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LIMIT LINEAR SERIES, AN INTRODUCTION

EDUARDO ESTEVES

ABSTRACT. Our goal is to introduce the technique of limit linear series by using the historic example of the proof of the Brill–Nöther theorem. In our approach, we employ a formula for limits of ramification points of linear systems along a family of curves degenerating to a nodal curve, also proved here.

1. INTRODUCTION

The technique of limit linear series was introduced by Eisenbud and Harris in the eighties. It originated from the proof by Griffiths and Harris [GH] of the Brill–Nöther theorem, and from subsequent work by Gieseker [Gi] on the Gieseker–Petri theorem. Eisenbud and Harris were able to obtain remarkable results from their technique. The reader may consult [EH2] for a description of some of these results and further references. In particular, they were able to give a shorter proof of the Brill–Nöther theorem [EH1].

Our aim in these notes is to illustrate the power of the technique of limit linear series by using it to give a proof of part of the Brill–Nöther theorem. We claim no originality though. In fact, the same goal was pursued by Harris and Morrison in [HM], where they actually prove the Gieseker–Petri theorem as well.

The approach in these notes is just slightly different from theirs, as we employ here a formula for limits of ramification points of linear systems, instead of the compatibility conditions on order sequences of limit linear series at nodes. To my knowledge, this formula appeared first in [Es], where it was derived for degenerations to nodal curves of every kind. To be more precise, Eisenbud and Harris produced the formula only for degenerations to curves of compact type, and only in the case the ramification points do not degenerate to nodes. And it is exactly the fact that the formula gives an effective 0-cycle at the nodes that we use in our approach.

The formula itself is important, so its presentation is also a goal of these notes. It can be used to approach a problem raised by Eisenbud and Harris in [EH3]: *What are the limits of Weierstrass points in families of curves degenerating to stable curves* not *of compact type?* This was done in [EM2] for nodal curves with just two irreducible components.

More details on the statement of the Brill–Nöther theorem and its history can be found in Section 2, which can be regarded as a second introduction. In Section 3 we present what we need from deformations of nodal curves, as the existence of regular smoothings of nodal curves, and how they behave under base change. In Section 4 we review the basic theory of ramification points of linear systems on smooth curves. In Section 5 we present the formula for computing limits of ramification points of linear systems along a family of curves degenerating to a nodal curve. Finally, in Section 6 we use the formula for proving the Brill–Nöther theorem.

These notes originated from two talks I gave at the Symposium on Algebraic Geometry and Topology at the Research Institute for Mathematical Sciences of Kyoto University in January, 2006. The notes follow the talks, giving many more details than I could give then. However, at the talks I gave a brief overview of the results in [EM2], about the determination of limits of Weierstrass points on nodal curves with two components. As I would have neither time nor space to give more than an overview here, and as this overview is given in [EM1] and in the introduction to [EM2], I decided to omit this part in the notes.

I would like to thank the organizers of the Symposium in Kyoto, Profs. Mizuho Ishizaka, Hajime Kaji and Kazuhiro Konno, for a very exciting meeting. I would also like to thank their hospitality, and that of the many Japanese mathematicians I met, which make a trip to Japan, as always, a very pleasant and productive experience. Finally, I would like to thank the participants of the seminar on moduli of curves run at IMPA in 2005. The seminar served as basis for the talks in Kyoto and these notes.

2. The Brill–Nöther theorem

2.1. *The Brill-Nöther property.* Let C be a nonsingular, connected, complex projective curve of genus g. A *linear system* on C is a nonzero vector space of sections of a line bundle on C. The degree of the line bundle is called the *degree* of the linear system, and the projective dimension of the vector space is called the *rank* of the linear system. For each pair of nonnegative integers (d, r), let

$$\rho(g, d, r) := (r+1)(d-r) - gr.$$

We call $\rho(g, d, r)$ the *Brill-Nöther number* associated to g, d and r. We say that C *satisfies the Brill-Nöther property* if for each pair (d, r) with $\rho(g, d, r) < 0$ there is no linear system on C of degree d and rank r.

Remark 2.2. It is not necessary to check each pair of nonnegative integers (d, r) to ascertain that C satisfies the Brill–Nöther property, but only a finite number of them. Indeed, if L is a line bundle of degree d on C that is *nonspecial*, i.e. $h^1(C, L) = 0$ or, equivalently,

$$h^0(C, L) = d + 1 - g,$$

then the rank r of any linear system of sections of L satisfies $r \leq d - g$, and hence $\rho(g, d, r) \geq g \geq 0$. Since $h^1(C, L) = 0$ if $d \geq 2g - 1$, and since at any rate $h^0(C, L) \leq d + 1$, we may restrict to pairs (d, r) with $d \leq 2g - 2$ and $r \leq d$. There are a finite number of those.

Remark 2.3. If C is a hyperelliptic curve of genus $g > 2$, then C does not satisfy the Brill–Nöther property. Indeed, a hyperelliptic curve is a degree-2 covering of the projective line, so admits a linear system of degree 2 and rank 1. But

$$\rho(g, 2, 1) = (1 + 1)(2 - 1) - g = 2 - g,$$

and hence $\rho(g, 2, 1) < 0$ if $g > 2$.

Theorem 2.4. (Brill–Nöther) *A general nonsingular, connected, complex projective curve of genus $g \geq 2$ satisfies the Brill–Nöther property.*

The proof will be given in Section 3, using Theorem 2.11.

Remark 2.5. Every rational or elliptic curve satisfies the Brill–Noether property, as it can easily be checked by considering their special linear systems. So we restrict our attention to $g \geq 2$.

2.6. *Generality.* What does "general" mean? The idea, when a statement is made for a "general" object, is that all objects of the same kind, but for particular cases, satisfy that statement. So, by this concept, the word "general" can only be used when there is a classifying space for the objects being considered. In the case of the Brill–Nöther theorem this space is the so-called *moduli space of smooth curves* of genus g, usually denoted by M_g. The precise statement of the Brill–Nöther theorem is thus:

There is an open dense subset of M_g, for each $g \geq 2$, such that any curve represented by a point on that open subset satisfies the Brill–Nöther property.

2.7. *Openness.* The Brill–Nöther theorem is also equivalent to the following statement:

For each $g \geq 2$ there is a nonsingular, connected, complex projective curve of genus g satisfying the Brill–Nöther property.

The point is that the Brill–Nöther property is "open." So, if there is a nonsingular curve satisfying it, then there is an open neighborhood of the point representing the curve in M_g such that all curves represented in that open set satisfy the Brill–Nöther property.

To explain this idea in more precise terms, we need to introduce a few objects. Let $f\colon X \to S$ be a smooth projective map between complex algebraic schemes with connected fibers of dimension 1. For each integer d, there is an S-scheme Pic_f^d parameterizing line bundles of degree d on the fibers of f, the so-called degree-d *relative Picard scheme* of f; see [Gr], Thm. 3.1 or [BLR], Thm. 1, p. 210. Since f is smooth, Pic_f^d is proper over S; see, for instance, [BLR], Thm. 3, p. 232 and Thm. 1, p. 252. For each nonnegative integer r, let $W_r^d(f) \subseteq \mathrm{Pic}_f^d$ be the closed subscheme parameterizing those line bundles on the fibers of f having at least $r+1$ linearly independent sections; see Subsection 3.5. (That $W_r^d(f)$ is indeed closed follows from the semicontinuity theorem.) Since Pic_f^d is proper over S, so is $W_r^d(f)$.

Now, suppose one of the fibers of f satisfies the Brill–Nöther property. Denote by s the point of S over which that curve lies. Let g denote the genus of every fiber of f, and let d and r be nonnegative integers such that $\rho(g,d,r) < 0$. By the Brill–Nöther property, $W_r^d(f)$ does not intersect the fiber of Pic_f^d over s. Since $W_r^d(f)$ is proper over S, its image in S is thus a closed subset not containing s. So there is an open neighborhood $U_s(d,r) \subseteq S$ of s such that $W_r^d(f)$ does not intersect any fiber of Pic_f^d over a point in $U_s(d,r)$. This means that no fiber of f over a point in $U_s(d,r)$ admits a linear system of degree d and rank r.

Intersecting all the $U_s(d,r)$ with $d \leq 2g-2$ and $r \leq d$ (and $\rho(g,d,r)$ negative) we get an open neighborhood of s such that all fibers of f over points of that neighborhood satisfy the Brill–Nöther property. We have just shown that the subset $U \subseteq S$ of points over which the fibers of f satisfy the Brill–Nöther property is open.

If M_g were a fine moduli space, then there would be a smooth projective map f as above with $S = M_g$ whose fiber over each $s \in S$ would be the curve represented by s in M_g. Then the above reasoning, and the irreducibility of M_g (see [DM]) would yield the Brill–Nöther statement of Subsection 2.6.

However, M_g is just a coarse moduli space. Anyway, there is a map f as above such that the induced "moduli map" $h : S \to M_g$, taking $s \in S$ to the point representing the fiber $f^{-1}(s)$ is surjective and proper, even finite; see [HM], Lemma 3.89, p. 142. Then $V := M_g - h(S-U)$ is open, and parameterizes curves satisfying the Brill–Nöther property.

Remark 2.8. Even though there are in a sense many more curves that satisfy the Brill–Nöther property than those that don't, it is very difficult to exhibit explicitly curves that satisfy the property. The reason is that most curves that we can think of, and those that appear in practice, are very particular, like plane curves, complete intersections, hyperelliptic, trigonal, tetragonal, etc.

2.9. *History.* What we are calling the Brill–Nöther theorem in these notes is actually just a part of the full statement of it. A more complete statement is:

A general nonsingular, connected, complex projective curve of genus $g \geq 2$ has a linear system of degree d and rank r if and only if $\rho(g,d,r) \geq 0$; and if so, then $\rho(g,d,r)$ is the dimension of the locus of those linear systems.

To make the addendum in the last statement more precise, let C be a nonsingular, connected, projective curve of genus g. As in Subsection 2.7, for each integer d, let $\mathrm{Pic}^d C$ be the degree-d Picard scheme of C, parameterizing line bundles of degree d on C. And, for each integer r, let $W^d_r C \subseteq \mathrm{Pic}^d C$ be the closed subset parameterizing line bundles with at least $r+1$ linearly independent sections; see Subsection 3.5. Then the addendum to the above Brill–Nöther statement says:

If C is general and $\rho(g,d,r) \geq 0$, then $\dim W^d_r C = \rho(g,d,r)$.

Brill and Nöther made their statement in [BN], p. 290, giving an incomplete proof. Severi, based on ideas of Castelnuovo [C], suggested a way of proving the statement, by using a degeneration argument; see [S], Anhang G, Section 8, p. 380. There are serious problems with his approach, but a variation of it eventually proved the statement, as we will comment in more detail below.

The "if" part of the Brill–Nöther statement was proved independently by Kempf [Ke] and by Kleiman and Laksov [KL1], [KL2]. It is not our goal in these notes to go through that proof. However, let us just sketch the argument. The argument is based on the fact that $W^d_r C$ is a determinantal variety, as explained in Subsection 3.5, and hence its class in the Chow ring of $\mathrm{Pic}^d C$ can be given by Porteous formula if $W^d_r C$ is either empty or of the right codimension. The idea is then to compute that class, and check that it is nonzero, and hence cannot be the class of the empty set. This argument, and hence the "if" part of the Brill–Nöther statement, is valid for any nonsingular, connected, projective curve C.

To prove the "only if" part, Severi suggested considering a family of smooth curves degenerating to a general rational nodal curve X_0, that is, a curve obtained from $\mathbf{P}^1$ by choosing $2g$ general points of $\mathbf{P}^1$, grouping them in g pairs, and identifying the two points in each pair, in such a way to produce an ordinary double point.

Severi's idea was that if linear systems of a certain rank and degree existed for the smooth curves in the family, then linear systems of the same kind would exist, by passage to the limit, on X_0. If so, one could consider the pullbacks of those linear systems on the $\mathbf{P}^1$ normalizing X_0. On $\mathbf{P}^1$ we would have linear systems of rank r and degree d that, being pullbacks, would be special in the sense that every section that is zero on a branch over a node of X_0 would have to vanish on the other branch as well. If the branches are in general position on $\mathbf{P}^1$, then one could hope that the locus of those linear systems on $\mathbf{P}^1$ has the "expected" dimension, and that is exactly $\rho(g,d,r)$; see [HM], Chapter 5 for more details.

It turns out that the above argument presents two problems. First, linear systems may not degenerate to linear systems, as line bundles may not degenerate to line bundles. The degree-d Picard scheme of X_0 is not complete! This problem was the first to be overcome, by Kleiman [Kl], by using torsion-free rank-1 sheaves.

The second problem is a major one. It is hard to exhibit a set of $2g$ points on $\mathbf{P}^1$ such that the locus of linear systems on $\mathbf{P}^1$ mentioned above has dimension $\rho(g,d,r)$, if nonempty. This seems to be as hard as exhibiting a nonsingular curve satisfying the Brill–Nöther property!

Despite this problem, Griffiths and Harris [GH] were able to "complete" Severi's argument by considering specializations of C_0, making the $2g$ points on $\mathbf{P}^1$ converge, in a certain way, to a single point.

Later, it was noticed by Eisenbud and Harris [EH1], following work by Gieseker [Gi], that the proof of the Brill–Nöther statement is simplified by considering a degeneration to a rational cuspidal curve, instead of a nodal one. And by considering a semistable model of that curve, where the cusps are replaced by elliptic curves attached to the normalization, a flag curve according to Definition 2.10 below, one would not even need to consider torsion-free rank-1 sheaves. The proof we give in these notes follows this idea.

Definition 2.10. A *nodal curve* is a connected complex projective curve whose only singularities are *nodes*, that is, ordinary double points. A *flag curve*, in these notes, is a nodal curve F satisfying the following three properties:

(1) It is of compact type or, equivalently, the number of nodes of F is smaller (by one) than the number of components.
(2) Each component of F is either $\mathbf{P}^1$ or an elliptic curve.

(3) Each elliptic component of F contains exactly one node of X.

Theorem 2.11. *Let $f\colon X \to S$ be a flat, projective map from a regular scheme X to $S := \operatorname{Spec}(\mathbb{C}[[t]])$. If the special fiber of f is a flag curve, then the general fiber satisfies the Brill–Nöther property.*

The proof will be given in Section 6. A clarification of the statement will be given in Subsection 3.6. Also, in Subsection 3.7 we will see how Theorem 2.11 implies Theorem 2.4.

3. Deformations of nodal curves

3.1. *Deformation theory.* The infinitesimal deformations of a nodal curve, as far as smoothening of the nodes go, is easy to describe.

Let X_0 be a nodal curve. Then there is a versal deformation of X_0 over a ring of power series over $\mathbb{C}$; see [DM], p. 79. In other words, there are a map $h\colon Y \to B$, where $B := \operatorname{Spec}(\mathbb{C}[[t_1, \dots, t_m]])$, and an isomorphism between X_0 and the closed fiber of h, satisfying certain universal properties.

The versal deformation space of X_0 is formally smooth over the versal deformation space of its singularities; see [DM], Prop. 1.5, p. 81. In other words, let $N_1, \dots, N_\delta$ denote the nodes of X_0. Then $m \geq \delta$ and, after a change of variables, we may assume that for each $i = 1, \dots, \delta$ there is an isomorphism of $\mathbb{C}[[t_1, \dots, t_m]]$-algebras:

$$\widehat{\mathcal{O}}_{Y,N_i} \xrightarrow{\sim} \frac{\mathbb{C}[[t_1, \dots, t_m, u, v]]}{(uv - t_i)}.$$

Let $S := \operatorname{Spec}(\mathbb{C}[[t]])$ and let $S \to B$ be the map given by sending t_i to t for each $i = 1, \dots, m$. Form the fibered product $X := Y \times_B S$, and let $f\colon X \to S$ denote the projection onto the second factor. Then f is flat and projective, being a base change of h. The closed fiber of f is naturally isomorphic to the closed fiber of h, which is identified with X_0. In addition, from the description of the map $S \to B$, for each $i = 1, \dots, \delta$ there is an isomorphism of $\mathbb{C}[[t]]$-algebras:

$$\widehat{\mathcal{O}}_{X,N_i} \cong \frac{\mathbb{C}[[t, u, v]]}{(uv - t)}.$$

In particular, X is regular at each N_i. Since in addition f is smooth on an open neighborhood of each nonsingular point of X_0, it follows that X is regular on an open neighborhood of X_0. But an open neighborhood of X_0 is X! So X is regular.

We have just proved that regular smoothings of X_0 exist, and this is everything we need in the sequel.

Definition 3.2. Let X_0 be a nodal curve. A *regular smoothing* of X_0 consists of two data: a flat, projective map $f\colon X \to S$ from a regular scheme X to $S := \mathrm{Spec}(\mathbb{C}[[t]])$ and an isomorphism between the closed fiber of f and X_0.

3.3. *Base changes of regular smoothings.* Let X_0 be a nodal curve, and $f\colon X \to S$ a regular smoothing of X_0. Identify X_0 with the closed fiber of f with the provided isomorphism. Let X_* denote the general fiber of f.

Since $X_* \subset X$ is open, X_* is regular. Moreover, since X_* is a scheme over the field of Laurent series, $\mathbb{C}((t))$, which has characteristic zero, X_* is smooth. In addition, since X_0 is connected, $h^0(X_0, \mathcal{O}_{X_0}) = 1$, and thus, by semicontinuity, $h^0(X_*, \mathcal{O}_{X_*}) = 1$. In particular, X_* is geometrically connected, that is, X_* is connected and any base extension of X_* is connected. Finally, since X_0 has dimension 1, by flatness so does X_*.

The fiber X_* is defined over $\mathbb{C}((t))$, which is not algebraically closed. In applications, it is often necessary to consider nonrational points of schemes derived from X_*, i.e. points defined over a finite field extension of $\mathbb{C}((t))$. At the cost of changing X_0 in a very controlled way, we may actually assume that the necessary field extension is trivial.

More precisely, let k be a finite field extension of $\mathbb{C}((t))$. Let R be the integral closure of $\mathbb{C}[[t]]$ in k. Since $\mathbb{C}[[t]]$ is Noetherian, R is a finite $\mathbb{C}[[t]]$-module by [M], Lemma 1, p. 262. So, by [Ei], Cor. 7.6, p. 190, the ring R is isomorphic to a finite product of complete local rings. Since R is a domain, R is itself a complete local ring. Let $P \subset R$ denote its maximal ideal. Since R is normal of dimension one, R is regular. Since R is finite over $\mathbb{C}[[t]]$, so is R/P over $\mathbb{C}$, and hence $\mathbb{C} \cong R/P$. So R is a complete, local, Noetherian $\mathbb{C}$-algebra of dimension 1 with residue field isomorphic to $\mathbb{C}$. By the Cohen structure theorem, [Ei], Thm. 7.7, p. 191, there is an isomorphism of $\mathbb{C}$-algebras $R \xrightarrow{\sim} \mathbb{C}[[s]]$. It follows that there is an integer $e \geq 1$ such that $tR = P^e$. Since every power series in $\mathbb{C}[[t]]$ with nonzero constant term has an e-th root, we may choose the isomorphism $R \xrightarrow{\sim} \mathbb{C}[[s]]$ such that t is sent to s^e.

Let $\epsilon\colon S \to S$ be the map given by sending t to t^e. To differentiate source from target, we will denote the source of ϵ by S_ϵ. The upshot is that the fibered product $X_\epsilon := X \times_S S_\epsilon$ has, as general fiber over S_ϵ, the base extension $X_* \times k$, and as special fiber, the same fiber X_0. The new scheme X_ϵ is flat and projective over S_ϵ, but fails to be regular if $e > 1$.

Indeed, let N be a node of X_0. Since X is regular, and flat over S with closed fiber of pure dimension 1, the dimension of X is 2. Using the

Cohen structure theorem again, there is an isomorphism of $\mathbb{C}$-algebras $\widehat{\mathcal{O}}_{X,N} \xrightarrow{\sim} \mathbb{C}[[u,v]]$. Since N is a node of X_0, the tangent space of X_0 at N is equal to that of X. Thus we may choose the isomorphism such that $uv\widehat{\mathcal{O}}_{X,N} = t\widehat{\mathcal{O}}_{X,N}$, and there is even a choice such that $t = uv$. So, as $\mathbb{C}[[t]]$-algebras,

$$\widehat{\mathcal{O}}_{X,N} \cong \frac{\mathbb{C}[[t,u,v]]}{(uv-t)}.$$

After the base change, we have that

$$\widehat{\mathcal{O}}_{X_\epsilon,N} \cong \frac{\mathbb{C}[[t,u,v]]}{(uv-t^e)}.$$

So X_ϵ fails to be regular at N if $e > 1$. A singularity of a surface whose complete local ring is isomorphic to the above local ring is called an A_{e-1}-*singularity.*

Suppose $e > 1$. We may resolve the singularities of X_ϵ by blowing up, at the cost of adding rational components to X_0. Indeed, let X'_ϵ be the blowup of X_ϵ at N. To describe X'_ϵ locally over N we may replace X_ϵ by $\mathrm{Spec}(\widehat{\mathcal{O}}_{X_\epsilon,N})$. The ideal of N in $\widehat{\mathcal{O}}_{X_\epsilon,N}$ is (t,u,v). Thus the blowup can be covered by three affine open subschemes, U_1, U_2 and U_3, the first two with rings of functions

$$\frac{\mathbb{C}[[u,v,t]][\xi_1,\xi_2]}{(u-\xi_1 t, v-\xi_2 t, \xi_1\xi_2 - t^{e-2})} \quad \text{and} \quad \frac{\mathbb{C}[[u,v,t]][\zeta_1,\zeta_2]}{(t-\zeta_1 u, v-\zeta_2 u, \zeta_2 - \zeta_1^e u^{e-2})},$$

respectively, and U_3 with a ring of functions very similar to that of U_2, but with u exchanged with v. The patching between U_1 and U_2 is given by $\xi_1\zeta_1 = 1$ and $\xi_1\zeta_2 = \xi_2$.

From the above local descriptions we see that the fiber of X'_ϵ over N consists of the union of two smooth rational curves, L_1 and L_2, meeting at a node, denoted N'. These curves are given by $\xi_1 = 0$ and $\xi_2 = 0$ in U_1. The node N' is the unique singular point of X'_ϵ, but is a milder singularity than N is with respect to X, as the power e drops to $e-2$. Actually, the above description works for $e > 3$ only. If $e = 2$, then X'_ϵ is regular, and the fiber over N is a unique smooth rational curve L, the conic given by $\xi_1\xi_2 = 1$ in U_1. From the descriptions of U_2 and U_3, we see that L_1 and L_2 (or just L) intersect transversally the rest of the closed fiber of X'_ϵ over S_ϵ. More precisely, the branches of X_0 at N are split in X'_ϵ, with one branch lying on U_2 and the other on U_3. Then L_2 passes through the branch lying on U_2 and L_1 through that on U_3. If $e = 2$, then both branches are in L.

The upshot is that, by blowing up at N, we produce a scheme X'_ϵ whose closed fiber over S_ϵ consists of the union of the partial normalization X_0^N of X_0 at N and a nodal curve E_N meeting transversally

X_0^N at the two branches over N. If $e = 2$, the curve E_N is smooth and rational, and X'_ϵ is regular on a neighborhood of E_N. If $e > 2$, then E_N is the union of two smooth, rational curves meeting transversally at a single point N', and X'_ϵ is regular on a neighborhood of E_N but at the point N', which, for $e > 3$, is an A_{e-3}-singularity of X'_ϵ. Also, the branches of X_0^N over N are distributed between the components of E_N.

If X'_ϵ is not regular on a neighborhood of E_N, that is, if $e > 3$, we proceed by blowing up X'_ϵ at N'. Since N' is an A_{e-3}-singularity, it is clear that this second blowup has a description similar to that given to X'_ϵ, with e replaced by $e - 2$.

By repeating the above process, and applying it to each node of X_0, it should be clear by now that we will end up with a regular surface $\widetilde{X}$, which is flat and projective over S_ϵ, and whose closed fiber is the union of the (total) normalization X_0^ν of X_0 with a collection of disjoint chains of $e - 1$ rational curves, one for each node of X_0. Each chain corresponds to a node of X_0, and intersects X_0^ν transversally at the two branches over that node, which become points on the outer components of the chain, one for each component.

From the above description, the general fiber of $\widetilde{X}$ over S_ϵ is the base extension $X_* \times k$, while the closed fiber is what we will call here an *avatar* of X_0, as explained below.

Definition 3.4. A *chain of n rational curves*, for $n \geq 2$, is a nodal curve with n irreducible components, all of them smooth and rational, and $n - 1$ nodes. In addition, it is required that the number of components containing only one node of the curve is 2. These two components are called the *outer* components of the chain. A smooth rational curve will eventually be called, for homogeneity, a chain of 1 rational curve.

Let X_0 be a nodal curve. Let $N_1, \dots, N_\delta$ be nodes of X_0, and X'_0 the partial normalization of X_0 along them. Let $E_1, \dots, E_\delta$ be chains of rational curves, not necessarily with the same number of components. Let X_1 be the union of X'_0 with $E_1, \dots, E_\delta$ in such a way that E_i and E_j are disjoint if $i \neq j$, and each E_i intersects X'_0 transversally at exactly two points: the branches of X'_0 over N_i on the side of X'_0, and two points lying each on a different outer component of E_i, on the side of E_i. We call all possible curves X_1 obtained from X_0 in this way *avatars* of X_0.

3.5. *Determinantal subschemes of the Picard scheme.* Let $f\colon X \to S$ be a smooth, projective map with geometrically connected fibers of dimension 1. Let g denote the genus of the fibers of f.

For each integer d, let Pic_f^d denote the degree-d relative Picard scheme of f, parameterizing invertible sheaves of degree d on the fibers of f. Assume f admits a section $\sigma\colon S \to X$, and let $\Sigma := \sigma(S)$. Then there is a Poincaré, or universal sheaf $\mathcal{L}$ on $X \times_S \mathrm{Pic}_f^d$, an invertible sheaf whose restriction to $X \times_S \{t\}$ for each $t \in \mathrm{Pic}_f^d$ is the invertible sheaf represented by t; see [BLR], Prop. 4, p. 211. The Poincaré sheaf is unique if we impose that it be rigidified by the section, i.e. that $\mathcal{L}|_{\Sigma \times_S \mathrm{Pic}_f^d}$ be trivial.

Since f is smooth, $\Sigma \subset X$ is an effective Cartier divisor. Denote by $p_1\colon X \times_S \mathrm{Pic}_f^d \to X$ and $p_2\colon X \times_S \mathrm{Pic}_f^d \to \mathrm{Pic}_f^d$ the projection maps. Set

$$\mathcal{M} := \mathcal{L} \otimes p_1^* \mathcal{O}_X(n\Sigma)$$

for an integer $n >> 0$. More precisely, we need that

$$h^1(X \times_S \{t\}, \mathcal{M}|_{X \times_S \{t\}}) = 0 \tag{3.5.1}$$

for each $t \in \mathrm{Pic}_f^d$. As $\mathcal{M}$ has relative degree $d+n$ over Pic_f^d, it is enough, by the Riemann–Roch theorem, to choose n with $n \geq 2g-1-d$.

Since f is smooth of relative dimension one, $n\Sigma \subset X$ is finite and flat over S with relative degree n. Set

$$\Sigma_n := n\Sigma \times_S \mathrm{Pic}_f^d \subset X \times_S \mathrm{Pic}_f^d.$$

Consider the derived long exact sequence of higher direct images under p_2 of the natural exact sequence

$$0 \longrightarrow \mathcal{L} \xrightarrow{\cdot \Sigma_n} \mathcal{M} \longrightarrow \mathcal{M}|_{\Sigma_n} \longrightarrow 0. \tag{3.5.2}$$

Since Equation (3.5.1) holds for each $t \in \mathrm{Pic}_f^d$, we have $R^1 p_{2*}\mathcal{M} = 0$. So we obtain an exact sequence:

$$0 \to p_{2*}\mathcal{L} \to p_{2*}\mathcal{M} \to p_{2*}\mathcal{M}|_{\Sigma_n} \to R^1 p_{2*}\mathcal{L} \to 0. \tag{3.5.3}$$

Let

$$\varphi\colon p_{2*}\mathcal{M} \longrightarrow p_{2*}\mathcal{M}|_{\Sigma_n}$$

denote the middle map in the above sequence.

Since $\mathcal{M}$ and $\mathcal{M}|_{\Sigma_n}$ are flat over Pic_f^d, and their restrictions to the fibers $X \times_S \{t\}$ for $t \in \mathrm{Pic}_f^d$ have zero higher cohomology, φ is a map of locally free sheaves. The rank of the source is $d+n+1-g$, by the Riemann–Roch theorem, while the rank of the target is n. For each integer $u \geq 0$ let

$$E_u := \{t \in \mathrm{Pic}_f^d \mid \varphi(t) \text{ has rank at most } u\}.$$

More precisely, E_u is the closed subscheme of Pic_f^d given locally by the vanishing of the minors of size $u+1$ of a matrix representing φ. Since different representing matrices are similar, the ideal generated by the minors is well defined. Because of the way it is defined, we call E_u a *determinantal scheme.*

What does E_u parameterize? To see this, let $h\colon T \to \mathrm{Pic}^d_f$ be any map of S-schemes, and put

$$h_1 := 1 \times h\colon X \times_S T \to X \times_S \mathrm{Pic}^d_f.$$

Let $q_2\colon X \times_S T \to T$ be the projection onto the second factor. Since $\mathcal{M}|_{\Sigma_n}$ is flat over Pic^d_f, applying h_1^* to (3.5.2) we end up with a short exact sequence of sheaves on $X \times_S T$. And, as before, the derived long exact sequence of higher direct images under q_2 truncates to the exact sequence:

$$(3.5.4) \qquad 0 \to q_{2*}h_1^*\mathcal{L} \to q_{2*}h_1^*\mathcal{M} \to q_{2*}h_1^*\mathcal{M}|_{\Sigma_n} \to R^1q_{2*}h_1^*\mathcal{L} \to 0.$$

There is a natural map of exact sequences from the pullback of (3.5.3) under h to (3.5.4):

$$\begin{array}{ccccccc}
h^*p_{2*}\mathcal{L} & \longrightarrow & h^*p_{2*}\mathcal{M} & \xrightarrow{h^*\varphi} & h^*p_{2*}\mathcal{M}|_{\Sigma_n} & \longrightarrow & h^*R^1p_{2*}\mathcal{L} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
q_{2*}h_1^*\mathcal{L} & \longrightarrow & q_{2*}h_1^*\mathcal{M} & \longrightarrow & q_{2*}h_1^*\mathcal{M}|_{\Sigma_n} & \longrightarrow & R^1q_{2*}h_1^*\mathcal{L}.
\end{array}$$

Since $\mathcal{M}$ and $\mathcal{M}|_{\Sigma_n}$ are flat over Pic^d_f, and their restrictions to the fibers $X \times_S \{t\}$ for $t \in \mathrm{Pic}^d_f$ have zero higher cohomology, the two middle vertical maps above are isomorphisms. Thus

$$(3.5.5) \qquad \mathrm{Ker}(h^*\varphi) \cong q_{2*}h_1^*\mathcal{L} \quad \text{and} \quad \mathrm{Coker}(h^*\varphi) \cong R^1q_{2*}h_1^*\mathcal{L}.$$

Because of this property, we say that φ represents universally the cohomology of $\mathcal{L}$ under p_2.

Applying (3.5.5) to the case $T = \{t\}$, for $t \in \mathrm{Pic}^d_f$, we see that

$$\mathrm{Ker}(\varphi(t)) \cong H^0(X \times_S \{t\}, \mathcal{L}|_{X \times_S \{t\}}).$$

So

$$E_u = \{t \in \mathrm{Pic}^d_f \mid h^0(X \times_S \{t\}, \mathcal{L}|_{X \times_S \{t\}}) \geq d + n + 1 - g - u\}.$$

Fix $u := d + n - g - r$. Then E_u parameterizes invertible sheaves with at least $r + 1$ linearly independent sections. We set $W^d_r(f) := E_u$.

In principle, it seems that $W^d_r(f)$ depends on the choice of the section σ and of the integer n. It does not. In fact, since φ is a presentation for $R^1p_{2*}\mathcal{L}$, from the exact sequence (3.5.3), we see that E_u is defined by the $(g+r-d-1)$-th Fitting ideal of $R^1p_{2*}\mathcal{L}$. (See [Ei], Section 22.2, p. 496 for the definition of Fitting ideals of modules, their independence of the choice of presentations, and their functoriality, which allows for their patching.) Being $\mathcal{L}$ rigidified by σ, it could still seem that $W^d_r(f)$ depends on the choice of σ. It does not. If $\mathcal{L}'$ is another Poincaré sheaf, rigidified by another section or not, then $\mathcal{L}' \cong \mathcal{L} \otimes p_2^*\mathcal{N}$ for an invertible sheaf $\mathcal{N}$ on Pic_f^d. Then $R^1p_{2*}\mathcal{L}' \cong R^1p_{2*}\mathcal{L} \otimes \mathcal{N}$, and hence $R^1p_{2*}\mathcal{L}'$ and $R^1p_{2*}\mathcal{L}$ have the same Fitting ideals.

What happens if f does not admit a section? Well, the projection onto the second factor, $b\colon X \times_S X \to X$, admits a section, the diagonal embedding. So we may construct a subscheme $W_r^d(b) \subset \mathrm{Pic}_b^d$ as before. Now, the formation of the relative Picard scheme is functorial, that is, commutes with base change. In addition, $W_r^d(b)$ does not depend on the choice of the section. Thus, since f is flat, $W_r^d(b)$ descends to a closed subscheme $W_r^d(f) \subset \mathrm{Pic}_f^d$. Moreover, the formation of $W_r^d(f)$ commutes with base change. More precisely, if $S' \to S$ is any map of schemes, and $f'\colon X \times_S S' \to S'$ is the projection onto the second factor, then $W_r^d(f) \times_S S' = W_r^d(f')$ as subschemes of $\mathrm{Pic}_f^d \times_S S' = \mathrm{Pic}_{f'}^d$.

If S is the spectrum of a field, we will use the notation $\mathrm{Pic}^d X := \mathrm{Pic}_f^d$ and $W_r^d X := W_r^d(f)$.

The above construction can be found in [ACGH], Chapter IV, Section 3, p. 176 for the case of a single curve.

3.6. *Clarification of the statement of Theorem 2.11.* Let X_* be the general fiber of the given map f. As we observed in Subsection 3.3, the fiber X_* is smooth and geometrically connected over $\mathbb{C}((t))$. Let k be an algebraic closure of $\mathbb{C}((t))$, and let $G := X_* \times k$ be the base extension of X_* over k. Let g be the genus of G.

Being more precise, Theorem 2.11 states that for each pair of non-negative integers (d, r) such that $\rho(g, d, r) < 0$ there is no invertible sheaf on G with degree d having at least $r + 1$ linearly independent sections, i.e. $W_r^d G = \emptyset$. Notice that, by what we saw in Subsection 3.5, we have $W_r^d G = W_r^d X_* \times k$. Thus, requiring that $W_r^d G = \emptyset$ is the same as requiring that $W_r^d X_* = \emptyset$.

3.7. *Proof of Theorem 2.4.* Let F be a flag curve of arithmetic genus g, i.e. with g elliptic components. Since F is nodal, as we observed in Subsection 3.1, there is a regular smoothing of F, i.e. there are a flat, projective map $f\colon X \to S$ from a regular scheme X to $S := \mathrm{Spec}(\mathbb{C}[[t]])$ and an isomorphism between the closed fiber and F. Let X_0 denote the closed fiber and X_* the generic fiber of f.

Since X_* is projective, hence given by a finite number of equations in projective space, there is a subfield $k \subseteq \mathbb{C}((t))$ finitely generated over $\mathbb{Q}$ such that X_* is actually defined over k, i.e. there is a projective curve G over k such that $X_* = G \times_k \mathbb{C}((t))$. Since $\mathbb{C}$ has infinite transcendence degree over $\mathbb{Q}$, we may embed k in $\mathbb{C}$, and thus consider an extension of G over $\mathbb{C}$ to a complex curve C, i.e. $C = G \times_k \mathbb{C}$. Since X_* is geometrically connected and smooth, so are G and C, and all of them have the same genus g. So C is a nonsingular, connected, complex projective curve of genus g. We claim that C satisfies the Brill–Nöther property, thus proving the Brill–Nöther statement in Subsection 2.7, from which Theorem 2.4 follows.

Indeed, let (d, r) be a pair of nonnegative integers such that $\rho(g, d, r)$ is negative. We need to show that $W_r^d C = \emptyset$. However, $W_r^d X_* = \emptyset$ by Theorem 2.11; see Subsection 3.6. Since

$$W_r^d C = W_r^d G \times_k \mathbb{C} \quad \text{and} \quad W_r^d X_* = W_r^d G \times_k \mathbb{C}((t)),$$

it follows that $W_r^d C = \emptyset$. The proof of Theorem 2.4 is complete.

4. Ramification points

4.1. *Ramification points of linear systems.* Let C be a nonsingular, connected, complex projective curve of genus g. Let L be a line bundle on C and $V \subseteq \Gamma(C, L)$ a nonzero vector subspace. Let $d := \deg L$ and $r := \dim V - 1$.

Let $P \in C$. We say that an integer ϵ is an *order* of the linear system (V, L) at P if there is a nonzero section of L in V vanishing at P with order ϵ. If two sections of L have the same order, a certain linear combination of them will be zero or have higher order. Thus there are exactly $r + 1$ orders of (V, L) at P. Putting them in increasing order we get a sequence,

$$\epsilon_0(P), \dots, \epsilon_r(P),$$

called the *order sequence* of (V, L) at P. Notice that $i \leq \epsilon_i(P) \leq d$ for each i. Put

$$\mathrm{wt}(P) := \sum_{i=0}^{r} \big(\epsilon_i(P) - i\big).$$

Then

$$0 \leq \mathrm{wt}(P) \leq (r+1)(d-r).$$

We call $\mathrm{wt}(P)$ the *ramification weight* of (V, L) at P. If $\mathrm{wt}(P) > 0$ we say that P is a *ramification point* of (V, L). Also, we call the cycle

$$[W(V, L)] := \sum_{P \in C} \mathrm{wt}(P)[P]$$

the *ramification cycle* of (V, L).

4.2. *The Plücker formula.* Keep the setup of Subsection 4.1. Since C is smooth, Ω^1_C is a line bundle. Let $U \subseteq C$ be an open subscheme such that Ω^1_U and $L|_U$ are trivial. Let $\mu \in \Gamma(U, \Omega^1_C)$ and $\sigma \in \Gamma(U, L)$ be sections generating Ω^1_U and $L|_U$.

Fix a basis $\beta = (s_0, \dots, s_r)$ of V. Then there are regular functions $f_0, \dots, f_r$ on U such that $s_i|_U = f_i\sigma$ for each i. Let ∂ be the $\mathbb{C}$-linear derivation of $\Gamma(U, \mathcal{O}_C)$ such that $dh = \partial(h)\mu$ for each $h \in \Gamma(U, \mathcal{O}_C)$. Form the Wronskian determinant:

$$w(\beta, \sigma, \mu) := \begin{vmatrix} f_0 & \dots & f_r \\ \partial f_0 & \dots & \partial f_r \\ \vdots & \ddots & \vdots \\ \partial^r f_0 & \dots & \partial^r f_r \end{vmatrix}.$$

If σ' and μ' are other bases of $L|_U$ and Ω^1_U then $\sigma' = a\sigma$ and $\mu' = b\mu$ for certain everywhere nonzero regular functions a and b on U. Then

$$w(\beta, \sigma', \mu') = \begin{vmatrix} af_0 & \dots & af_r \\ b\partial(af_0) & \dots & b\partial(af_r) \\ \vdots & \ddots & \vdots \\ (b\partial)^r(af_0) & \dots & (b\partial)^r(af_r) \end{vmatrix} = a^{r+1}b^{\binom{r+1}{2}}w(\beta, \sigma, \mu),$$

where the first equality follows from the definition, and the second from the multilinearity of the determinant and the product rule of derivations.

Thus the $w(\beta, \sigma, \mu)$ patch up to a section of

$$L^{\otimes r+1} \otimes (\Omega^1_C)^{\otimes \binom{r+1}{2}}.$$

Denote the zero scheme of this section by $W(V, L)$. We call $W(V, L)$ the *ramification divisor* of (V, L).

The multilinearity of the determinant, and the fact that ∂ is $\mathbb{C}$-linear, imply that $W(V, L)$ does not depend on the choice of basis β of V.

Given any effective divisor D of C and any $P \in C$ we let

$$\operatorname{mult}_{P,C}(D)$$

denote the multiplicity of D at P, and consider the associated cycle:

$$[D] := \sum_{P \in C} \operatorname{mult}_{P,C}(D)[P].$$

The cycle associated to $W(V, L)$ is the ramification cycle $[W(V, L)]$.

This statement justifies the notation used in Subsection 4.1. Since L has degree d and Ω^1_C has degree $2g - 2$, it follows that

$$\deg[W(V, L)] = (r + 1)\big(d + r(g - 1)\big),$$

a formula known as the *Plücker formula.*

To prove the statement, let $P \in C$. Let t be a local parameter of C at P. Then t is a regular function on an open neighborhood $U \subset C$ of P.

Shrinking U around P if necessary, we may assume that dt generates Ω^1_U. Also, we may assume there is $\sigma \in \Gamma(U, L)$ generating $L|_U$.

There are $s_0, \dots, s_r \in V$ vanishing at P with orders $\epsilon_0(P), \dots, \epsilon_r(P)$. Shrinking U around P if necessary, we may assume that there are everywhere nonzero regular functions $u_0, \dots, u_r$ on U such that

$$s_i|_U = u_i t^{\epsilon_i(P)} \sigma$$

for each i. Since the orders of vanishing are distinct, $\beta := (s_0, \dots, s_r)$ is a basis of V.

The Wronskian determinant $w(\beta, \sigma, dt)$ has the form:

$$w(\beta, \sigma, dt) = \begin{vmatrix} u_0 t^{\epsilon_0(P)} & \dots & u_r t^{\epsilon_r(P)} \\ \frac{d}{dt}(u_0 t^{\epsilon_0(P)}) & \dots & \frac{d}{dt}(u_r t^{\epsilon_r(P)}) \\ \vdots & \ddots & \vdots \\ \frac{d^r}{dt^r}(u_0 t^{\epsilon_0(P)}) & \dots & \frac{d^r}{dt^r}(u_r t^{\epsilon_r(P)}) \end{vmatrix}.$$

Using the multilinearity of the determinant, the product rule of derivations, and the fact that $\frac{d}{dt}(t^j) = jt^{j-1}$ for each integer $j \geq 1$, we get

$$w(\beta, \sigma, dt) = t^{\mathrm{wt}(P)} v,$$

where v is a regular function on U whose value at P satisfies

$$v(P) = \prod_{i=0}^{r} \binom{\epsilon_i(P)}{i} \prod_{i=0}^{r} u_i(P).$$

In particular, $v(P) \neq 0$, and thus $w(\beta, \sigma, dt)$ vanishes at P with order $\mathrm{wt}(P)$. This order of vanishing is, by definition, the multiplicity of $W(V, L)$ at P. Since this is valid for every $P \in C$, we get that the cycle associated to $W(V, L)$ is indeed $[W(V, L)]$.

5. Limit linear series

5.1. *Setup.* Let $S := \mathrm{Spec}(\mathbb{C}[[t]])$. Let X_0 be a nodal curve, and $f \colon X \to S$ a regular smoothing of X_0. Let X_* denote the general fiber of f, and identify the closed fiber with X_0. Let $C_1, \dots, C_n$ denote the irreducible components of X_0. Though not really necessary, for simplicity we will assume in these notes that $C_1, \dots, C_n$ are nonsingular.

5.2. *Twists.* Keep Setup 5.1. Since X is regular, every invertible sheaf on X_* can be extended to an invertible sheaf on the whole X. But the extension is not unique. Indeed, since X is regular and two-dimensional, $C_1, \dots, C_n$ are Cartier divisors of X. So, for each invertible sheaf $\mathcal{L}$ on X, and each n-tuple of integers $\alpha = (\alpha_1, \dots, \alpha_n)$, we may define

$$\mathcal{L}^\alpha := \mathcal{L} \otimes \mathcal{O}_X(\alpha_1 C_1 + \dots + \alpha_n C_n).$$

Then $\mathcal{L}^\alpha$ is invertible and satisfies $\mathcal{L}^\alpha|_{X_*} = \mathcal{L}|_{X_*}$. We say that $\mathcal{L}^\alpha$ is the *α-twist* of $\mathcal{L}$, or simply a *twist* of $\mathcal{L}$.

Let $\mathcal{L}$ be an invertible sheaf on X. Notice that, since f is flat, the endomorphism of $\mathcal{L}$ given by multiplication by t is injective. Thus $t\Gamma(X,\mathcal{L})$ is the kernel of the restriction map $\Gamma(X,\mathcal{L}) \to \Gamma(X_0, \mathcal{L}|_{X_0})$. We say that $\mathcal{L}$ has *focus* on C_i if the restriction map

$$\Gamma(X,\mathcal{L}) \longrightarrow \Gamma(C_i, \mathcal{L}|_{C_i})$$

has kernel $t\Gamma(X,\mathcal{L})$ as well. Equivalently, $\mathcal{L}$ has focus on C_i if every global section of $\mathcal{L}$ that vanishes on C_i vanishes on the whole X_0.

Proposition 5.3. *Keep Setup 5.1. Let $\mathcal{L}$ be an invertible sheaf on X. Then for each C_i there is a twist of $\mathcal{L}$ that has focus on C_i.*

Proof. It is enough to exhibit a twist of $\mathcal{L}$ whose restrictions to C_j for $j \neq i$ have negative degree.

Without loss of generality, we may assume that $i = 1$, and that the components C_j are ordered in the following way. First, $C_2, \dots, C_{i_1}$ intersect C_1. Then $C_{i_1+1}, \dots, C_{i_2}$ intersect $C_2 \cup \dots \cup C_{i_1}$ but not C_1. Next, $C_{i_2+1}, \dots, C_{i_3}$ intersect $C_{i_1+1} \cup \dots \cup C_{i_2}$ but not $C_2 \cup \dots \cup C_{i_1}$. Go on like this, until all components are exhausted. At the end, $C_{i_m+1}, \dots, C_n$ intersect $C_{i_{m-1}+1} \cup \dots \cup C_{i_m}$ but not $C_{i_{m-2}+1} \cup \dots \cup C_{i_{m-1}}$. That all components are exhausted follows from the fact that X_0 is connected.

Now, choose $m+1$ integers $\ell_m, \dots, \ell_0$ in this order satisfying the following conditions. First, choose ℓ_m such that

$$\mathcal{L}_m := \mathcal{L} \otimes \mathcal{O}_X\big(-\ell_m(C_{i_{m-1}+1} + \dots + C_{i_m})\big)$$

has negative degree on each $C_{i_m+1}, \dots, C_n$. This is possible because each of these curves intersects $C_{i_{m-1}+1} \cup \dots \cup C_{i_m}$. Second, choose ℓ_{m-1} such that

$$\mathcal{L}_{m-1} := \mathcal{L}_m \otimes \mathcal{O}_X\big(-\ell_{m-1}(C_{i_{m-2}+1} + \dots + C_{i_{m-1}})\big)$$

has negative degree on each $C_{i_{m-1}+1}, \dots, C_{i_m}$. As before, this is possible because each of these curves intersect $C_{i_{m-2}+1} \cup \dots \cup C_{i_{m-1}}$. Also, $\mathcal{L}_{m-1}$ has the same degree as $\mathcal{L}_m$ on each $C_{i_m+1}, \dots, C_n$, as none of these curves intersect $C_{i_{m-2}+1} \cup \dots \cup C_{i_{m-1}}$. Go on like this, choosing integers $\ell_{m-2}, \dots, \ell_1$ and obtaining sheaves $\mathcal{L}_{m-2}, \dots, \mathcal{L}_1$. The sheaf $\mathcal{L}_1$ has negative degree on $C_{i_1+1}, \dots, C_n$.

Finally, choose an integer ℓ_0 such that $\mathcal{L}_0 := \mathcal{L}_1 \otimes \mathcal{O}_X(-\ell_0 C_1)$ has negative degree on each $C_2, \dots, C_{i_1}$. Then $\mathcal{L}_0$ has negative degree on each $C_2, \dots, C_n$, and hence is a desired twist of $\mathcal{L}$. $\square$

Proposition 5.4. *Keep Setup 5.1. Let $\mathcal{L}$ be an invertible sheaf on X. Then $\mathcal{L}^\alpha \cong \mathcal{L}^\beta$ if and only if $\alpha - \beta \in \mathbb{Z}(1, \dots, 1)$.*

Proof. We may assume that $\mathcal{L} = \mathcal{O}_X$ and $\beta = 0$.

First, since X_0 is reduced, $\mathrm{div}_X(t) = C_1 + \cdots + C_n$. Thus

$$\mathcal{O}_X \cong \mathcal{O}_X(C_1 + \cdots + C_n).$$

Iterating, we get that $\mathcal{O}_X^\alpha \cong \mathcal{O}_X$ if $\alpha \in \mathbb{Z}(1, \ldots, 1)$.

Now, suppose $\mathcal{O}_X^\alpha \cong \mathcal{O}_X$ for a certain n-tuple α. Using the already proved part, we may assume that α is the unique representative of $\alpha + \mathbb{Z}(1, \ldots, 1)$ such that $\alpha_j \geq 0$ for each j, with equality for at least one j. We will show that $\alpha = 0$.

Without loss of generality, we may assume that $\alpha_1 = 0$. We may also assume that $C_1, \ldots, C_n$ are ordered as in the proof of Proposition 5.3. Now, since $\mathcal{O}_X^\alpha \cong \mathcal{O}_X$, in particular $\mathcal{O}_X^\alpha|_{C_1}$ has degree 0. Since $C_2, \ldots, C_{i_1}$ intersect C_1, and $\alpha_1 = 0$, we get $\alpha_2 = \cdots = \alpha_{i_1} = 0$. Also, $\mathcal{O}_X^\alpha$ has degree 0 on each $C_2, \ldots, C_{i_1}$. Since $C_{i_1+1}, \ldots, C_{i_2}$ intersect $C_2 \cup \cdots \cup C_{i_1}$, and $\alpha_2 = \cdots = \alpha_{i_1} = 0$, we must also have $\alpha_{i_1+1} = \cdots = \alpha_{i_2} = 0$. Go on like this, and, since X_0 is connected, we will get at the end that $\alpha = 0$. $\square$

5.5. *Connecting numbers.* Keep Setup 5.1. Let $\mathcal{L}$ be an invertible sheaf on X, and $\mathcal{L}^\alpha$ and $\mathcal{L}^\beta$ twists of $\mathcal{L}$. For each pair of distinct components C_i and C_j let

$$\ell_{i,j}(\mathcal{L}^\alpha, \mathcal{L}^\beta) := \alpha_j - \alpha_i + \beta_i - \beta_j.$$

We call $\ell_{i,j}(\mathcal{L}^\alpha, \mathcal{L}^\beta)$ the *connecting number* between $\mathcal{L}^\alpha$ and $\mathcal{L}^\beta$ with respect to C_i and C_j. It follows from Proposition 5.4 that the connecting number depends only on $\mathcal{L}^\alpha$ and $\mathcal{L}^\beta$, and not on the choices of α and β. In addition, from the definition,

$$\ell_{i,j}(\mathcal{L}^\alpha, \mathcal{L}^\beta) = \ell_{j,i}(\mathcal{L}^\beta, \mathcal{L}^\alpha).$$

5.6. *The relative ramification divisor.* Keep Setup 5.1. Since X is a regular surface, Ω_X^1 is locally free of rank 2. Consider the natural presentation of the sheaf of relative differentials:

$$f^*\Omega_S^1 \to \Omega_X^1 \to \Omega_{X/S}^1 \to 0. \tag{5.6.1}$$

Taking exterior product with f^*dt gives a natural map $\Omega_X^1 \to \Omega_X^2$. As dt is a basis of Ω_S^1, this map factors through a map $\eta : \Omega_{X/S}^1 \to \Omega_X^2$. Let $D\colon \mathcal{O}_X \to \Omega_X^2$ denote the induced $\mathcal{O}_S$-derivation.

The map η is bijective on the smooth locus of f, i. e. off the nodes of X_0. Indeed, the natural pullback map $f^*\Omega_S^1 \to \Omega_X^1$ is injective, because it is so on the generic fiber. So the presentation (5.6.1) is a short exact sequence. The map η is bijective where $\Omega_{X/S}^1$ is locally free (and hence where (5.6.1) is locally split), that is, off the nodes of X_0.

Let $\mathcal{L}$ be an invertible sheaf on X. Since f is flat, the associated points of $\mathcal{L}$ lie on X_*, and hence the restriction $\Gamma(X, \mathcal{L}) \to \Gamma(X_*, \mathcal{L}|_{X_*})$ is injective. Thus $\Gamma(X, \mathcal{L})$ is a torsion-free $\mathbb{C}[[t]]$-module, whence free.

Let $V \subseteq \Gamma(X, \mathcal{L})$ be a $\mathbb{C}[[t]]$-submodule. Assume V is *saturated*, that is, the quotient module is free. Since $\Gamma(X, \mathcal{L})$ is free, so is V. Assume V is nonzero, of rank $r+1$ for a certain nonnegative integer r. Let $\beta = (s_0, \dots, s_r)$ be a $\mathbb{C}[[t]]$-basis of V.

For each open subscheme $U \subseteq X$ such that $\mathcal{L}|_U$ and Ω^2_U are trivial, let $\sigma \in \Gamma(U, \mathcal{L})$ and $\mu \in \Gamma(U, \Omega^2_X)$ such that $\mathcal{L}|_U = \mathcal{O}_U \sigma$ and $\Omega^2_U = \mathcal{O}_U \mu$. Then $s_i|_U = f_i \sigma$ for a regular function f_i on U for each $i = 0, \dots, r$. Also, there is a $\mathbb{C}[[t]]$-derivation ∂ of $\Gamma(U, \mathcal{O}_X)$ such that $D|_U(\cdot) = \partial(\cdot)\mu$. Form the Wronskian determinant:

$$w(\beta, \sigma, \mu) := \begin{vmatrix} f_0 & \dots & f_r \\ \partial f_0 & \dots & \partial f_r \\ \vdots & \ddots & \vdots \\ \partial^r f_0 & \dots & \partial^r f_r \end{vmatrix}.$$

As in Subsection 4.2, the $w(\beta, \sigma, \mu)$ patch up to a section of

$$\mathcal{L}^{\otimes r+1} \otimes (\Omega^2_X)^{\otimes \binom{r+1}{2}}.$$

Denote the zero scheme of this section by $W(V, \mathcal{L})$. We call $W(V, \mathcal{L})$ the *relative ramification divisor* associated to $(V, \mathcal{L})$. As in Subsection 4.2, this divisor does not depend on the choice of the basis β.

Let $R_* := W(V, \mathcal{L}) \cap X_*$. Since X_* is smooth, $\eta|_{X_*}$ is bijective, and it follows from Subsection 4.2 that R_* is a Cartier divisor of X_*. So $W(V, \mathcal{L})$ is indeed a divisor of X. But $W(V, \mathcal{L})$ may contain the components C_i in its support. Let $\overline{W}(V, \mathcal{L}) \subset X$ be the Cartier divisor obtained by removing from $W(V, \mathcal{L})$ the components C_i with their multiplicities. Then $\overline{W}(V, \mathcal{L})$ is S-flat, and restricts to R_* on X_*, whence

$$\overline{W}(V, \mathcal{L}) = \overline{R_*}.$$

If $\mathcal{L}$ has focus on C_i, the sections $s_0, \dots, s_r$ restrict to a basis of a vector subspace $V_i \subseteq \Gamma(C_i, \mathcal{L}|_{C_i})$. Since η is bijective off the nodes of X_0, it follows that

$$W(V, \mathcal{L}) \cap C'_i = \overline{W}(V, \mathcal{L}) \cap C'_i = W(V_i, \mathcal{L}|_{C_i}) \cap C'_i, \tag{5.6.2}$$

where $C'_i := X_0 - \bigcup_{j \neq i} C_j$.

5.7. *Twists of modules.* Keep Setup 5.1. Let $\mathcal{L}$ be an invertible sheaf on X, and $V \subseteq H^0(X, \mathcal{L})$ a saturated $\mathbb{C}[[t]]$-submodule.

Let α be a n-tuple of integers. Using the natural identification $\mathcal{L}^\alpha|_{X_*} = \mathcal{L}|_{X_*}$, define

$$V^\alpha := \{s \in \Gamma(X, \mathcal{L}^\alpha) \mid s|_{X_*} = v|_{X_*} \text{ for some } v \in V\}.$$

We call the submodule $V^\alpha \subseteq \Gamma(X, \mathcal{L}^\alpha)$ the *α-twist* of the submodule $V \subseteq \Gamma(X, \mathcal{L})$.

It follows directly from the definition that V^α is a saturated submodule of the same rank as V. In addition, since the sections of V^α and V coincide over X_*, we have that

$$W(V, \mathcal{L}) \cap X_* = W(V^\alpha, \mathcal{L}^\alpha) \cap X_*,$$

and hence $\overline{W}(V, \mathcal{L}) = \overline{W}(V^\alpha, \mathcal{L}^\alpha)$.

5.8. *The limit ramification divisor.* Keep Setup 5.1. Let $\mathcal{L}$ be an invertible sheaf on X and $V \subseteq H^0(X, \mathcal{L})$ a saturated $\mathbb{C}[[t]]$-submodule. Let $W(V, \mathcal{L})$ be the corresponding relative ramification divisor, and $\overline{W}(V, \mathcal{L})$ the divisor obtained by removing from $W(V, \mathcal{L})$ the components C_i with their multiplicities. Then

$$\lim W(V, \mathcal{L}) := \overline{W}(V, \mathcal{L}) \cap X_0$$

is a Cartier divisor, called the *limit ramification divisor* of $(V, \mathcal{L})$.

Theorem 5.9. *Keep Setup 5.1. Let $\mathcal{L}$ be an invertible sheaf on X and $V \subseteq \Gamma(X, \mathcal{L})$ a saturated submodule. For each C_i, let α_i be a n-tuple such that $\mathcal{L}^{\alpha_i}$ has focus on C_i, and let $V_i \subseteq \Gamma(C_i, \mathcal{L}^{\alpha_i}|_{C_i})$ be the vector subspace generated by V^{α_i}. For each pair of distinct C_i and C_j, let $\ell_{i,j}$ be the connecting number between $\mathcal{L}^{\alpha_i}$ and $\mathcal{L}^{\alpha_j}$ with respect to C_i and C_j. For each $i = 1, \ldots, n$ let W_i be the ramification divisor of $(V_i, \mathcal{L}^{\alpha_i}|_{C_i})$. Then*

$$[\lim W(V, \mathcal{L})] = \sum_{i=1}^{n} [W_i] + \sum_{i<j} \sum_{P \in C_i \cap C_j} (r+1)(r - \ell_{i,j})[P]. \tag{5.9.1}$$

Proof. Let $P \in X_0$. Suppose first that P is not a node of X_0. So $P \in C_i'$ for some i, where

$$C_i' := X_0 - \bigcup_{j \neq i} C_j.$$

By (5.6.2),

$$\operatorname{mult}_{P, C_i}\big(\lim W(V, \mathcal{L})\big) = \operatorname{mult}_{P, C_i}(W_i).$$

So the coefficients of P on both sides of Equation (5.9.1) are equal.

Assume now that P is a node of X_0. We may assume, without loss of generality, that $P \in C_1 \cap C_2$. Let

$$b_j := \operatorname{mult}_{P, C_j}(\overline{W}(V, \mathcal{L}) \cap C_j)$$

for $j = 1, 2$. Since $\overline{W}(V, \mathcal{L}) \cap X_0$ is a Cartier divisor of X_0, the coefficient of P in $[\lim W(V, \mathcal{L})]$ is $b_1 + b_2$.

Now, for each $j = 1, 2$, there is a n-tuple of nonnegative integers μ_j such that

$$\overline{W}(V, \mathcal{L}) = W(V^{\alpha_j}, \mathcal{L}^{\alpha_j}) - \sum_{i=1}^{n} \mu_{j,i} C_i. \tag{5.9.2}$$

Notice that $\mu_{j,j} = 0$ because $\mathcal{L}^{\alpha_j}$ has focus on C_j. Then the intersection $W(V^{\alpha_j}, \mathcal{L}^{\alpha_j}) \cap C_j$ is a Cartier divisor of C_j. Let

$$a_j := \mathrm{mult}_{P,C_j}\big(W(V^{\alpha_j}, \mathcal{L}^{\alpha_j}) \cap C_j\big).$$

Then

$$b_1 = a_1 - \mu_{1,2} \quad \text{and} \quad b_2 = a_2 - \mu_{2,1}. \tag{5.9.3}$$

Comparing Equations (5.9.2) for $j = 1, 2$, we get

$$W(V^{\alpha_1}, \mathcal{L}^{\alpha_1}) - W(V^{\alpha_2}, \mathcal{L}^{\alpha_2}) = \sum_{i=1}^{n} (\mu_{1,i} - \mu_{2,i}) C_i.$$

Now,

$$\mathcal{O}_X\big(W(V^{\alpha_j}, \mathcal{L}^{\alpha_j})\big) \cong (\mathcal{L}^{\alpha_j})^{\otimes r+1} \otimes (\Omega_X^2)^{\otimes \binom{r+1}{2}}$$

for $j = 1, 2$. Thus

$$(\mathcal{L}^{\alpha_1})^{\otimes r+1} = (\mathcal{L}^{\alpha_2})^{\otimes r+1} \otimes \mathcal{O}_X^{\mu_1 - \mu_2}.$$

Then, by Proposition 5.4,

$$(r+1)(\alpha_1 - \alpha_2) - (\mu_1 - \mu_2) \in \mathbb{Z}(1, \dots, 1).$$

Since $\mu_{1,1} = \mu_{2,2} = 0$, it follows that

$$\mu_{1,2} + \mu_{2,1} = (r+1)(\alpha_{1,2} - \alpha_{1,1} + \alpha_{2,1} - \alpha_{2,2}) = (r+1)\ell_{1,2}.$$

Thus, using Equations (5.9.3) we get

$$b_1 + b_2 = a_1 + a_2 - (r+1)\ell_{1,2}. \tag{5.9.4}$$

Now, by adjunction, for each $j = 1, 2$,

$$\Omega^1_{C_j} \cong \big(\Omega^2_X \otimes \mathcal{O}_X(C_j)\big)|_{C_j}.$$

Since $C_1 + \cdots + C_n = \mathrm{div}_X(t)$, it follows that

$$\Omega^2_X|_{C_j} = \Omega^1_{C_j} \otimes \mathcal{O}_{C_j}(\textstyle\sum Q),$$

where Q runs through all nodes of X_0 on C_j. We claim that the restriction of the map $\eta\colon \Omega^1_{X/S} \to \Omega^2_X$ of Subsection 5.6 to C_j factors through the natural map

$$\Omega^1_{C_j} \xrightarrow{\sum Q} \Omega^1_{C_j} \otimes \mathcal{O}_{C_j}(\textstyle\sum Q).$$

Indeed, as we saw in Subsection 3.3, for each Q we have $f^*t = uv$ in $\widehat{\mathcal{O}}_{X,Q}$, where u and v are equations of C_j and $\overline{C - X_j}$, respectively. So

$$f^*dt = udv + vdu \equiv vdu \mod (u).$$

Now, since Q is a node, v restricts to a local parameter of C_j at Q. As η was defined by taking the exterior product with f^*dt, the claim follows.

The upshot is that the $\mathcal{O}_S$-derivation $\mathcal{O}_X \to \Omega^2_X$ induced by η restricts on a neighborhood of P in C_j to $z\frac{d}{dz}$ where z is a local parameter for C_j at P. Since the $\mathbb{C}[[t]]$-submodule $V^{\alpha_j} \subseteq \Gamma(X, \mathcal{L}^{\alpha_j})$ generates $V_j \subseteq \Gamma(C_j, \mathcal{L}^{\alpha_j}|_{C_j})$, using the multilinearity of the determinant, and the product rule of derivations, we get

$$a_j = \mathrm{mult}_{P,C_j}(W_j) + \binom{r+1}{2}. \tag{5.9.5}$$

Combining (5.9.4) with (5.9.5) for $j = 1, 2$, we get

$$b_1 + b_2 = \mathrm{mult}_{P,C_1}(W_1) + \mathrm{mult}_{P,C_2}(W_2) + (r+1)(r - \ell_{1,2}).$$

Since $b_1 + b_2$ is the coefficient with which P appears in $[\lim W(V, \mathcal{L})]$, the coefficients of P on both sides of Equation (5.9.1) are equal. □

6. Application

Proposition 6.1. *Keep Setup 5.1. Let $\mathcal{L}$ be an invertible sheaf on X, and $\mathcal{L}_1, \dots, \mathcal{L}_n$ twists of $\mathcal{L}$. For each pair of distinct C_i and C_j, let $\delta_{i,j}$ be the number of points of $C_i \cap C_j$, and $\ell_{i,j}$ the connecting number between $\mathcal{L}_i$ and $\mathcal{L}_j$ with respect to C_i and C_j. Let d be the degree of $\mathcal{L}$ on the general fiber, and $d_i := \deg \mathcal{L}_i|_{C_i}$ for each $i = 1, \dots, n$. Then*

$$d = \sum_{i=1}^{n} d_i - \sum_{i<j} \delta_{i,j}\ell_{i,j}.$$

Proof. By Proposition 5.4, for each $i = 1, \dots, n$ there is a n-tuple α_i such that $\mathcal{L}_i \cong \mathcal{L}^{\alpha_i}$ and $\alpha_{i,i} = 0$. Restricting to C_i and taking degrees, we get

$$\deg \mathcal{L}|_{C_i} = d_i - \sum_{j \neq i} \alpha_{i,j}\delta_{i,j}.$$

Summing up for $i = 1, \dots, n$, we get

$$d = \sum_{i=1}^{n} \deg \mathcal{L}|_{C_i} = \sum_{i=1}^{n} d_i - \sum_{i<j} (\alpha_{i,j} + \alpha_{j,i})\delta_{i,j}.$$

Now, it is enough to observe that the connecting number between $\mathcal{L}_i$ and $\mathcal{L}_j$ with respect to C_i and C_j is

$$\alpha_{i,j} - \alpha_{i,i} + \alpha_{j,i} - \alpha_{j,j} = \alpha_{i,j} + \alpha_{j,i}.$$

□

6.2. *Proof of Theorem 2.11.* Use the notation in Setup 5.1. Let g be the genus of X_*. Fix a pair of nonnegative integers (d,r). Suppose $W^d_r X_* \neq \emptyset$. We will show that $\rho(g,d,r) \geq 0$.

Since $W^d_r X_*$ is of finite type over $\mathbb{C}((t))$, there are a finite field extension k of $\mathbb{C}((t))$ and a k-point of $W^d_r X_*$. As we saw in Subsection 3.3, up to replacing the special fiber by an avatar, which is also a flag curve, we may assume that $k = \mathbb{C}((t))$. In other words, we may assume we are given an invertible sheaf L on X_* of degree d such that $\dim \Gamma(X_*, L) \geq r+1$.

Let $r' := \dim \Gamma(X_*, L) - 1$. It is enough to show that $\rho(g,d,r') \geq 0$. Indeed, $r' \geq d-g$ by the Riemann–Roch theorem. Thus

$$\begin{aligned}\rho(g,d,r) =& \rho(g,d,r') + (r'-r)(r+r'+1+g-d)\\ \geq& \rho(g,d,r') + (r'-r)(r+1)\\ \geq& \rho(g,d,r').\end{aligned}$$

So we may assume that $\dim \Gamma(X_*, L) = r+1$.

Since X is regular, L extends to an invertible sheaf $\mathcal{L}$ on X. So we may apply the theory of limit linear series developed in Section 5.

Let $V := H^0(X, \mathcal{L})$. For each C_i, let $\mathcal{L}_i$ be a twist of $\mathcal{L}$ with focus on C_i, and $d_i := \deg \mathcal{L}_i|_{C_i}$. Let $V_i \subseteq H^0(C_i, \mathcal{L}_i|_{C_i})$ be the vector subspace generated by $H^0(X, \mathcal{L}_i)$, and let $W_i \subset C_i$ be the ramification divisor of $(V_i, \mathcal{L}_i|_{C_i})$.

Let s_i be the sum of the multiplicities of the nodes of X_0 in W_i. By the Plücker formula, if C_i is rational, $\deg W_i = (r+1)(d_i - r)$, and hence

$$s_i \leq (r+1)(d_i - r). \tag{6.2.1}$$

On the other hand, suppose C_i is not rational. Since X_0 is a flag curve, C_i is elliptic, and contains only one node of X_0. Call this node Q. Let $\epsilon_0, \ldots, \epsilon_r$ be the order sequence of $(V_i, \mathcal{L}_i|_{C_i})$ at Q.

We claim that $\epsilon_r \leq d_i$, with equality only if $\epsilon_{r-1} \leq d_i - 2$. That ϵ_r is bounded by d_i follows from $d_i = \deg \mathcal{L}_i|_{C_i}$. Now, suppose $\epsilon_r = d_i$. Then $\mathcal{L}_i|_{C_i} \cong \mathcal{O}_{C_i}(d_i Q)$. If $\epsilon_{r-1} = d_i - 1$, then $\mathcal{L}_i|_{C_i} \cong \mathcal{O}_{C_i}((d_i-1)Q + Q')$ for some $Q' \in C_i - \{Q\}$. It would follow that $\mathcal{O}_{C_i}(Q) \cong \mathcal{O}_{C_i}(Q')$, and hence that C_i is rational. So our claim is proved.

From our claim, the ramification weight of $(V_i, \mathcal{L}_i|_{C_i})$ at Q is at most $(r+1)(d_i - r) - r$, and hence

$$s_i \leq (r+1)(d_i - r) - r. \tag{6.2.2}$$

Since there are exactly g elliptic components of X_0, combining (6.2.1), valid for C_i rational, and (6.2.2), valid for C_i elliptic, we get

$$\sum_{i=1}^{n} s_i \leq (r+1)\Big(\sum_{i=1}^{n} d_i - nr\Big) - gr.$$

Now, by Formula (5.9.1), since $\lim W(V, \mathcal{L})$ is an effective divisor,

$$\sum_{i=1}^{n} s_i + \sum_{i<j} (r+1)(r - \ell_{i,j})\delta_{i,j} \geq 0.$$

In particular,

$$(r+1)\Big(\sum_{i=1}^{n} d_i - nr\Big) - gr + \sum_{i<j} (r+1)(r - \ell_{i,j})\delta_{i,j} \geq 0.$$

Using Proposition 6.1, the left-hand side of the inequality becomes

$$(r+1)(d - nr) - gr + (r+1)r \sum_{i<j} \delta_{i,j}.$$

Furthermore, since X_0 is a flag curve, $\sum \delta_{i,j} = n - 1$. Then the inequality becomes

$$(r+1)(d-r) - gr \geq 0,$$

that is, $\rho(g, d, r) \geq 0$. The proof of Theorem 2.11 is complete.

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