}_g \cong T^{1,0}(\mathrm{A}_g/\mathfrak{S}_g) \otimes \mathcal{F}^*T^{1,0} \mathfrak{S}_g.
$$

Let $g^{\mathfrak{S}_g}$ be the Bergman metric on $\mathfrak{S}_g$ with Kähler form

$$
\omega_{\mathfrak{S}_g} = -2\sqrt{-1} \partial \overline{\partial} \log \det \mathrm{Im} \tau.
$$

Then $g^{\mathfrak{S}_g}$ is $\Gamma_g$-invariant. Using the $\Gamma_g$-equivariant isomorphism (3), we define the $\Gamma_g$-invariant Hermitian metric $g^{\mathfrak{A}_g}$ on $T \mathfrak{S}_g$ by

$$
g^{\mathfrak{A}_g} := g^{\mathfrak{S}_g} \otimes f^* g^{\mathfrak{S}_g}.
$$

**Theorem 4.2.** The Hermitian metric $g^{\mathfrak{A}_g}$ is Kähler.

**Proof.** See [15] \qed

We put

$$
A_k(\Gamma_g, \chi) = \{ f \in \mathcal{O}(\mathfrak{S}_g) \mid f(\gamma \cdot \tau) = j(\tau, \gamma)^k \chi(\gamma) f(\tau), \ \gamma \in \Gamma_g \}
$$

where $\chi$ is a character of $\Gamma_g$ and $j(\tau, \gamma) = \det(CT + D)$ for $\gamma \in \left( \begin{array}{cc} A & B \\ C & D \end{array} \right)$. An element of $A_k(\Gamma_g, \chi)$ is called a Siegel modular form of weight $k$ with character $\chi$. In particular, an element of $A_k(\Gamma_g, 1)$ is called a Siegel modular form. Let $\mathcal{F}^k_g := \mathfrak{S}_g \times \mathbb{C}^g$ be the trivial holomorphic line bundle over $\mathfrak{S}_g$ with the $\Gamma_g$-action

$$
\gamma \cdot (\tau, \xi) := (\gamma \cdot \tau, j(\tau, \gamma)^k \xi).
$$

A Siegel modular form of weight $k$ is regarded as a $\Gamma_g$-invariant holomorphic section of $\mathcal{F}^k_g$. Define the Peterson metric on $\mathcal{F}^k_g$ by

$$
||\xi||^2_{\mathcal{F}^k_g} := (\det \mathrm{Im} \tau)^k ||\xi||^2, \quad (\tau, \xi) \in \mathcal{F}^k_g.
$$

By the automorphic property of $\det \mathrm{Im}(\gamma \cdot \tau) = |j(\tau, \gamma)|^{-2} \det \mathrm{Im} \tau$, we see that $||\cdot||_{\mathcal{F}^k_g}$ is $\Gamma_g$-invariant.

Let $\mathcal{N}_g := \{ \tau \in \mathfrak{S}_g \mid \text{Sing} \Theta_{\tau} \neq \emptyset \}$ be the Andreotti-Mayer locus, which is the locus of Abelian varieties whose theta divisors is singular. The followings are known for the locus $\mathcal{N}_g$. 


Theorem 4.3 ([12]). \( N_g \) is a divisor of \( \mathcal{G}_g \), consisting of two irreducible components as a divisor of the modular variety \( \Gamma_g \setminus \mathcal{G}_g \):

\[
N_g = \theta_{\text{null}, g} + 2N'_g.
\]

Here \( \theta_{\text{null}, g} \) is the zero divisor of Igusa's modular form \( \chi_g(\tau) \) which is the Siegel modular form of weight \( 2g^2 - 2(2g + 1) \) defined as the product of all even theta constants and \( N'_g = 0 \) for \( g = 2, 3 \). For a generic point \( \tau \in \theta_{\text{null}, g} \), \( \text{Sing}(\Theta) \) consists of one ordinary double point.

Theorem 4.4 ([25]). There is a Siegel cusp form \( \Delta_g(\tau) \) of weight \( \frac{(g+3)g!}{2} \) with zero divisor \( N_g \). By the Proposition 4.3, this implies that there exists \( J_g(\tau) \) which is a Siegel modular form of weight \( \frac{(g+3)g!}{4} - 2g^2 - 2g + 1 \) with zero divisor \( N'_g \) such that

\[
\Delta_g := \chi_g(\tau)J_g(\tau)^2.
\]

We put \( \mathcal{G}'_g := \mathcal{G}_g - N_g \), \( \Theta'_g := \Theta|_{\mathcal{G}'_g} \). Then \( f : \Theta' \rightarrow \mathcal{G}'_g \) is a family of smooth theta divisors. Endow \( T^{1,0}(\Theta'/\mathcal{G}'_g) \) the Hermitian metric \( g^{\Theta'/\mathcal{G}'_g} : = g^{\mathcal{A}_g}|_{\Theta}' \). Let \( g^{\Theta'} : = g^{\mathcal{A}_g}|_{\Theta}' \) be the restriction of the Kähler metric \( g^{\mathcal{A}_g} \). Consider \( g^{\Theta'/\mathcal{G}'_g} \) as Riemannian metric on \( T(\Theta'/\mathcal{G}'_g) \) and \( T\Theta' \).

Let

\[
T_{H}\Theta' := (T(\Theta'/\mathcal{G}'_g))_{1} = \nabla^{H}_{\Theta'/\mathcal{G}'_g} - \nabla^{h}_{\Theta'/\mathcal{G}'_g}
\]

be the orthogonal complement of \( T(\Theta'/\mathcal{G}'_g) \) in \( T\Theta' \), which induces a connection \( P_{\Theta'} : T\Theta' \rightarrow T\Theta'/\mathcal{G}'_g \).

Hence we obtain the connection \( \nabla^{\Theta'/\mathcal{G}'_g} \) on \( T(\Theta'/\mathcal{G}'_g) \) by using \( g^{\Theta'/\mathcal{G}'_g} \) and \( P_{\Theta'} \) as in Section 2.2. Let \( \nabla^{h} \) be the holomorphic Hermitian connection on \( T^{1,0}(\Theta'/\mathcal{G}'_g) \) with respect to the Hermitian metric \( g^{\Theta'/\mathcal{G}'_g} \).

Lemma 4.5. Under the \( C^{\infty} \)-isomorphism \( T(\Theta'/\mathcal{G}'_g) \otimes C \cong T^{1,0}(\Theta'/\mathcal{G}'_g) \otimes T^{0,1}(\Theta'/\mathcal{G}'_g) \), the following equality of connections holds.

\[
\nabla^{\Theta'/\mathcal{G}'_g} \otimes C = \nabla^{h} \otimes \nabla^{h}
\]

Proof. Let \( \nabla^{L} \) be the Levi-Civita connection on \( TA_{g} \) and let \( \nabla^{H} \) be the holomorphic Hermitian connection on \( T^{1,0}A_{g} \). Since \( g^{\mathcal{A}_g} \) is Kähler, the following equality holds ([18])

\[
\nabla^{L} \otimes C = \nabla^{H} \otimes \nabla^{H}
\]

under the isomorphism \( TA_{g} \otimes C = T^{1,0}A_{g} \otimes T^{0,1}A_{g} \).

By (2), we get

\[
\nabla^{\Theta'/\mathcal{G}'_g} \otimes C = (P_{\rho} \nabla^{L} P_{\rho}) \otimes C = (P_{\rho} \nabla^{L} \otimes C) P_{\rho} = P_{\rho}^{1,0} \nabla^{H} P_{\rho}^{1,0} + P_{\rho}^{0,1} \nabla^{H} P_{\rho}^{0,1}.
\]

Since \( P_{\rho}^{1,0} \nabla^{H} P_{\rho}^{1,0} = \nabla^{h} \) (see [18] Capter I, Section 6), we get the result. \( \square \)

Let \( g_{1_{g}} \) be the restriction of the Hermitian metric \(|dz|^{2} \) on \( TA_{g} / \mathcal{G}_g \) to the relative tangent bundle \( T\Theta'/\mathcal{G}_g \). Let \( F(T\Theta'/\mathcal{G}_g, g_{1_{g}}) \) be the corresponding Chern-Weil form for \( F(x) \) and the holomorphic Hermitian connection of \( (T\Theta'/\mathcal{G}_g, g_{1_{g}}) \).

Proposition 4.6 ([24], Proposition 2.1). The following equality holds:

\[
[F(T\Theta'/\mathcal{G}_g, g_{1_{g}})]^{(g, g)} \equiv 0.
\]

In particular one has

\[
[f,F(T\Theta'/\mathcal{G}_g, g_{1_{g}})]^{(1,1)} \equiv 0.
\]
Let \( \|\Delta_{2g}(\tau)\|^2 := (\det \text{Im}\tau)^{(2g+3)(2g+2)!} \|\Delta_{2g}(\tau)\|^2 \) denote the Peterson norm of the Siegel modular form \( \Delta_{2g}(\tau) \) and let \( B_k \) be the \( k \)-th Bernoulli number, i.e.,

\[
\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \sum_{k=1}^{\infty} (-1)^{k+1} B_k \frac{x^{2k}}{(2k)!}.
\]

**Theorem 4.7.** The following equality holds:

\[
\left[ f_\ast L(T(\mathcal{G}/\mathcal{S}_{g'}), \nabla^{g}/\mathcal{S}_{g'}) \right]^{(2)} = \frac{(-1)^{g_2 g_1 + 1} (2g_2 + 2 - 1)}{(2g + 1)(g + 1)} B_{g+1} dd^c \log \det \text{Im}\tau
\]

\[
= \frac{(-1)^{g_2 g_1 + 3} (2g_2 + 2 - 1)}{(2g + 3)!} B_{g+1} dd^c \log \|\Delta_{2g}(\tau)\|^2.
\]

By Lemma 4.5 and the fact that \((\nabla^h)^2\) is a \((1, 1)\)-form, we see that the left-hand side is equal to \([ f_\ast L(T^{1,0}(\mathcal{G}/\mathcal{S}_{g}), \nabla^h)]^{(1,1)}\). By Proposition 4.6 we obtain

\[
[L(T^{1,0}(\mathcal{G}/\mathcal{S}_{g}), \nabla^h)]^{(1,1)} = -dd^c f_\ast [\tilde{L}(T^{1,0}(\mathcal{G}/\mathcal{S}_{g}), g_{12}, g^{0}/\mathcal{S}_{g})]^{2g-1,2g-1}.
\]

Hence we deduced the proof to the computation of the Bott-Chern form and we can compute it by using the same idea in [25]. Since this is rather complicated, we omit the proof.

**Remark 4.8.** In Section 5, it will be crucial that \(dd^c \|\Delta_{g}(\tau)\|^2\) is \( \Gamma_g \)-invariant and that \(dd^c \|\Delta_{g}(\tau)\|^2\) is an exact form as a \((2, 0)\)-form on \( \Gamma'_g \backslash \mathcal{S}_g \).

5. The signature cocycle for smooth theta divisors

Since \( \Gamma_g \) acts on \( \mathcal{S}_g \) properly discontinuously the space \( \Gamma_g \backslash \mathcal{S}_g \) has naturally orbifold structure and can be regarded as the moduli space of smooth theta divisors. We shall consider the orbifold fundamental group of \( \Gamma_g \backslash \mathcal{S}_g \) and construct a 2-cocycle of this group.

In the rest of this section we fix a generic base point \(* \in \mathcal{S}_g\), i.e., \(*\) satisfies \(\{ \gamma \in \Gamma_g | \gamma \ast = \ast\} = \{ \pm 1_{2g}\} \). Let \((B, b)\) be a topological space with a base point and let \(\pi : \tilde{B} \rightarrow B\) be the universal covering. Then the fundamental group \(\pi_1(B, b)\) acts on \(\tilde{B}\) as the deck transformation. Fix a lift \(\tilde{b} \in \tilde{B}\) of \(b \in B\). We set

\[
[B, \Gamma_g \backslash \mathcal{S}_g^{\text{orb}}] := \{(p, \beta) | p : \tilde{B} \rightarrow \mathcal{S}_g, \beta : \pi_1(B, b) \rightarrow \Gamma_g, \text{ s.t. } p(\tilde{b}) = \ast, \ p(\gamma \cdot x) = \beta(\gamma) \cdot p(x) / \sim \}. \]

Here the relation \((p_0, \beta_0) \sim (p_1, \beta_1)\) holds if and only if \(\beta_0 = \beta_1\) and there is a map \(\tilde{p} : \tilde{B} \times [0, 1] \rightarrow \mathcal{S}_g\) such that \(\tilde{p}(x, 0) = p_0, \tilde{p}(x, 1) = p_1\) and \(\tilde{p}(\gamma \cdot x, t) = \beta(\gamma) \cdot \tilde{p}(x, t)\) for any \(\gamma \in \Gamma_g, x \in \tilde{B}, t \in [0, 1]\).

**Definition 5.1.** We define the orbifold fundamental group of \( \Gamma_g \backslash \mathcal{S}_g \) by

\[
S_g := \{ S^1, \Gamma_g \backslash \mathcal{S}_g \}^{\text{orb}} = \{ (\alpha, \gamma) | \gamma \in \Gamma_g, \alpha : \mathbb{R} \rightarrow \mathcal{S}_g, \text{s.t. } \alpha(0) = \ast, \ \alpha(t) = \gamma \cdot \alpha(t + 1), \ t \in \mathbb{R} / \sim \}. \]

Then \(S_g = \{ (\alpha, \gamma) | \gamma \in \Gamma_g, \alpha : [0, 1] \rightarrow \mathcal{S}_g, \text{s.t. } \alpha(0) = \gamma \cdot \alpha(1) = \ast / \sim \}. \)

Here \((\alpha_0, \gamma_0) \sim (\alpha_1, \gamma_1)\) if and only if \(\gamma_0 = \gamma_1\) and there exists a homotopy \(\alpha(s, t) : [0, 1] \times [0, 1] \rightarrow \mathcal{S}_g\) connecting \(\alpha_0\) and \(\alpha_1\), such that \(\alpha(s, 0) = \gamma_0 \cdot \alpha(s, 1) = \ast \) for \(s \in [0, 1]\).

The group law of \(S_g\) is defined as follows. Let \([(\alpha_1, \gamma_1)], [(\alpha_2, \gamma_2)] \in S_g\). Then \(\gamma_2^{-1} \cdot \alpha_1\) is a path path from \(\gamma_2^{-1} \cdot \ast\) to \((\gamma_2) \gamma_1^{-1} \cdot \ast\). We define the new path \(\alpha : [0, 1] \rightarrow \mathcal{S}_g\) by \(\alpha(t) := \alpha_2(2t)\) for \(0 \leq t \leq \frac{1}{2}, \ \alpha(t) := \gamma_2^{-1} \cdot \alpha_1(2t - 1)\) for \(\frac{1}{2} \leq t \leq 1\). Then we define \([(\alpha_1, \gamma_1) \cdot (\alpha_2, \gamma_2)] := [(\alpha, \gamma \gamma_2)] \in S_g\).
Let $p : S_g \to \Gamma_g$ be the projection to the second factor. Since the kernel of $p$ is isomorphic to $\pi_1(S'_g, \ast)$, we have an exact sequence
\[(5) \quad 1 \to \pi_1(S'_g, \ast) \to S_g \to \Gamma_g \to 1.\]

**Remark 5.2.** When $g = 1$, $\Gamma_1 \setminus S'_1 = SL_2\mathbb{Z}\setminus \mathfrak{S}_1$ is the moduli space of curves of genus 1 and $S_1 = \mathcal{M}_1$. When $g = 2$, $\Gamma_2 \setminus S'_2$ is the moduli space of curves of genus 2 by the Torelli theorem and $S_2 = \mathcal{M}_2$.

Recall that a $\pi_1(B, b)$-equivariant map $f : (\tilde{B}, \tilde{b}) \to (\mathfrak{S}'_g, \ast)$ induces the homomorphism of groups $f_* : \pi_1(B, b) \to S_g$ such that $f_*([c]) = [f \circ c]$ for $[c] \in \pi_1(B, b)$.

**Proposition 5.3.** Let $(B, b)$ be a compact orientable surface with base point and with non empty boundary. Then the map
\[ [B, \Gamma_g \setminus \mathfrak{S}'_g]^{\text{orb}} \ni [f] \mapsto f_* \in \text{Hom}(\pi_1(B, b), S_g). \]

is a bijection.

**Proof.** It is known that $B$ is homotopy equivalent to an $n$-bouquet $\vee_{k=1}^n S^1_k$ for some $n$ and the fundamental group $\pi_1(B, b) \cong \pi_1(\vee_{k=1}^n S^1_k, o)$ is isomorphic to the free group of rank $n$. Hence we get
\[ [B, \Gamma_g \setminus \mathfrak{S}'_g]^{\text{orb}} \cong [\vee_{k=1}^n S^1_k, \Gamma_g \setminus \mathfrak{S}'_g]^{\text{orb}} \cong \text{Hom}(\pi_1(\vee_{k=1}^n S^1_k, o), S_g) \cong \text{Hom}(\pi_1(B, b), S_g). \]

which completes the proof.

In the rest of this section we assume that $B = S^2 - \Pi_{k=1}^3 D_k$, where $D_1, D_2, D_3$ are mutually disjoint open discs. Since $B$ is homotopy equivalent to a 2-bouquet $\pi_1(B, b)$ is the free group of rank 2. Let $g_1, g_2$ be generators of $\pi_1(B, b)$ represented by the loops which are mutually homotopy equivalent to $\partial D_1, \partial D_2$. By Proposition 5.3 we have a bijection
\[(6) \quad [B, \Gamma_g \setminus \mathfrak{S}'_g]^{\text{orb}} \cong S_g \times S_g, \]

which is given by $[f] \mapsto (f_*(g_1), f_*(g_2)) \in S_g \times S_g$ for $[f] \in [B, \Gamma_g \setminus \mathfrak{S}'_g]^{\text{orb}}$.

For $[f] \in [B, \Gamma_g \setminus \mathfrak{S}'_g]^{\text{orb}}$ the fiber product $\pi : \tilde{B} \times_f \Theta \to \tilde{B}$ is a $\pi_1(B, b)$-equivariant fiber bundle because $f : \tilde{B} \to \mathfrak{S}'_g$ is a $\pi_1(B, b)$-equivariant map. We get the fiber bundle $\pi : (\tilde{B} \times_f \Theta)/\pi_1(B, b) \to B$, which is uniquely determined by $[f] \in [B, \Gamma_g \setminus \mathfrak{S}'_g]^{\text{orb}}$ up to an isomorphism and which is 2g-dimensional compact oriented manifold with boundary. For $(\sigma_1, \sigma_2) \in S_g \times S_g$, Let $\pi : X(\sigma_1, \sigma_2) \to B$ denote the corresponding fiber bundle under the isomorphism (6).

**Definition 5.4.** Define the map $c_{2g} : S_{2g} \times S_{2g} \to \mathbb{Z}$ by
\[ c_{2g}(\sigma_1, \sigma_2) := \text{Sign}(X(\sigma_1, \sigma_2)). \]

We call $c_{2g}$ the signature cocycle for smooth theta divisors.

**Remark 5.5.** We only consider the case of an even genus because in the case of an odd genus $\text{Sign}(X(\sigma_1, \sigma_2))$ always vanishes.

**Lemma 5.6.** The following relation holds:
\[ c_{2g}(\sigma_1, \sigma_2) + c_{2g}(\sigma_1 \sigma_2, \sigma_3) = c_{2g}(\sigma_2, \sigma_3) + c_{2g}(\sigma_2 \sigma_3, \sigma_1), \]

for any $\sigma_1, \sigma_2, C \in S_{2g}$. In particular, $c_{2g}$ is a 2-cocycle of the group $S_{2g}$ ([11]).

**Proof.** By the same argument in [1], we obtain the assertion.

Let $[c_{2g}] \in H^2(S_{2g}, \mathbb{Z})$ be the cohomology class of $c_{2g}$. When $g = 1$, $c_2$ is the Meyer cocycle.
6. Construction of the Meyer function

Let \( \sigma = [(\alpha, \gamma)] \) be an element of \( S_{2g} \), where \( \alpha : \mathbb{R} \to \mathfrak{S}_{2g}' \) and \( \gamma \in \Gamma_{2g} \). Let \( \mathbb{R} \times _{\alpha} \Theta' \) be the fiber product, which has a natural \( \pi_{1}(S^{1}) \)-action. We define the mapping torus \( M_{\sigma} \) for \( \sigma \) by

\[
\pi : M_{\sigma} := (\mathbb{R} \times _{\alpha} \Theta')/\pi_{1}(S^{1}) \to S^{1}.
\]

Since the metric \( g^{\Theta'}/\mathcal{G}_{2g}' \) on \( T(\Theta'/\mathcal{G}_{2g}') \) and the connection \( P_{\theta}' \) on \( \Theta' \) are \( \Gamma_{2g} \)-invariant and the map \( p : \mathcal{S}^{1} = \mathbb{R} \to \mathcal{G}_{2g}' \) is \( \pi_{1}(S^{1}) \)-equivariant, the metric \( g^{\Theta'}/\mathcal{S}^{1} \) on \( T(\Theta'/\mathcal{S}^{1}) \) and the connection on \( P_{\sigma} \) on \( M_{\sigma} \) are naturally induced via the map \( p \). Using the connection \( P_{\sigma} \) we define the 1-parameter family of Riemannian metrics \( \{ g^{\Theta', \epsilon} \}_{\epsilon > 0} \) on \( M_{\sigma} \) by

\[
g^{\Theta', \epsilon} := g^{\Theta'}/\mathcal{S}^{1} \oplus \epsilon^{-1} \pi^{*} dt^{2}, \quad \epsilon \in \mathbb{R}_{>0}.
\]

Here we regard \( S^{1} \) as \( \mathbb{R}/\mathbb{Z} \) and \( t \in \mathbb{R} \) as a coordinate of \( S^{1} \). By the theorem 3.3, the adiabatic limit

\[
\eta^{0}(M_{\sigma}, g^{\Theta', \epsilon}) := \lim_{\epsilon \to 0} \eta(M_{\sigma}, g^{\Theta', \epsilon})
\]

exists. Recall that the Siegel modular form \( \Delta_{2g}(\tau) \) with zero divisors \( \mathcal{N}_{2g} \) (see Section 3.3.). Since the 1-form \( d\log \| \Delta_{2g}(\tau) \|^{2} \) is \( \Gamma_{2g} \)-invariant the pull-back \( p^{*} d\log \| \Delta_{2g}(\tau) \|^{2} \) can be regarded as a 1-form on \( S^{1} \).

**Definition 6.1.** For \( \sigma \in S_{2g} \) we fix \( (p, \gamma) \) which represents \( \sigma = [(\alpha, \gamma)] \), where \( \gamma \in \Gamma_{2g} \) and \( p : \mathbb{R} \to \mathfrak{S}_{2g}' \) we set

\[
\Phi_{2g}(p, \gamma) := \eta^{0}(M_{\sigma}, g^{\Theta', \epsilon}) + \frac{(-1)^{g_{2g}+3} (2g+2) - 1}{(2g + 3)!} \int_{S^{1}} p^{*} d\log \| \Delta_{2g}(\tau) \|^{2}.
\]

The following theorem is the main result of this paper.

**Theorem 6.2.** (a) The value \( \Phi_{2g}(p, \gamma) \) is independent of a choice of \( (p, \gamma) \) which represents \( \sigma \in S_{2g} \). In particular \( \Phi_{2g} \) is a function on \( S_{2g} \).

(b) The cocycle \( -c_{2g} \) is the coboundary of the function \( \Phi_{2g} \). In particular \( [c_{2g}] \otimes \mathbb{Q} = 0 \in H^{2}(S_{2g}, \mathbb{Z}) \).

As a corollary of the Theorem 6.2, it follows that \( \Phi_{2} = \Phi_{2} \) by the uniqueness of Meyer's function of genus 2. On the other hand, \( \Delta_{2}(\tau) \) coincides with the Igusa's modular form \( \chi_{2}(\tau) \) ([25]), which is the product of all even theta constants. Then we can derive the following formula:

**Corollary 6.3 ([15]).** Let \( \sigma = [(\alpha, \gamma)] \) be an element of \( S_{2} = M_{2} \) as before. Then we have

\[
\Phi_{2}(\sigma) = \eta^{0}(M_{\sigma}, g^{\Theta', \epsilon}) - \frac{2}{15} \int_{S^{1}} p^{*} d\log \| \chi_{2}(\tau) \|^{2}.
\]

**Proof of Theorem 6.2.** (a) Assume that \( (p_{0}, \gamma) \) and \( (p_{1}, \gamma) \) represents the same element \( \sigma \in S_{2g} \). Put \( I := [0, 1] \). There is a map

\[
\tilde{\phi} : I \times \mathbb{R} \to \mathfrak{S}_{2g}'
\]

which satisfies \( \tilde{\phi}(s, 0) = * \) for \( s \in I \) and \( \tilde{\phi}(s, t) = \gamma \tilde{\phi}(s, t + 1) \) for \( (s, t) \in I \times \mathbb{R} \) and the following condition

\[
(7) \quad \tilde{\phi}(s, t) = p_{0}(t), \quad s \in [0, \frac{1}{3}) \quad \text{and} \quad \tilde{\phi}(s, t) = p_{1}(t), \quad s \in [\frac{2}{3}, 1].
\]

Since \( \tilde{\phi} \) is \( \pi_{1}(I \times \mathbb{R}) \)-equivariant, the fiber product \( (I \times \mathbb{R}) \times _{\phi} \Theta' \) has the \( \pi_{1}(I \times S^{1}) \)-action and the quotient space

\[
\tilde{\pi} : \tilde{M}_{\sigma} := (I \times \mathbb{R}) \times _{\phi} \Theta' / \pi_{1}(I \times S^{1}) \to I \times S^{1}
\]
has the induced metric $g^{M_{\sigma}/I\times S^{1}}$ on $T(M_{\sigma}/I\times S^{1})$ from the metric $g^{\Theta'/\mathcal{E}'}$ and the connection $P_{\sigma}$ on $M_{\sigma}$ from the connection $P_{\Theta'}$ mutually via the map $p$. Using the connection $\tilde{P}_{\sigma}$ we set

$$g^{M_{\sigma},\epsilon} := g^{M_{\sigma}/I\times S^{1}} \oplus \epsilon^{-1} \pi^{*}(ds^{2} \oplus dt^{2}), \quad \epsilon \in \mathbb{R}_{>0}.$$ 

Let $g^{M_{\sigma},\epsilon}$ be the metrics on $M_{\sigma}$, induced from the map $p_{i}$ for $i = 0, 1$ as above. The condition (7) implies that

$$g^{M_{\sigma},\epsilon}|_{(0,\frac{1}{\epsilon})\times S^{1}} = g^{M_{\sigma},\epsilon}|_{(0,\frac{1}{\epsilon})\times S^{1}} = g^{M_{\sigma},\epsilon}|_{(\frac{3}{4},1)\times S^{1}} = g^{M_{\sigma},\epsilon}|_{(\frac{3}{4},1)\times S^{1}}.$$

Then we can apply the Atiyah-Patodi Singer's index theorem to $(M_{\sigma}, g^{M_{\sigma},\epsilon})$:

$$\text{Sign}(M_{\sigma}) = \int_{I\times S^{1}} \pi_{*}L(T(M_{\sigma}, g^{M_{\sigma},\epsilon})) - (\eta(M_{\sigma}, g^{M_{\sigma},\epsilon}) - \eta(M_{\sigma}, g^{M_{\sigma},\epsilon})).$$

Since $M_{\sigma}$ is isomorphic to the product $M_{\sigma}\times I$, we have (see [3]),

$$\text{Sign}(M_{\sigma}) = \text{Sign}(M_{\sigma}) \times \text{Sign}(I) = 0.$$

By Proposition 2.4 and the Proposition 2.5, we get

$$\lim_{\epsilon \to 0} \int_{I\times S^{1}} \pi_{*}L(T(M_{\sigma}, g^{M_{\sigma},\epsilon})) = \int_{I\times S^{1}} \pi_{*}L(T(M_{\sigma}/(I\times S^{1})))\pi_{*}L(T(I\times S^{1}))$$

$$= \int_{I\times S^{1}} \left( \pi_{*}L(T(M_{\sigma}/(I\times S^{1})), \nabla^{M_{\sigma}/(I\times S^{1}))} \right)^{(2)}$$

$$= \int_{I\times S^{1}} \left( \pi_{*}L(T(\Theta'/\mathcal{E}'), \nabla^{\Theta'/\mathcal{E}'}) \right)^{(2)}$$

$$= \int_{I\times S^{1}} \tilde{p}_{*}L(T(\Theta'/\mathcal{E}'), \nabla^{\Theta'/\mathcal{E}'})^{(2)}$$

where $\nabla^{M_{\sigma}/(I\times I)}$ is the connection on the relative tangent bundle $T(M_{\sigma}/(I\times I))$ associated with $g^{M_{\sigma}/(I\times I)}$ and $\tilde{P}_{\sigma}$ and we used the commutativity of fiber integrals and base changes in the last equality. By the Proposition 4.7, we have

$$\int_{I\times S^{1}} \tilde{p}_{*}L(T(\Theta'/\mathcal{E}'), \nabla^{\Theta'/\mathcal{E}'})^{(2)}$$

$$= -\frac{2^{2g+3}(2^{2g+2} - 1)B_{2g+2}}{(2g + 3)!} \int_{I\times S^{1}} \tilde{p}_{*}d\bar{c}d\log||\Delta_{2g}(\tau)||^{2}$$

$$= -\frac{2^{2g+3}(2^{2g+2} - 1)B_{2g+2}}{(2g + 3)!} \int_{\{1\}\times S^{1}} \pi_{*}d\bar{c}d\log||\Delta_{2g}(\tau)||^{2}$$

$$= -\frac{2^{2g+3}(2^{2g+2} - 1)B_{2g+2}}{(2g + 3)!} \int_{\{0\}\times S^{1}} \pi_{*}d\bar{c}d\log||\Delta_{2g}(\tau)||^{2},$$

where we used the $\Gamma_{2g}$-invariance of the 1-form $d\bar{c}d\log||\Delta_{2g}(\tau)||^{2}$ in the last equality. By (25) ~ (12) and the Definition 6.1, we obtain

$$0 = \Phi_{2g}(p_{1}, \gamma) - \Phi_{2g}(p_{0}, \gamma),$$

which completes the proof of (a).

(b) Let $\sigma_{1} = [(p_{1}, \gamma_{1})], \sigma_{2} = [(p_{2}, \gamma_{2})], \sigma_{3} := (\sigma_{1}\sigma_{2})^{-1} = (p_{3}, (\gamma_{1}\gamma_{2})^{-1}) \in S_{2g}$. Set $B := S^{2} - \bigcup_{k=1}^{3} D_{k}$. Recall that the fiber bundle $\pi : X(\sigma_{1}, \sigma_{2}) \to B$ for $\sigma_{1}, \sigma_{2}$ defined at the Section 3.2. By the definition of $\Phi_{2g}$, we have $\Phi_{2g}(\sigma^{-1}) = -\Phi_{2g}(\sigma)$ for any $\sigma \in S_{2g}$. Therefore to show that $-c_{2g}$ is the coboundary of $\Phi$, we have to show that

$$\text{Sign}(X(\sigma_{1}, \sigma_{2})) = -\sum_{i=1}^{3} \Phi_{2g}(\sigma_{i}).$$
Let $U_i$ be the neighborhood of $\partial D_i$ in $B$ such that $U_i \cong [0,1) \times \partial D_i$. Let $\beta_i : \tilde{U}_i \cong [0,1) \times \mathbb{R} \to \tilde{B}$ be the lift of the map $U_i \to B$. Let $g_1, g_2 \in \pi_1(B, b)$ be the generators represented by the loops $\partial D_1, \partial D_2$. Let $[(p, \alpha)] \in \Gamma \Theta_{2g} \cap \Sigma_{2g}$ be the corresponding element for $(\sigma_1, \sigma_2) \in S_{2g} \times S_{2g}$ under the isomorphism (6) where $\alpha : \pi_1(B, b) \to \Gamma_{2g}$ is a group homomorphism and $p : \tilde{B} \to \Sigma_{2g}$ is a $\pi_1(B, b)$-equivariant homomorphism preserving the basepoint. Since $\partial D_1, \partial D_2$ and $\partial D_3$ are homotopy equivalent to the loops which represent $g_1, g_2$ and $(g_1g_2)^{-1} \in \pi_1(B, b)$ we can assume that

$$p \beta_i |_{\tilde{U}_i}(s, t) = p_i(t), \quad (s, t) \in \tilde{U}_i \cong [0,1) \times \mathbb{R}, \quad i = 1 \sim 3.$$ 

Let $g^{X(\sigma_1, \sigma_2)}/B$ and $P_{X(\sigma_1, \sigma_2)}$ be the metric on $TX(\sigma_1, \sigma_2)$ and the connection on $X(\sigma_1, \sigma_2)$ induced from the metric $g^\Theta/\Sigma_{2g}$ and the connection $P_{\Theta}$ via the map $p$. Let $g^B$ be the metric on $TB$ such that $g^B |_{U_i} = ds^2 \oplus dt^2$. Using the connection $P_{X(\sigma_1, \sigma_2)}$ we define the metric on $TX(\sigma_1, \sigma_2)$ by

$$g^{X(\sigma_1, \sigma_2)}/e := g^{X(\sigma_1, \sigma_2)}/B \oplus e^{-1} \pi^* g^B, \quad e \in \mathbb{R}_{>0}.$$ 

Let $g^{M_{\sigma_i}, \epsilon}$ be the metric on $M_{\sigma_i}$ induced from $p_i$ for $i = 1 \sim 3$ as above. Let $\nabla^{X(\sigma_1, \sigma_2)}/B$ be the connection on $T(X(\sigma_1, \sigma_2))$ defined by the metric $g^{X(\sigma_1, \sigma_2)}/B$ and the connection $P_{X(\sigma_1, \sigma_2)}$. Since the condition (13) implies that the metric $g^{X(\sigma_1, \sigma_2)}/e$ is a product metric near the boundary of $X(\sigma_1, \sigma_2)$ we can apply the Atiyah-Patodi-Singer's index theorem to $(X(\sigma_1, \sigma_2), g^{X(\sigma_1, \sigma_2)}/e)$:

$$\text{Sign}(X(\sigma_1, \sigma_2)) = \int_{X(\sigma_1, \sigma_2)} L(TX(\sigma_1, \sigma_2), g^{X(\sigma_1, \sigma_2)}/e) - \sum_{i=1}^{3} \eta(M_{\sigma_i}, g^{M_{\sigma_i}, \epsilon})$$

$$= \int_B \pi_* L(T(X(\sigma_1, \sigma_2)/B), \nabla^{X(\sigma_1, \sigma_2)/B}) - \sum_{i=1}^{3} \eta^0(M_{\sigma_i}, g^{M_{\sigma_i}, \epsilon})$$

$$= \int_B p^* \left[ f_i L(T(\Theta'/\Sigma_{2g}), \nabla^{\Theta'/\Sigma_{2g}}) \right]^{(2)} - \sum_{i=1}^{3} \eta^0(M_{\sigma_i}, g^{M_{\sigma_i}, \epsilon})$$

$$= - \sum_{i=1}^{3} \int_{\partial D_i} -\frac{2^{2g+3}(2^{2g+2}-1)B_{2g+2}}{(2g+3)!} d^2 \log \|\Delta_{2g}(\tau)\|^2$$

$$- \sum_{i=1}^{3} \eta^0(M_{\sigma_i}, g^{M_{\sigma_i}, \epsilon})$$

$$= - \sum_{i=1}^{3} \Phi_{2g}(\sigma_i)$$

which completes the proof of (b).

$$\square$$

7. The first cohomology of $S_g$

The uniqueness of a 1-cocycle that cobounds the 2-cocycle $c_{2g}$ is equivalent to the vanishing of $H^1(S_{2g}, \mathbb{Z})$. In deed, if there is another 1-cocycle $\Phi_{2g} : S_{2g} \to \mathbb{R}$ that cobounds $c_{2g}$, the difference $\Phi_{2g} - \Phi_{2g}'$ is an element of $\text{Hom}(S_{2g}, \mathbb{R}) \cong H^1(S_{2g}, \mathbb{R})$. While $H^1(S_1, \mathbb{Z}) = H^1(S_2, \mathbb{Z}) = 0$, the uniqueness no longer valid for higher genus.

Theorem 7.1. The following holds:

$$H^1(S_g, \mathbb{Z}) = \begin{cases} 0 & 1 \leq g \leq 3, \\ \mathbb{Z} & g \geq 4. \end{cases}$$
In particular, the cochain cobounding the signature cocycle \( c_{2g} \) is not unique when \( g \geq 2 \).

By (5) and (11), we have the 5-term exact sequence

\[
1 \rightarrow H^{1}(\Gamma_{g}, \mathbb{Z}) \rightarrow H^{1}(S_{g}, \mathbb{Z}) \rightarrow H^{1}(\pi_{1}(\mathfrak{S}_{g}', \ast), \mathbb{Z})^{\Gamma_{\ast}} \xrightarrow{\delta} H^{2}(\Gamma_{g}, \mathbb{Z}) \rightarrow H^{2}(S_{g}, \mathbb{Z}).
\]

We have \( H^{1}(\Gamma_{g}, \mathbb{Z}) = 0 \) for \( g \geq 1 \) and \( H^{2}(\Gamma_{g}, \mathbb{Z}) = \mathbb{Z} \) for \( g \geq 3 \). By the Hurwitz theorem we see that

\[
H^{1}(\pi_{1}(\mathfrak{S}_{g}', \ast), \mathbb{Z}) \cong H^{1}(\mathfrak{S}_{g}', \mathbb{Z}).
\]

**Lemma 7.2.** Let \( X \) be a connected complex manifold of \( \dim_{\mathbb{C}} X \geq 2 \). Assume that

\[
H^{1}(X, \mathbb{Z}) = H^{2}(X, \mathbb{Z}) = 0.
\]

Let \( D = \sum_{\lambda \in \Lambda} n_{\lambda} D_{\lambda} \) be a divisor on \( X \) such that \( n_{\lambda} \neq 0 \) and \( D_{\lambda} \) is irreducible for all \( \lambda \in \Lambda \). Then

\[
H^{1}(X - D, \mathbb{Z}) \cong \mathbb{Z}^{\lambda}.
\]

The generator of the cohomology \( H^{1}(X - D, \mathbb{Z}) \) corresponding to \( \lambda \in \Lambda \) is represented by the map \( l_{\lambda} \rightarrow 1 \) and \( l_{\mu} \rightarrow 0 \) for \( \mu \neq \lambda \in \Lambda \), where \( l_{\mu} \) denotes the loop around a small disk and intersecting \( D_{\mu} \) transversally.

**Proof.** Since the real codimension of \( \text{Sing} D \) in \( X \) is greater than or equal to 4, we have \( \pi_{k}(X, X - \text{Sing} D, *) = 0 \) for \( 1 \leq k \leq 3 \). The relative Hurwitz theorem asserts that \( H_{k}(X, X - \text{Sing} D, \mathbb{Z}) = 0 \) for \( k \leq 3 \). Hence \( H^{k}(X, X - \text{Sing} D, \mathbb{Z}) = 0 \) for \( k \leq 3 \), which together with the cohomology exact sequence for the triple \((X, X - \text{Sing} D, X - D)\), yields that

\[
H^{2}(X, X - D, \mathbb{Z}) \cong H^{2}(X - \text{Sing} D, X - D, \mathbb{Z}).
\]

By the cohomology exact sequence for the pair \((X, X - D)\) and (16), we obtain

\[
H^{1}(X - D, \mathbb{Z}) \cong H^{2}(X, X - D, \mathbb{Z}).
\]

Since \( X - \text{Sing} D \) is a closed submanifold in \( X - \text{Sing} D \) and \( X - D = (X - \text{Sing} D) - (D - \text{Sing} D) \), the Thom isomorphism asserts that

\[
H^{2}(X - \text{Sing} D, X - D, \mathbb{Z}) \cong H^{0}(D - \text{Sing} D, \mathbb{Z}).
\]

By the irreducibility of \( D_{\lambda} \), \( D_{\lambda} - \text{Sing} D_{\lambda} \) is path connected so that

\[
H^{0}(D - \text{Sing} D, \mathbb{Z}) \cong \mathbb{Z}^{\lambda}.
\]

The result follows from (17)~(20).

**Lemma 7.3.** The following holds:

\[
H^{1}(\pi_{1}(\mathfrak{S}_{g}', \ast), \mathbb{Z})^{\Gamma_{\ast}} = \begin{cases} 
\mathbb{Z} & 1 \leq g \leq 3 \\
\mathbb{Z}^{\oplus 2} & g \geq 4.
\end{cases}
\]

By regarding \( H^{1}(\mathfrak{S}_{g}', \mathbb{C}) \) as the de Rham cohomology group, the image of the generators under the natural map \( H^{1}(\mathfrak{S}_{g}', \mathbb{Z}) \rightarrow H^{1}(\mathfrak{S}_{g}', \mathbb{C}) \) are represented by the 1-forms \( \frac{1}{2 \pi \sqrt{-1}} \log \chi_{g}(\tau) \) and \( \frac{1}{2 \pi \sqrt{-1}} \log J_{g}(\tau) \). Here \( J_{g}(\tau) \equiv 1 \) and hence \( \log J_{g}(\tau) = 0 \) for \( 1 \leq g \leq 3 \).

**Proof.** By Proposition 4.3, Proposition 4.4, the isomorphism (15) and Lemma 7.2, we get the assertion.
Recall that the automorphic factor $j(t, \gamma)$ is a nowhere vanishing holomorphic function on $\mathfrak{g}$. Since $\mathfrak{g}$ is simply connected, the logarithm of $j(t, \gamma)$ makes sense. Choose a branch of the logarithm of $j(t, \gamma)$ and denote it by $\log_{\sigma}j(t, \gamma)$ for $\gamma \in \Gamma_g$. Define the function $\lambda_{\sigma}: \Gamma_g \times \Gamma_g \rightarrow \mathbb{Z}$ by

$$\lambda_{\sigma}(A, B) := \frac{1}{2\pi \sqrt{-1}} \{ \log_{\sigma}j(\tau, AB) - \log_{\sigma}j(B \cdot \tau, A) - \log_{\sigma}j(\tau, B) \}, \quad (A, B) \in \Gamma_g \times \Gamma_g.$$  

**Lemma 7.4.** The function $\lambda_{\sigma}$ is a $2$-cocycle of $\Gamma_g$, whose cohomology class generates $H^2(\Gamma_g, \mathbb{Z})$.

**Proof.** For $g = 1$ see [4]. When $g \geq 1$, we follow [4]. Let $G := \text{Sp}(2g, \mathbb{R})$ be the symplectic group and let $G^d$ be the same group endowed with the discrete topology. Let $u \in H^2(G^d, \mathbb{Z})$ be the cohomology class corresponding to the universal covering $0 \rightarrow \mathbb{Z} \rightarrow \tilde{G} \rightarrow G \rightarrow 1$.

We choose the branch $\log_{\sigma}j(t, \gamma)$ satisfying

$$\text{Im} \log_{\sigma}j(\sqrt{-1} \cdot 1_{2g}, \gamma) \in [0, 2\pi).$$

Since the function $\lambda_{\sigma}$ is measurable, the cohomology class $[\lambda_{\sigma}]$ is a constant multiple of $u$ by [20]. Therefore it suffices to determine the restriction of the cohomology class $[\lambda_{\sigma}]$ to the maximal compact subgroup of $G$. We shall identify the unitary group $U(g)$ with the maximal compact subgroup of $G$ by the inclusion map defined as

$$\iota: U(g) \ni Z \mapsto \begin{pmatrix} \text{Re} \, Z & \text{Im} \, Z \\ -\text{Im} \, Z & \text{Re} \, Z \end{pmatrix} \in G, \quad Z \in U(g).$$

Since $j(\sqrt{-1} \cdot 1_{2g}, \iota(Z)) = \det(Z)^{-1}$ for $Z \in U(g)$ and the isotropy subgroup at $\sqrt{-1} \cdot 1_{2g} \in \mathfrak{g}$ is just $U(g)$, we have

$$2\pi \sqrt{-1} \lambda_{\sigma}(Z_1, Z_2) = -\log_{\sigma} \det(Z_1 Z_2) + \log_{\sigma} \det(Z_1) + \log_{\sigma} \det(Z_2)$$

for $(Z_1, Z_2) \in U(g) \times U(g)$. By (23), the restriction of the cohomology class $[\lambda_{\sigma}]$ to $U(g)$ is the pull-back of the cohomology class corresponding to the universal covering $0 \rightarrow \mathbb{Z} \rightarrow \tilde{U}(1) \cong \mathbb{R} \rightarrow U(1) \rightarrow 1$, via the map $\det: U(g) \rightarrow U(1)$. Since the induced map $(\det)_*: \pi_1(U(g)) \rightarrow \pi_1(U(1))$ is an isomorphism, we obtain $[\lambda_{\sigma}] = u$. Since the cohomology class $[\lambda_{\sigma}]$ is independent of the choice of the branch of $\log_{\sigma}j(t, \gamma)$ and since the restriction of $u$ to $\Gamma_g$ is the generator of the cohomology $H^2(\Gamma_g, \mathbb{Z})$ we obtain the assertion. \qed

**Lemma 7.5.** Let $g \geq 2$. The map $\delta: H^1(\pi_1(\mathfrak{g}', \star), \mathbb{Z}) \rightarrow H^2(\Gamma_g, \mathbb{Z})$ is given by

$$(m, n) \mapsto (k_1(g)m + k_2(g)n) \in H^2(\Gamma_g, \mathbb{Z}) \cong \mathbb{Z}$$

for $(m, n) \in H^1(\pi_1(\mathfrak{g}', \star), \mathbb{Z}) \cong \mathbb{Z}^{2g}$. Here,

$$k_1(g) = 2g^{-2}(2g+1), \quad k_2(g) = \frac{(g+3)g!}{4} - 2g^{-3}(2g+1)$$

are the weights of Siegel modular forms $\chi_g(t), J_g(t)$, respectively.

**Proof.** Let $\sigma: \Gamma_g \rightarrow S_g$ be a section, and write $\sigma(\gamma) = [(l_{\gamma}, \gamma)] \in S_g$ for $\gamma \in \Gamma_g$. We can assume that $l_{\gamma^{-1}} = -\gamma \cdot l_{\gamma}$, where $l(t) := l(1-t)$, $t \in [0, 1]$ for a path $l(t)$. Hence $\sigma(\gamma^{-1}) = \sigma(\gamma)^{-1}$. Let $\alpha$ be an element of $H^1(\pi_1(\mathfrak{g}', \star), \mathbb{Z})^{\star} \cong \text{Hom}(\pi_1(\mathfrak{g}', \star), \mathbb{Z})^{\star}$. Then $\delta(\alpha): \Gamma_g \times \Gamma_g \rightarrow \mathbb{Z}$ is given by

$$(A, B) \mapsto \alpha(\sigma(A) \sigma(B) \sigma(AB)^{-1}) \in \mathbb{Z}, \quad (A, B) \in \Gamma_g \times \Gamma_g,$$

where we identify $\sigma(\sigma(A) \sigma(B) \sigma(AB)^{-1}) \in \text{Im}(\pi_1(\mathfrak{g}', \star) \rightarrow S_g)$ with the corresponding preimage of $\pi_1(\mathfrak{g}', \star)$ under the inclusion $\pi_1(\mathfrak{g}', \star) \rightarrow S_g$. Write $\sigma(\sigma(A) \sigma(B) \sigma(AB)^{-1}) = [(l_{(A,B)}, 1)] \in \pi_1(\mathfrak{g}', \star)$.}


Here \( l_{(A,B)} \) is a loop on \( \mathcal{S}_{g}' \), which is the composition of the paths \( l_{B}, B^{-1}l_{A} \) and \(-l_{AB} \). Under the identification \( H^{1}(\pi_{1}(\mathcal{S}_{g}', \ast), \mathbb{Z})^{\Gamma_{g}} \cong \mathbb{Z}^{\oplus 2} \) given in Lemma 7.3, the cochain \( \delta(m, n) \) is given by
\[
\delta(m, n)(A, B) = \frac{1}{2\pi \sqrt{-1}} \int_{l_{(A,B)}} d\log\chi_{g}(\tau)^{m}J_{g}(\tau)^{n} \in \mathbb{Z}, \quad (A, B) \in \Gamma_{g} \times \Gamma_{g},
\]
for \((m, n) \in H^{1}(\pi_{1}(\mathcal{S}_{g}', \ast), \mathbb{Z})^{\Gamma_{g}} \cong \mathbb{Z}^{\oplus 2} \). Using \( \sigma \), we choose the branch \( \log_{\sigma}j(\tau, \gamma) \) for \( \gamma \in \Gamma_{g} \) such that
\[
\log_{\sigma}j(\ast, \gamma) := \frac{1}{k_{1}(g)} \int_{l_{-1}} d\log\chi_{g}(\tau).
\]
Then we get
\[
2\pi \sqrt{-1}\delta(1,0)(A, B) = \int_{l_{(A,B)}} d\log\chi_{g}(\tau)
= \int_{AB-l_{(A,B)}} d\log\chi_{g}(AB \cdot \tau)
= \int_{AB-l_{(A,B)}} [k_{1}(g)\log_{\sigma}j(\tau, AB) + d\log\chi_{g}(\tau)]
= \int_{AB-l_{(A,B)}} d\log\chi_{g}(\tau) + \int_{A-l_{A}} d\log\chi_{g}(\tau) - \int_{A-l_{A}} d\log\chi_{g}(\tau)
= -k_{1}(g)\log_{\sigma}j(\ast, A) + k_{1}(g)\log_{\sigma}j(\ast, AB)
= k_{1}(g)\log_{\sigma}j(\ast, A) + k_{1}(g)\log_{\sigma}j(\ast, AB)
= k_{1}(g)[\log_{\sigma}j(\ast, A) - \log_{\sigma}j(\ast, B)]
= k_{1}(g)[\log_{\sigma}j(\ast, A) - \log_{\sigma}j(\ast, B)]
\]
By Lemma 7.4 we get \( \delta(1,0) = k_{1}(g) \in H^{2}(\Gamma_{g}, \mathbb{Z}) \cong \mathbb{Z} \). Similarly, \( \delta(0,1) = k_{2}(g) \in H^{2}(\Gamma_{g}, \mathbb{Z}) \cong \mathbb{Z} \).

This completes the proof.

\( \square \)

**Proof of Theorem 7.1.** Since \( H^{1}(\Gamma_{g}, \mathbb{Z}) \) in the exact sequence (5), we get \( H^{1}(S_{g}, \mathbb{Z}) = \ker \delta \). By Lemma 7.5, we get \( \ker \delta = 0 \) for \( 1 \leq g \leq 3 \) and \( \ker \delta \cong \mathbb{Z} \) for \( g \geq 4 \). This completes the proof of Theorem 7.1.

\( \square \)

8. **The value for the Dehn twist**

In this section, we shall compute the value of \( \Phi_{2g} \) for the **Dehn twist**, which is defined as follows (cf. [16]). Let \( \Delta \subset C \) be the unit disk. Recall that the Andreotti-Mayer locus \( N_{2g} \) has two irreducible components \( \theta_{null,2g} \) and \( N_{2g} \) by Theorem 4.3. Let \( \rho : \Delta \to \mathcal{S}_{2g} \) be a \( C_{10} \)-map such that \( \rho(0) \in \theta_{null,2g} \) is a generic point, \( \rho(z) \not\in \mathcal{N}_{2g} \) for \( z \in \Delta \setminus \{0\} \) and \( \rho(\Delta) \) intersects with \( \theta_{null,2g} \) at \( \rho(0) \) transversally. For simplicity we assume that the base point \( * \) lies in \( \rho(\partial \Delta) \) and we denote the monodromy corresponding to the loop \( \rho|_{\partial \Delta} : \partial \Delta \to \mathcal{S}_{g} \) by \( \sigma_{2g} \in S_{2g} \). The element \( \sigma_{2g} \) is called the Dehn twist. We put
\[
\omega : X_{2g} := \Delta \times_{\rho} \Theta \to \Delta,
\]
which is smooth family of theta divisors over \( \Delta \) induced from the universal family \( \pi : \Theta \to \mathcal{S}_{2g} \) by \( \rho \). Let \( \tilde{\rho} : X_{2g} \to \Theta \) be the lift of the map \( \rho \) defined as the projection to the second factor. By
the assumption of $\rho$ and the Theorem 4.3, $\text{Sing}(\omega^{-1}(0))$ consists of one ordinary double point and $\omega^{-1}(z)$ is a smooth theta divisor for $z \in \Delta \setminus \{0\}$. Notice that $\partial X_{2g}$ endowed with the orientation induced from $X_{2g}$ is diffeomorphic to the mapping torus $M_{\sigma_{2g}}$ endowed with the natural orientation, i.e., $\partial X_{2g} = -M_{\sigma_{2g}}$.

Theorem 8.1. The following equality holds:

$$\Phi_{2g}(\sigma_{2g}) = \begin{cases} -\frac{g}{2} & \text{if } g = 1, \\ -\left(1\right)^{g+1}(2g+1)(2g+2)(2g+3)(2g+4)B_{g+1} & \text{if } g > 1. \end{cases}$$

Proof. Put $\Delta_{r} := \{z \in \Delta \mid |z| < r\} \subset \Delta$ for $0 < r < 1$. We choose $\rho$ such that the restriction $\rho|_{\Delta_{1/3}} : \Delta_{1/3} \rightarrow \rho(\Delta_{1/3}) \subset \mathfrak{S}_{2g}$ is a holomorphic embedding that

$$\rho(re^{-i\theta}) = \rho \left( \frac{2}{3} e^{-i\theta} \right), \quad 0 \leq \theta < 2\pi.$$  

Let $g^{\Delta}$ be the metric on $T\Delta$ which is a product metric near the boundary $\partial\Delta$ and coincides with the metric $\rho^{*}g^{\Theta}$ on $\Delta_{1/3}$. Let $p \in X_{2g}$ be the unique singular point on the singular fiber $X_{0}$. Let $g^{X_{2g}/\Delta}$ be the metric on $T(X_{2g}/\Delta)|_{X_{2g} - \{p\}}$ induced from the metric $g^{\Theta}e^{\phi}$ via the map $\rho$. Let $g^{X_{2g}}$ be the metric on $TX_{2g}$ which coincides with $g^{X_{2g}/\Delta} \oplus \omega^{*}g^{\Delta}$, where we used the connection induced from the connection $P_{\rho}$ on $\Theta^{*}$ via the map $\rho$, on $X_{2g} - \{p\}$ and coincides with the metric induced from the metric $g^{\rho^{*}}$ via the map $\rho$ on a neighborhood of $p$. Set

$$g^{X_{2g},\varepsilon} := g^{X_{2g}} \oplus \varepsilon^{-1}\omega^{*}g^{\Delta}, \quad \varepsilon \in \mathbb{R}_{>0}.$$  

By the assumption of $g^{\Delta}$ and the condition (24), $g^{X_{2g},\varepsilon}$ is the product metric near the boundary $\partial X_{2g}$ for $\varepsilon \in \mathbb{R}_{>0}$. By the Atiyah-Patodi-Singer index theorem,

$$(25) \quad \text{Sign}(X_{2g}) = \int_{X_{2g}} L(TX_{2g}, g^{X_{2g},\varepsilon}) + \eta(M_{\sigma_{2g}}, g^{M_{\sigma_{2g}},\varepsilon}).$$

Here $\partial X_{2g}$ is identified with $-M_{\sigma_{2g}}$, and $g^{M_{\sigma_{2g}},\varepsilon}$ is the restriction of $g^{X_{2g},\varepsilon}$ to the boundary $\partial X_{2g} \cong -M_{\sigma_{2g}}$. By the formula in [26], the first term of the right-hand side of (25):

$$(26) \quad \lim_{\varepsilon \rightarrow 0} L(TX_{2g}, g^{X_{2g},\varepsilon}) = L(T(X_{2g}/\Delta), \nabla^{X_{2g}/\Delta}) + P(-t, \cdots, (-t)^{2g})|_{t^{2g}} \cdot \mu(p) \delta_{p}$$

Here $L(T(X_{2g}/\Delta), \nabla^{X_{2g}/\Delta})$ is only defined on $X_{2g} - \{p\}$ but has the natural smooth extension on whole $X_{2g}$. The constant $\mu(p)$ is the Milnor number of the singular point $p$, $\delta_{p}$ is the Dirac delta current supported at $p$ and $P(x_{1}, \cdots, x_{2g}) \in \mathbb{C}[x_{1}, \cdots, x_{2g}]$ is defined by

$$\prod_{k=1}^{2g} L(x_{k}) = P(\sigma_{1}, \cdots, \sigma_{2g}),$$

where $L(x) = x/\tanh(x)$ and $\sigma_{1} = \sum_{k} x_{k}, \sigma_{2} = \sum_{i>j} x_{i}x_{j}, \cdots, \sigma_{2g} = \prod_{k} x_{k}$ are the fundamental symmetric polynomials. Notice that

$$P(-t, \cdots, (-t)^{2g})|_{t^{2g}} = L^{-1}(t)|_{t^{2g}}.$$  

Since $p$ is a non-degenerate critical point of $\pi : X \rightarrow \Delta$, we get $\mu(p) = 1$, which together with (25), (26) and Theorem 4.7, yields that

$$(27) \quad \text{Sign}(X_{2g}) = \frac{(-1)^{g}2^{g+1}(2g+2)!}{B_{g+1}} \int_{\Delta} \rho^{*}dd^{*}\log\text{det}Im\tau + \frac{(-1)^{g}2^{g+2}(2g+3)!}{B_{g+1}} \eta^{0}(M_{\sigma_{2g}}, g^{M_{\sigma_{2g}},\varepsilon}).$$
By (27) and Definition 6.1, we get

\[
\Phi_{2g}(\sigma_{2g}) = \eta^{0}(M_{\sigma_{2g}}, G^{M_{\sigma_{2g}, \tau}}) + \frac{(-1)^{g}2^{2g+3}(2^{2g+2} - 1)}{(2g + 3)!} B_{g+1} \int_{\Delta} p^{*} d^{c}(\log|\Delta_{2g}(\tau)|^{2}(\det \text{Im} \tau)^{(2g+3)(2g)!})
\]

where we used the Poincaré-Lelong formula and Theorem 4.4 to get the last equality. When \( g = 1 \), since the singular fiber has two irreducible components and \( \text{Sign}(X_{2}) = -1 \), we obtain the proof for the case \( g = 1 \). We complete the proof by the following Lemma in the case \( g > 1 \).

**Lemma 8.2.** Let \( \pi : X \to \Delta \) be a Lefschetz degeneration of relative dimension \( 2n - 1 \), i.e., \( \pi \) is a proper holomorphic surjective map from a \( 2n \)-dimensional complex manifold \( X \) to the unit disk \( \Delta \) and there is a point \( p \in X \) and an open neighbourhood \( p \in U \cong \{ (z_{1}, \cdots, z_{2n}) \in \mathbb{C}^{2n} | \sum_{k=1}^{2n} |z_{k}|^{2} < 1 \} \) such that

\[
\pi(z_{1}, \cdots, z_{2n}) = \sum_{k=1}^{2n} z_{k}^{2}, \quad (z_{1}, \cdots, z_{2n}) \in U
\]

and \( \pi_{*} \) has maximal rank on \( X \setminus p \). Assume that \( n > 1 \). Then \( \text{Sign}(X) = 0 \).

**Proof.** For \( \in \Delta \), we set \( U_{t} := X_{t} \cap U \). Then a sequence of inclusions

\[
X_{0} \setminus U_{0} \subset X_{0} \setminus \{p\} \subset X_{0} \subset X
\]

induces a sequence of isomorphisms:

\[
(28) \quad H_{2n}(X_{0} \setminus U_{0}, Z) \cong H_{2n}(X_{0} \setminus \{p\}, Z) \cong H_{2n}(X_{0}, Z) \cong H_{2n}(X, Z).
\]

Here the first isomorphism follows from the homotopy equivalence of \( X_{0} \setminus U_{0} \) and \( X \setminus \{p\} \), the second isomorphism follows from the fact \( \text{codim}_{X}(p) / X_{0} = 4n - 2 > 2n + 1 \), and the third isomorphism follows from the fact that the inclusion \( X_{0} \hookrightarrow X \) is a deformation retraction. By Ehresman's Theorem, \( X \setminus U \) is diffeomorphic to \((X_{0} \setminus U_{0}) \times \Delta\) as a fiber bundle over an open subset \( \Delta \). Since \( X_{0} \) is contractible, the inclusion \( \pi_{*} : X_{0} \setminus U_{0} \hookrightarrow X \setminus U \) induces an isomorphism \( H_{2n}(X_{0} \setminus U_{0}, Z) \cong H_{2n}(X \setminus U, Z) \). By (28), the inclusion \( \pi_{*} : X_{0} \setminus U_{0} \hookrightarrow X \) induces an isomorphism \( H_{2n}(X_{0} \setminus U_{0}, Z) \cong H_{2n}(X, Z) \). Hence, for any \( t \in \Delta \), any element of \( H_{2n}(X, Z) \) can be represented by a cycle contained in \( X_{t} \). Therefore the intersection matrix of \( H_{2n}(X, Z) \) is trivial and \( \text{Sign}(X) = 0 \). This completes the proof.

**Remark 8.3.** When \( g = 1 \), \( \sigma_{2} \in \Delta \) is the Dehn twist along a separating simple closed curve on a Riemann surface of genus two. Since \( \text{Sign}(X_{2}) = -1 \) and \( B_{2} = \frac{3}{35} \), we obtain \( \phi_{2}(\sigma_{2}) = \Phi_{2}(\sigma_{2}) = -\frac{4}{5} \), which confirms a result of Matsumoto (199).

**References**


