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THE ORDER OF FINITE ALGEBRAIC FUNDAMENTAL GROUPS OF SURFACES WITH $K^2 \leq 3\chi - 2$

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ABSTRACT. In this note we study the structure of $\pi_1^{\mathrm{alg}}(S)$ for minimal surfaces of general type S satisfying $K_S^2 \leq 3\chi - 2$ and not having any irregular étale cover. We show that, if $K_S^2 \leq 3\chi - 2$, then $|\pi_1^{\mathrm{alg}}(S)| \leq 5$, and equality only occurs if S is a Godeaux surface. We also show that if $K_S^2 \leq 3\chi - 3$ and $\pi_1^{\mathrm{alg}}(S) \neq \{1\}$, then $\pi_1^{\mathrm{alg}}(S) = \mathbb{Z}_2$, or $\pi_1^{\mathrm{alg}}(S) = \mathbb{Z}_2^2$ or $\pi_1^{\mathrm{alg}}(S) = \mathbb{Z}_3$. Furthermore in this last case one has: $2 \leq \chi \leq 4$, $K^2 = 3\chi - 3$ and these possibilities do occur.

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1. INTRODUCTION

In this note we study the structure of $\pi_1^{\mathrm{alg}}(S)$ for minimal surfaces of general type S satisfying $K_S^2 \leq 3\chi - 2$ and not having any irregular étale cover.

In [MP2] we have shown, among other things, that if S has no irregular étale cover and $K_S^2 \leq 3\chi - 1$ then the order of $\pi_1^{\mathrm{alg}}(S)$ is less than or equal to 9 and equality is only possible if $\chi(S) = 1$. In this note we show some more results on the structure of $\pi_1^{\mathrm{alg}}(S)$.

We want to remark that most of the present results are somehow implicit in [X], but we think it is worthwhile spelling them out.

We prove the following:

Theorem 1.1. *Let S be a minimal algebraic surface of general type such that $K^2 \leq 3\chi - 2$ not having any irregular étale cover. Then the order of $\pi_1^{\mathrm{alg}}(S)$ is at most 5, and equality only occurs if $\chi = 1$ and $K^2 = 1$ (i.e. S is a numerical Godeaux surface).*

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Theorem 1.2. *Let S be a minimal algebraic surface of general type such that $K^2 \leq 3\chi-3$ not having any irregular étale cover. If $\pi_1^{\mathrm{alg}}(S) \neq \{1\}$, then there are the following posibilities:*

(i) $\pi_1^{\mathrm{alg}}(S) = \mathbb{Z}_2$;
(ii) $\pi_1^{\mathrm{alg}}(S) = \mathbb{Z}_2^2$;
(iii) $\pi_1^{\mathrm{alg}}(S) = \mathbb{Z}_3$. *In this case one has:* $2 \leq \chi \leq 4$, $K^2 = 3\chi - 3$.

Remark 1.3. We remark that there are examples of surfaces with $\pi_1^{\mathrm{alg}} = \mathbb{Z}_3$ for all the values of χ and K^2 given in (iii) of Theorem 1.2.

For $\chi(S) = 2$ see [Mur], whilst for $\chi(S) = 3$ it is enough to take S as an étale triple cover of the Campedelli surfaces with fundamental group of order 9 described in [MP1].

An example for $\chi = 4$ is described in 4.1.

Remark 1.4. For details on Godeaux surfaces see for instance the introduction and the references in [CCM].

Notation We work over the complex numbers. All varieties are projective algebraic. All the notation we use is standard in algebraic geometry. We just recall the definition of the numerical invariants of a smooth surface S: the self-intersection number K_S^2 of the canonical divisor K_S, the *geometric genus* $p_g(S) := h^0(K_S) = h^2(\mathcal{O}_S)$, the *irregularity* $q(S) := h^0(\Omega_S^1) = h^1(\mathcal{O}_S)$ and the *holomorphic Euler characteristic* $\chi(S) := 1 + p_g(S) - q(S)$.

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2. Some useful facts

We will use the following fundamental facts:

Proposition 2.1. ([Be, Cor. 5.8], cf. Proposition 4.1 of [MP2]) *Let Y be a surface of general type such that the canonical map of Y has degree 2 onto a rational surface. If G is a group that acts freely on Y, then $G = \mathbb{Z}_2^r$, for some r.*

Lemma 2.2. *Let Y be a regular surface of general type, let $G \neq \{1\}$ be a finite group that acts freely on Y and let $|F|$ be a G-invariant free pencil $|F|$ of curves of genus $g(F) \leq 4$. Then only the following possibilities can occur:*

(i) $G = \mathbb{Z}_2^2$, $g(F) = 3$ *and G acts faithfully on $|F|$;*
(ii) $G = \mathbb{Z}_3$, $g(F) = 4$;
(iii) $G = \mathbb{Z}_2$, $g(F) = 3$.

Proof. Assume that such a pencil $|F|$ exists, let H be the subgroup of G consisting of the elements that act trivially on $|F|$ and let h be the order of H. Set $Y' := Y/H$ and $S := Y/G$. The pencil $|F|$ induces a free pencil $|F'|$ on Y' with general fibre $F' := F/H$. Denote by $f'\colon Y' \to \mathbb{P}^1$ the morphism given by $|F'|$. There is a cartesian diagram:

$$\begin{array}{ccc} Y' & \longrightarrow & S \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbb{P}^1 & \xrightarrow{p} & \mathbb{P}^1 \end{array} \tag{2.1}$$

where p is a G/H-cover and the general fibre of f is also equal to F'. The fibres of f over the branch points of p are multiple fibres, of multiplicity equal to the branching order. Hence, if G/H is nontrivial, then f has multiple fibres.

Since $g(F') = 1 + (g(F) - 1)/h$ (recall that H acts freely), $g(F) \leq 4$ and S is of general type, we get $h \leq 3$.

Assume that $|G| \geq 4$. Then G/H is not the trivial group and f has multiple fibres, hence $g(F') > 2$ by the adjunction formula. So the only possibility is $h = 1$, $g(F) = 3$ or 4 and G is isomorphic to a subgroup of $\operatorname{Aut}|F| = \operatorname{Aut}\mathbb{P}^1$.

If $g(F) = 3$, then by the adjunction formula the multiple fibres of f are double fibres. Since every automorphism of $\mathbb{P}^1$ has fixed points, this implies that every element of G has order 2. Hence $G = \mathbb{Z}_2^r$ for some r. Since $\operatorname{Aut}\mathbb{P}^1$ does not contain a subgroup isomorphic to $\mathbb{Z}_2^3$, we have $r \leq 2$.

If $g(F) = 4$, then by the adjunction formula the multiple fibres of f are triple fibres. Since every automorphism of $\mathbb{P}^1$ has fixed points, this implies that every element of G has order 3. It is well known (cf. [Bl] or [EC]) that a finite subgroup of $\operatorname{Aut}\mathbb{P}^1$ is isomorphic to one of the following: $\mathbb{Z}_n$, $\mathbb{Z}_2^2$, the dihedral group D_n, the symmetric group S_4, the alternating groups A_4 and A_5. It follows that $G = \mathbb{Z}_3$ in this case. So we have proven that the only possibility for $|G| \geq 4$ is (i).

Statements (ii) and (iii) now follow from the adjunction formula. □

3. The proof of Theorem 1.1

In this section we denote by S a surface satisfying the assumptions of Theorem 1.1, i. e., a surface such that $K_S^2 = 3\chi - m$, where $m \geq 2$, having no irregular étale cover. Note that, by Theorem 1.3 of [MP2], $|\pi_1^{\mathrm{alg}}(S)| \leq 8$.

We divide the proof of Theorem 1.1 in two steps.

Step 1: $|\pi_1^{\mathrm{alg}}(S)| \leq 5$.

Assume by contradiction that $d := |\pi_1^{\mathrm{alg}}(S)| > 5$ and let $Y \to S$ be the corresponding étale G-cover of degree d.

Since $K_Y^2 = dK_S^2 = 3d\chi(S) - dm = 3\chi(Y) - dm < 3\chi(Y) - 10$, by [Be] the canonical map of Y is generically finite of degree 2 and its image is a ruled surface Σ. The surface Y is regular and so Σ is a rational surface. Since $|\pi_1^{\mathrm{alg}}(S)| \leq 8$, by Proposition 2.1 we conclude that $G = \mathbb{Z}_2^3$.

So $d = 8$ and $K_Y^2 = 3\chi(Y) - 8m$.

Assume that $m \geq 3$. Then $K_Y^2 \leq 3\chi - 24 < 3(\chi - \frac{43}{8})$ and so, by Theorem 1.1 of [X], Y has a unique free pencil of hyperelliptic curves of genus $g \leq 3$. Since this pencil is necessarily G-invariant we have a contradiction to Lemma 2.2.

Suppose that $m = 2$. Then $K_Y^2 = 3\chi(Y) - 16 < 3(\chi(Y) - 5)$. If $\chi(Y) \geq 24$ (i. e. $\chi(S) \geq 3$), again by Theorem 1.1 of [X], Y has a unique free pencil of hyperelliptic curves of genus $g \leq 3$ and again we have the same contradiction. If $\chi(Y) = 16$ (i. e. $\chi(S) = 2$), then, by the Example on page 133 of [X], either Y has a unique pencil of hyperelliptic curves of genus ≤ 3 or Y is a double cover of $\mathbb{P}^2$ with branch locus a curve of degree 14 with at most non-essential singularities. The first case can be excluded as before whilst the second case is excluded because Y admits no free involution (see the proof of Lemma 3 of [X]).

If $\chi(S) = 1$ then by Noether's inequality S cannot have an étale cover of degree 8 (the resulting Y would satisfy $K_Y^2 = 8 < 2\chi - 6 = 10$).

Step 2: *If* $|\pi_1^{\mathrm{alg}}(S)| = 5$, *then* S *is a Godeaux surface.*

Assume $|\pi_1^{\mathrm{alg}}(S)| = 5$ and let $Y \to S$ be the corresponding étale G-cover.

Then $K_Y^2 = 5K_S^2 = 15\chi(S) - 5m = 3\chi(Y) - 5m$. If $m \geq 3$, then $K_Y^2 = 3\chi(Y) - 5m < 3\chi(Y) - 10$ and so we obtain a contradiction as in Step 1, because $\mathbb{Z}_2^n$ has not order 5 for any n.

If $m = 2$, then the regular surface Y satisfies $K_Y^2 = 3\chi - 10 = 3p_g - 7$ and by [AK] either the canonical map φ of Y is birational or it is a generically finite map of degree 2 on a rational surface. This last case is immediately excluded by Proposition 2.1.

So the canonical map φ of Y is birational. To show the claim we only need to show that $\chi(S) \geq 2$ does not occur in this situation. If $\chi(S) \geq 2$ then $\chi(Y) \geq 10$. So, by [AK], Y again has an unique free pencil of genus 3 curves and therefore, by Lemma 2.2, Y does not admit a free action by a group of order 5.

4. The proof of Theorem 1.2

Here we prove Theorem 1.2. We continue assuming that S is a surface having no irregular étale cover and now we suppose that $K^2 \leq 3\chi - 3$.

By Theorem 1.1 the order of $\pi_1^{\mathrm{alg}}(S)$ is less than or equal to 4. If the order of $\pi_1^{\mathrm{alg}}(S)$ is 4, then the corresponding étale cover Y satisfies $K_Y^2 \leq 3\chi(Y) - 12$. Since by [Be] the canonical map of Y is 2-1 onto a rational surface, by Proposition 2.1 we have $\pi_1^{\mathrm{alg}}(S) = \mathbb{Z}_2^2$.

Assume that $\pi_1^{\mathrm{alg}}(S) = \mathbb{Z}_3$. Then by the same argument one has $K_S^2 = 3\chi(S) - 3$, and the corresponding étale cover $Y \to S$ satisfies $K_Y^2 = 3\chi(Y) - 9 = 3p_g(Y) - 6$.

Surfaces with these invariants have been classified by Konno ([Ko]). If $\chi(Y) \geq 13$ and $q(Y) = 0$, then either the canonical map of Y is generically finite of degree 2 onto a rational surface, or the canonical map is birational and Y has a unique free pencil of genus 3 non-hyperelliptic curves. The first possibility does not occur by Proposition 2.1 and the second one does not occur by Lemma 2.2. Therefore, if the order of $\pi_1^{\mathrm{alg}}(S)$ is 3, then necessarily $\chi(Y) \leq 12$, i. e. $\chi(S) \leq 4$. Finally notice that $\chi(S) = 1$ does not occur, since by assumption $3\chi(S) - 3 \geq K_S^2$ and $K_S^2 > 0$ since S is of general type. This finishes the proof of Theorem 1.2.

Example 4.1. As explained in Remark 1.3, examples of surfaces with $K^2 = 3\chi - 3$ and $\pi_1^{\mathrm{alg}} = \mathbb{Z}_3$ are known in the literature for $\chi = 2, 3$. We describe here an example with $\chi = 4$.

Consider homogeneous coordinates (x_0, x_1, x_2) on $\mathbb{P}^2$ and let $1 \in \mathbb{Z}_3$ act on $\mathbb{P}^2$ by: $(x_0, x_1, x_2) \mapsto (x_0, \omega x_1, \omega^2 x_2)$, where $\omega \neq 1$ is a cube root of 1. Consider $\mathbb{P}^{10}$ with homogeneous coordinates $z_1, \ldots z_{10}$ and identify $\mathbb{P}^9$ with the hyperplane of $\mathbb{P}^{10}$ defined by $z_0 = 0$. Denote by $m_1, \ldots m_{10}$ the homogeneous monomials of degree 3 in x_0, x_1, x_2. The Veronese embedding of degree 3 $v\colon \mathbb{P}^2 \to \mathbb{P}^9$ is defined by letting $z_i = m_i$, $i = 1, \ldots 10$. Denote by Σ the image of v and by $K \subset \mathbb{P}^{10}$ the cone over Σ with vertex the point $P := (1, 0, \ldots 0)$. The $\mathbb{Z}_3$-action on $\mathbb{P}^2$ induces a compatible action on $\Sigma \subset \mathbb{P}^9$, which is diagonal with respect to the coordinates $z_1, \ldots z_{10}$. We extend this action to $\mathbb{P}^{10}$ by letting $1 \in \mathbb{Z}_3$ act on z_0 as multiplication by ω. Clearly this action fixes the point P and maps the cone K to itself. We claim that the only fixed points of $\mathbb{Z}_3$ on K are P and the image points $Q_0, Q_1, Q_2 \in \Sigma$ of the coordinate points of $\mathbb{P}^2$. In fact, let $Q \in K$, $Q \neq P$ be a fixed point. Then the line $< P, Q >$ meets Σ in a point fixed by $\mathbb{Z}_3$, namely Q lies on one of the lines $< P, Q_i >$, for some $i \in \{0, 1, 2\}$. So it is enough to show that $\mathbb{Z}_3$ acts non trivially on the line $< P, Q_i >$ for

$i = 0, 1, 2$. This is easy to check, since the Q_i are coordinate points in $\mathbb{P}^9$ and the only nonzero coordinate of Q_i is the one corresponding to the monomial x_i^3, hence $\mathbb{Z}_3$ acts on it as multiplication by 1.

Let now $T_1 \subset H^0(\mathcal{O}_{\mathbb{P}^{10}}(3))$ be the subspace of elements which are fixed by $\mathbb{Z}_3$. Notice that $z_0^3, \dots z_{10}^3$ are elements of T_1, hence the system $|T_1|$ is free. Let V be the intersection of K with a general hypersurface of $|T_1|$. Then by Bertini's theorem V is smooth, $\mathbb{Z}_3$-invariant and the $\mathbb{Z}_3$-action on V is free. One has $K_V = \mathcal{O}_V(1)$, $K_V^2 = 27$, $p_g(V) = 11$, $q(V) = 0$ (cf. [Ko, §4]). The quotient surface $S := V/\mathbb{Z}_3$ is smooth, minimal of general type with $K_S^2 = 9$, $q(S) = 0$, $\chi(S) = 4$. By Theorem 1.2, V is the universal cover of S and S is an example of case (iii) of Theorem 1.1. (One can also check directly that V is simply connected).

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