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Trigonal Algebraic Surfaces and Triple Covers

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ABSTRACT. We will survey the applications of our method on triple covers to the study of trigonal surfaces, the bounds on the slopes of trigonal fibrations and the cubic defining equations of rational triple points.

1. Gonality of curves and surfaces

The gonality of an algebraic curve is defined to be the smallest degree of a morphism from the curve to the projective line $\mathbb{P}^1$. It is known that a curve C of genus g admits a map to $\mathbb{P}^1$ of degree at most $[(g+3)/2]$. Gonality is an old invariant which measures how complicated the curve is. So curves of genus $g \geq 1$ are divided into subclasses according to their gonality: hyperelliptic, trigonal, and d-gonal.

d	1	2	3	......
C	$\mathbb{P}^1$	**Hyperelliptic**	**Trigonal**	......
g	0	1, 2	3, 4	......

In general, the gonality $d \leq \left[\frac{g+3}{2}\right]$. We are interested in curves of genus $g \geq 2$.

(I) Curves C of genus 2 are hyperelliptic, $\pi : C \xrightarrow{2:1} \mathbb{P}^1$, and the double cover π is exactly the canonical map Φ_{K_C} of C.

(II) Curves C of genus 3 or 4 are hyperelliptic or trigonal, i.e., non-hyperelliptic curves are trigonal.

One can define the gonality $d(X)$ of a projective complex surface X as the minimal degree of a generically finite map to some *ruled surface*.

$$d(X) := \min\{\, d \mid X \overset{d:1}{\dashrightarrow} C \times \mathbb{P}^1 \text{ for some curve } C\}.$$

$d(X)$ is well defined because any projective surface is a generic cover of $\mathbb{P}^2$. According to the gonality $d(X)$, algebraic surfaces are divided into subclasses:

$d(X)$	1	2	3	$\cdots$
X	**Ruled**	**Hyperelliptic**	**Trigonal**	$\cdots$

The analogue of curves of genus ≥ 2 is the minimal surfaces X of general type. In this case, the Chern numbers of X satisfies Neother's inequality: $K_X^2 \geq 2p_g(X) - 4$. By Castelnuovo-Beauville Theorem ([5]), we have

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(I) Surfaces X with $2p_g - 4 \le K_X^2 < 3p_g - 7$ are hyperelliptic, and the double cover over some ruled surface is exactly the canonical map Φ_{K_X} of X.

So surfaces in the range $2p_g - 4 \le K_X^2 < 3p_g - 7$ are analogue of curves of genus 2. Curves C of genus 3 or 4 are hyperelliptic or trigonal. The analogue of this result is the following conjecture (due to Horikawa, Reid and Xiao):

(II) There are two numbers $3 < a \le 4$ and $b \ge 8$ such that surfaces with $3p_g - 7 \le K_X^2 < ap_g - b$ are hyperelliptic or trigonal.

Denote by Σ the image of Φ_{K_X}. If Σ is a curve, then $K_X^2 \ge 4p_g - 7$ ([31]). If Σ is a surfrace and $\deg \Phi_{K_X} \ge 4$, or Σ is a non-ruled surface and $\deg \Phi_{K_X} \ge 2$, then $K_X^2 \ge 4p_g - 8$ ([5]). So if $ap_g - b < 4p_g - 8$, then the canonical map is a birational map or a generically finite cover of degree 2 or 3 over a ruled surface. Therefore, the conjecture is equivalent to the following:

(II') Canonical surfaces (i.e., $\deg \Phi_{K_X} = 1$) with $K_X^2 < ap_g - b$ are trigonal.

The second natural generalization of gonality of curves is the *irrationality* $e(X)$ of surfaces, introduced by by T. T. Moh and W. Heinzer [18],

$$e(X) := \min\{\ d \mid X \overset{d:1}{\dashrightarrow} \mathbb{P}^2\ \},$$

equivalently, $e(X)$ is the minimal degree of the field extension $\mathbb{C}(x_1, x_2) \subset \mathbb{C}(X)$, where x_1 and x_2 are two algebraically independent rational functions on X. If $q(X) = \dim H^1(X, \mathcal{O}_X) = 0$, then $d(X) = e(X)$. In general,

$$d(X) \le e(X).$$

It is obviously that $d(X)$ and $e(X)$ are two birational invariants of surfaces. For surfaces of non-general type, we have

(A) $\kappa(X) = -\infty$: Ruled surface $f : X \to C$ or $\mathbb{P}^2$. $\quad d = 1, \quad e = d(C)$.

(B) $\kappa(X) = 0$:

$$\begin{cases} \text{Enriques}, & d = e = 2 \text{ (see [20])} \\ \text{K3}, & d = e = 2, 3 \text{ (Conjecture)} \\ \text{Bielliptic}, & d \le e = 2, 3, 4 \text{ see [30]} \\ \text{Abelian}, & d \le e,\ e \ge 3. \end{cases}$$

(C) $\kappa(X) = 1$: Elliptic Surfaces $f : X \to C$. If f has a section Γ, then $d(X) = 2$.

Conjecture: The gonality of a $K3$ surface is 2 or 3.

For surfaces X with a fibration $f : X \to C$ of genus $g \ge 2$, if the generic fiber is a hyperelliptic curve and $\kappa(X) \ge 0$, then $d(X) = 2$, and the double cover is given by the relative canonical map.

If the generic fiber of f is a non-hyperellptic curve of genus 3, and f has a section, then X admits a generically finite triple cover on a ruled surface over C. So $d(X) \le 3$.

In general, we need base changes $\pi : \widetilde{C} \to C$ to get an upper bound on the gonality. Denote by $\widetilde{f} : \widetilde{X} \to \widetilde{C}$ the pullback fibration. Then for any f, there is a base change π such that $d(\widetilde{X})$ is less than or equal to the gonality of a generic fiber of f.

Hyperelliptic surfaces play an important role in the classification of surfaces. Due to the theory of double covers, the structure of hyperelliptic surfaces are relatively clear. For example, one knows how to compute the global invariants of X from the branch locus by using Horikawa's canonical resolution of singularities.

Trigonal surfaces are the next simple classes of surfaces which may have a nice classification. Assume that X is a trigonal surface (so $\kappa(X) \ge 0$), i.e., there is a generically finite triple cover $\phi_0 : X \dashrightarrow \Sigma$ over some ruled surface (not necessarily smooth), $\eta_0 : Y_0 \to \Sigma$ is the desingularization of Σ,

$$\begin{tikzcd}
\widehat{X} \arrow[r, "\widehat{\phi}"] \arrow[d, equal] & X_0 \arrow[d, "\pi_0"'] & \widetilde{X} \arrow[l, "\text{can. resol.}"'] \arrow[d, "\pi"] \\
\widehat{X} \arrow[r, "\phi"] \arrow[d, "\varepsilon_0"'] & Y_0 \arrow[d, "\eta_0"] & Y \arrow[l] \\
X \arrow[r, dashed, "\phi_0"] & \Sigma &
\end{tikzcd}$$

where ε_0 is the composition of blowing-ups such that ϕ is a morphism. Assume that $\phi = \pi_0 \circ \widehat{\phi}$ is the Stein factorization of ϕ, i.e., X_0 is normal, π_0 is a finite triple cover and $\widehat{\phi}$ is birational. Then X is the unique minimal nonsingular model of X_0.

So the essential part of the classification of trigonal surfaces is to understand triple covers. Therefore, many authors have established new theories on triple covers, (see [17], [27]). We start from the cubic defining equations of triple covers so that the computation of the normalization can be applied. The advantage of this point of view is that we can see globally the branch locus, we have the canonical resolution $\widetilde{X} \to X_0$ of the singularities, and we have formulas to compute the global invariants. So triple covers are quite similar to double covers. Note that finite covers of degree higher than 3 do not admit the canonical resolution.

In §2, we will recall the basic facts on triple covers. Then we will apply our method on triple covers to study trigonal fibrations and rational triple points of dimension two.

2. Basic facts on triple covers

In this section we recall some facts about triple covers. The details are referred to [26] or [8].

2.1. Triple cover data. Let X be a smooth algebraic surface over $\mathbb{C}$, and let $\pi : Y \longrightarrow X$ be a normal triple cover. The following lemma is standard.

LEMMA 2.1. *We can find an invertible sheaf $\mathcal{L}$, and two global sections $s \in H^0(X, \mathcal{L}^2)$ and $0 \neq t \in H^0(X, \mathcal{L}^3)$, such that Y is the normalization of the surface defined by $z^3 + sz + t = 0$ in the line bundle of $\mathcal{L}$, and π is induced by the bundle projection.*

PROOF. The extension of function fields $\pi^* : \mathbb{C}(X) \hookrightarrow \mathbb{C}(Y)$ has degree 3. The field extension is generated by one element $\theta \in \mathbb{C}(Y) \setminus \mathbb{C}(X)$ satisfying

$$\theta^3 + \pi^* a \cdot \theta + \pi^* b = 0, \quad \text{for some } a, b \in \mathbb{C}(X). \tag{2.1}$$

$b \neq 0$ because the equation is irreducible. Without lose of generality, we assume that $a \neq 0$. Let L be the minimal divisor on X such that

$$2L + \operatorname{div}(a) \geq 0, \qquad 3L + \operatorname{div}(b) \geq 0,$$

and let $\mathcal{L} = \mathcal{O}_X(L)$. Note that L is not necessarily effective, and L is defined by a rational section ℓ of $\mathcal{L}$. Now consider the following sections of $\mathcal{L}^2$, $\mathcal{L}^3$ and $\pi^*\mathcal{L}$ respectively,

$$s = a\ell^2, \quad t = b\ell^3, \quad \widetilde{\theta} = \pi^*\ell \cdot \theta.$$

By the choice of L, we see that $s \in H^0(X, \mathcal{L}^2)$, $t \in H^0(X, \mathcal{L}^3)$, and as a section of $\pi^*\mathcal{L}^3$,

$$\widetilde{\theta}^3 + \pi^*(s)\widetilde{\theta} + \pi^*(t) = 0. \tag{2.2}$$

Because of this equation, $\widetilde{\theta} = \pi^*(\ell)\theta$ has no pole when viewed as a section of $\pi^*\mathcal{L}$ on Y. So $\widetilde{\theta} \in H^0(Y, \pi^*\mathcal{L})$.

On the other hand, we denote by $p : V(L) \to X$ the line bundle associated to L, and by $z \in H^0(V(L), p^*\mathcal{L})$ the fiber coordinate of $V(L)$. Then $z^3 + p^*sz + p^*t$ is a section of $H^0(V(L), p^*(\mathcal{L})^3)$ whose zero set is a surface $\Sigma \subset V(L)$. We say simply that Σ is defined by

$$z^3 + sz + t = 0 \tag{2.3}$$

in $V(L)$. $\widetilde{\theta}$ defines a section of the line bundle $\widetilde{p}: V(\pi^*L) \to Y$ which is the pullback line bundle of $p: V(L) \to X$ under the base change $\pi: Y \to X$.

$$\begin{tikzcd}
V(\pi^*L) \arrow[r, "\widetilde{\pi}"] \arrow[d, shift right, "\widetilde{p}"'] & V(L) \arrow[d, "p"] \\
Y \arrow[u, shift right, "\widetilde{\theta}"'] \arrow[ur, "\nu"] \arrow[r, "\pi"] & X
\end{tikzcd}$$

So π is lifted to a map $\nu = \widetilde{\pi} \circ \widetilde{\theta} : Y \to V(L)$. Locally, $\nu(y) = \widetilde{\pi}(y, \widetilde{\theta}(y)) = (\pi(y), \widetilde{\theta}(y))$, the fiber coordinate of $\nu(y)$ is $\widetilde{\theta}(y)$, i.e., $z(\nu(y)) = \widetilde{\theta}(y)$ and $\nu^*(z) = \widetilde{\theta}$ as sections of $\pi^*\mathcal{L}$. Hence (2.2) is the pullback of (2.3) under ν^*, namely,

$$z(\nu(y))^3 + s(\pi(y))z(\nu(y)) + t(\pi(y)) = 0, \quad \text{for all } y \in Y.$$

Hence the image of ν is obviously Σ which is a (non-normal) triple cover of X induced by p. Now we see that the birational finite map ν is nothing but the normalization of Σ and $\pi := p \circ \nu$. $\square$

The triplet $(s, t, \mathcal{L})$ in the lemma is called the triple cover data of π. Any triple cover π is determined by some triple cover data $(s, t, \mathcal{L})$. Because X is smooth, we can talk about the factorization of a section according to its divisor.

If $s = 0$, then the triple cover is cyclic and everything is known. So we always assume that $s \neq 0$. Let

$$a = \frac{4s^3}{\gcd(s^3, t^2)}, \qquad b = \frac{27t^2}{\gcd(s^3, t^2)}, \qquad c = \frac{4s^3 + 27t^2}{\gcd(s^3, t^2)}.$$

Then a, b and c are coprime sections of an invertible sheaf such that $a + b = c$.

Conversely, any coprime triples (a, b, c) with $a + b = c$ can determine a triple cover over X. Assume that we have decompositions (according to the decompositions of their divisors)

$$a = 4a_1a_2^2a_0^3, \qquad b = 27b_1b_0^2, \qquad c = c_1c_0^2,$$

where a_1, a_2, b_1, c_1 are square-free and $\gcd(a_1, a_2) = 1$. Then the data (s, t) determined by (a, b, c) is given as follows:

$$s = a_1a_2^2b_1a_0, \quad t = a_1a_2^2b_1^2b_0.$$

Denote the corresponding divisors by

$$A_i = \mathrm{Div}(a_i), \qquad B_i = \mathrm{Div}(b_i), \qquad C_i = \mathrm{Div}(c_i).$$

Let $D_1 = B_1 + C_1$, $D_2 = A_1 + A_2$. Then the branch locus of the triple cover π is $2D_2 + D_1 = 2A_2 + 2A_2 + B_1 + C_1$. π is totally ramified over $D_2 = A_1 + A_2$, hence D_2 is called the *totally ramified branch locus*. D_1 is called the *simply ramified branch locus*. Let $\mathcal{E}_\pi$ denote the trace-free subsheaf of $\pi_*\mathcal{O}_Y$, then $c_1(\mathcal{E}_\pi) = -D_2 - \frac{1}{2}D_1$.

It is proved that X is smooth if and only if D_2 is smooth, D_2 and D_1 have no common points, and all of the singular points of D_1 are cusps (i.e., locally defined by $y^2 + f(x, y)^3 = 0$, $f(0, 0) = 0$) where π is totally ramified.

2.2. Canonical resolution. The *canonical resolution* $\tau : \widetilde{Y} \to Y$ of the singularities of Y is the following commutative diagrams.

$$\begin{tikzcd}
\widetilde{Y} = Y_k \arrow[r, "\tau_k"] \arrow[d, "\widetilde{\pi} = \pi_k"'] & Y_{k-1} \arrow[r, "\tau_{k-1}"] \arrow[d, "\pi_{k-1}"] & \cdots \arrow[r] & Y_2 \arrow[r, "\tau_2"] \arrow[d, "\pi_2"] & Y_1 \arrow[r, "\tau_1"] \arrow[d, "\pi_1"] & Y_0 = Y \arrow[d, "\pi_0 = \pi"] \\
\widetilde{X} = X_k \arrow[r, "\sigma_k"] & X_{k-1} \arrow[r, "\sigma_{k-1}"] & \cdots \arrow[r] & X_2 \arrow[r, "\sigma_2"] & X_1 \arrow[r, "\sigma_1"] & X_0 = X
\end{tikzcd}$$

(1) σ_{i+1} is the blowing-up of X_i at a singular point p_i of the branch locus of π_i. Y_{i+1} is the normalization of $X_{i+1} \times_{X_i} Y_i$.

(2) The corresponding data $(a^{(i)}, b^{(i)}, c^{(i)})$ of π_i is obtained from $(\sigma_i^* a^{(i-1)}, \sigma_i^* b^{(i-1)}, \sigma_i^* c^{(i-1)})$ by eliminating the common factors. (This is due to the computation of the normalization (see [25])).

(3) $\widetilde{\pi} = \pi_k$ has a smooth branch locus. So $\widetilde{Y} = Y_k$ is smooth.

The idea to prove the exitance of k in step (3) is quite simple. Consider the curve $\mathcal{D}^{(i)}=\mathrm{Div}(a^{(i)}b^{(i)}c^{(i)})$. We see from step (2) that

$$\mathcal{D}^{(i+1)} \leq \sigma_i^*(\mathcal{D}^{(i)}) \leq \cdots \leq (\sigma_1 \circ \cdots \circ \sigma_i)^*(\mathcal{D}^{(0)}).$$

By the embedded resolution of the singularities of $\mathcal{D}^{(0)}_{\mathrm{red}}$, we can assume that $\mathcal{D}^{(i)}$ is a normal crossing divisor. This implies that any two of the sections $a^{(i)}$, $b^{(i)}$ and $c^{(i)}$ have no common zero points because $a^{(i)} + b^{(i)} = c^{(i)}$. The next step is just the canonical resolution of cyclic triple covers or double covers (locally).

2.3. Determination of the new branch locus. Put

$$d_i = \min\left\{\mu_{p_i}(A^{(i)}),\ \mu_{p_i}(B^{(i)}),\ \mu_{p_i}(C^{(i)})\right\},$$

where $\mu_p(D)$ is the multiplicity of a divisor D at p. Let

$$m_i = \left[\frac{\mu_{p_i}(D_1^{(i)})}{2}\right], \tag{2.4}$$

$$n_i = \begin{cases} \mu_{p_i}(D_2^{(i)}), & \text{if } d_i \equiv \mu_{p_i}(A^{(i)}) \pmod 3; \\ \mu_{p_i}(D_2^{(i)}) - 1, & \text{otherwise.} \end{cases} \tag{2.5}$$

Let E_i be the exceptional curve of σ_i, $\mathcal{E}_i$ be the total transform of E_i in $\widetilde{X}$, and let $\sigma = \sigma_1 \cdots \sigma_k$. Then

$$\widetilde{D}_1 = \sigma^*(D_1) - 2\sum_{i=0}^{k-1} m_i \mathcal{E}_{i+1}, \tag{2.6}$$

$$\widetilde{D}_2 = \sigma^*(D_2) - \sum_{i=0}^{k-1} n_i \mathcal{E}_{i+1}. \tag{2.7}$$

We use also E_i to denote the strict transform of E_i in $\widetilde{X}$.

(i) $E_i \subset \widetilde{D}_1 \iff \mu_{p_i}(D_1^{(i)})$ is odd;
(ii) $E_i \not\subset \widetilde{D}_1$ and $E_i \not\subset \widetilde{D}_2 \iff \mu_{p_i}(D_1^{(i)})$ is even and $d_i \equiv \mu_{p_i}(A^{(i)}) \pmod 3$;
(iii) $E_i \subset \widetilde{D}_2 \iff \mu_{p_i}(D_1^{(i)})$ is even and $d_i \not\equiv \mu_{p_i}(A^{(i)}) \pmod 3$. Furthermore,
  (a) if $\mu_{p_i}(A^{(i)}) - d_i \equiv 1 \pmod 3$, then E_i is a component of $\widetilde{A}_1$;
  (b) if $\mu_{p_i}(A^{(i)}) - d_i \equiv 2 \pmod 3$, then E_i is a component of $\widetilde{A}_2$.

LEMMA 2.2 ([8], Lemma 2.2). *The local intersection multiplicity $(D_1D_2)_p$ of D_1 with D_2 at any point p is an even number.*

2.4. Computation of invariants. Now we have the formulas for the canonical resolution:

$$\chi(\mathcal{O}_{\widetilde{Y}}) = 3\chi(\mathcal{O}_X) + \frac{1}{8}D_1^2 + \frac{1}{4}D_1K_X + \frac{5}{18}D_2^2 + \frac{1}{2}D_2K_X - \sum_{i=0}^{k-1}\frac{m_i(m_i-1)}{2} - \sum_{i=0}^{k-1}\frac{n_i(5n_i-9)}{18}, \tag{2.8}$$

$$K_{\widetilde{Y}}^2 = 3K_X^2 + \frac{1}{2}D_1^2 + 2D_1K_X + \frac{4}{3}D_2^2 + 4D_2K_X - \sum_{i=0}^{k-1}2(m_i-1)^2 - \sum_{i=0}^{k-1}\frac{4n_i(n_i-3)}{3} - k, \tag{2.9}$$

3. On trigonal fibrations

Let $f : S \longrightarrow C$ be a fibration of genus g, where S is a relatively minimal smooth projective surface over complex number field, C is a smooth projective curve of genus b. If the general fibre of f is trigonal, i.e. is a triple cover of $\mathbb{P}^1$, f is called a *trigonal fibration.*

For any relatively minimal fibration $f : S \longrightarrow C$, we have the following basic relative numerical invariants:

$$K_f^2 \overset{\text{def}}{=} K_{S/C}^2 = K_S^2 - 8(g-1)(b-1),$$
$$\chi_f = \chi(\mathcal{O}_S) - (g-1)(b-1).$$

Whenever $\chi_f \neq 0$, the slope of the fibration f can be defined as

$$\lambda_f = K_f^2/\chi_f.$$

And it is known that

$$4 - \frac{4}{g} \leq \lambda_f \leq 12.$$

$\lambda_f = 12$ if and only if f is a Kodaira fibration.

The slope λ_f is an important invariant for a fibration. In 1987, G. Xiao [32] proved that for a relatively minimal fibration f of genus $g \geq 2$ (see also [10] for semistable fibrations), one has

$$4 - 4/g \leq \lambda_f \leq 12,$$

and $\lambda_f = 12$ if and only if every fibre of f is smooth and reduced, i.e., f is a Kodaira fibration. For a genus 2 fibration f, Xiao [31] proved that

$$2 \leq \lambda_f \leq 7.$$

In general, if f is a hyperelliptic fibration of genus g, Xiao [33] obtained an upper bound:

$$4 - 4/g \leq \lambda_f \leq \begin{cases} 12 - (8g+4)/g^2, & g \text{ even}, \\ 12 - (8g+4)/(g^2-1), & g \text{ odd}. \end{cases}$$

In particular, for a hyperelliptic fibration f of genus 3, we have

$$8/3 \leq \lambda_f \leq 17/2$$

As for the relatively minimal non-hyperelliptic fibration f of genus g, one has:

$$\lambda_f \geq \begin{cases} 3, & g = 3, \text{ E. Horikawa [15] and [12]}, \\ 24/7, & g = 4, \text{ Z. Chen [6] and K. Konno [13]}; \\ 4, & g = 5, \text{ K. Konno [13]}; \\ 96/25, & g = 6, \text{ K. Konno [14]}. \end{cases}$$

Stankova-Frenkel [21] proved that if f is a semistable trigonal fibration, then

$$\lambda_f \geq \frac{24(g-1)}{5g+1}.$$

See [21, 24] for some other lower bounds. [3] is a very good survey on the study of slopes.

Let α be the number of total ramification points in a general fibre of f. Then by Hurwitz's theorem, we have $2\alpha \leq 2g + 4$, i.e. $\alpha \leq g + 2$. It is obvious that α is invariant under base changes. A trigonal fibration may have several α's, we will denote its maximum by $\alpha(f)$.

In [8, 9] the following 2 theorems are obtained.

THEOREM 3.1 ([8]). *Let $f : S \longrightarrow C$ be a trigonal fibration of genus 3. If f is not locally trivial, then*

$$\lambda_f \leq \begin{cases} 9, & \text{if } \alpha(f) = 3, 4, 5; \\ 21/2, & \text{if } \alpha(f) = 2. \end{cases}$$

Therefore, Kodaira fibrations occur only when $\alpha(f) \leq 1$.

THEOREM 3.2 ([9]). *Let $f : S \longrightarrow C$ be a trigonal fibration of genus $g \geq 4$. If $\alpha(f) = g + 2$, then*

$$\lambda_f \leq \begin{cases} 12 - \dfrac{6(g+1)}{g^2} & \text{if } g \text{ is even}, \\ 12 - \dfrac{6}{g-1} & \text{if } g \text{ is odd}. \end{cases}$$

If $g/2 < \alpha(f) \leq g+1$, then

$$\lambda_f \leq \begin{cases} 12 - \dfrac{24\alpha(f)}{g(g^2 - g(\alpha(f)+1) + 6(\alpha(f)-1))} & \text{if } g \text{ is even,} \\ 12 - \dfrac{24\alpha(f)}{g^3 - g^2(\alpha(f)+3) + g(8\alpha(f)-1) - 7\alpha(f) + 3} & \text{if } g \text{ is odd.} \end{cases}$$

If $2 \leq \alpha(f) \leq \frac{g}{2}$, then

$$\lambda_f < 12.$$

Thus Kodaira fibration only occurs when $\alpha(f) \leq 1$.

Here we will give a sketch of the proof. Firstly, we have the following propositions about base change:

PROPOSITION 3.3 ([**23**], Corollary 4.3). *Let f be a non-semistable fibration with $\lambda_f > 8$, then the slope will incease through any non-trivial stabilizing base change.*

COROLLARY 3.4 ([**23**], Corollary 4.4). *Let f be a fibration with maximal slope. If $\lambda_f > 8$, then f is semistable.*

Hence in the theorems 3.1 and 3.2 we may assume the fibration is semistable.

For a trigonal fibration $f : S \longrightarrow C$, after some base change, we have the following commutative diagram:

$$\begin{tikzcd} \widetilde{P} \arrow[d, "\widetilde{\sigma}"'] & \widetilde{S} \arrow[l, "\widetilde{\pi}"'] \arrow[d, "\widetilde{\tau}"] & \\ P_0 \arrow[dr, "\varphi_0"'] & S' \arrow[l, dashed, "\phi"'] \arrow[r] \arrow[d, "f'"] & S \arrow[d, "f"] \\ & C' \arrow[r] & C \end{tikzcd}$$

where $f' : S' \longrightarrow C'$ is the base change of f, $\widetilde{\sigma}$, $\widetilde{\tau}$ are birational morphisms, $\varphi_0 : P_0 \longrightarrow C'$ is a minimal ruled surface, $\widetilde{\pi} : \widetilde{S} \longrightarrow \widetilde{P}$ is a smooth triple cover. Since $(P_0, \widetilde{\sigma})$ is not unique, we may choose a suitable contraction such that the singularities are not too bad.

LEMMA 3.5 (Cf. [8], Lemma 5.2). *$\widetilde{P}$ can be contracted to a relatively minimal model P_0 with a ruling $\varphi_0 : P_0 \to C$ satisfying the following conditions.*

$$\begin{tikzcd} \widetilde{P} \arrow[rr, "\widetilde{\sigma}"] \arrow[dr, "\widetilde{\varphi}"'] & & P_0 \arrow[dl, "\varphi_0"] \\ & C & \end{tikzcd}$$

(1) *Let $\widetilde{R}$ be the branch locus of $\widetilde{\pi}$, and R be the image of $\widetilde{R}$ in P_0. Then $\widetilde{\sigma} : \widetilde{P} \to P_0$ is the canonical resolution of R.*
(2) *Let R_h be the horizontal part of R (i.e., R_h does not contain any fibres of φ_0 and $R_v = R - R_h$ is the sum of some fibres), then the orders of the singular points of R_h (resp. R) are less or equal to $g+2$ (resp. $g+4$).*

Such a geometrically ruled surface $\varphi_0 : P_0 \to C$ with the branch locus R will be called normalized.

LEMMA 3.6 (Cf. [8], Lemma 5.5). *Let f be a trigonal fibration with maximal slope. Then we can assume that R has no vertical fibres, and that each component of D_1 or D_2 is a section of $\varphi_0 : P_0 \longrightarrow C$.*

Let $\widetilde{R}$ be the branch locus of $\widetilde{\pi}$, $R = \widetilde{\sigma}(\widetilde{R})$. Then $\widetilde{\sigma}$ is the embedded resolution of singularities of the branch locus R, $\widetilde{\pi}$ is a smooth triple cover. Let C_0 be a section of the ruled surface $\varphi_0 : P_0 \longrightarrow C$ such that the self-intersection number $C_0^2 = -e$ is minimal. Let

$$R = D_1 + 2D_2, \qquad D_1 = B_1 + C_1, \qquad D_2 = A_1 + A_2.$$

Here D_1 is the simply ramified branch locus, D_2 is the totally ramified branch locus. Since the genus of a general fibre is equal to g, $RF = D_1F + 2D_2F = 2g + 4$. Let

$$D_2 \sim \alpha(f)C_0 + \beta F,$$

$$D_1 \sim (2g+4-2\alpha(f))C_0 + 2\gamma F.$$

By (2.8), (2.9), we have

$$\chi_f = \left(\frac{5\alpha(f)}{9}-1\right)\left(\beta-\frac{\alpha(f)}{2}e\right)+(g+1-\alpha(f))\left(\gamma-\frac{g+2-\alpha(f)}{2}e\right)$$
$$-\sum_{i=0}^{k-1}\frac{m_i(m_i-1)}{2}-\sum_{i=0}^{k-1}\frac{n_i(5n_i-9)}{18},$$
$$K_f^2 = 8\left(\frac{\alpha(f)}{3}-1\right)\left(\beta-\frac{\alpha(f)}{2}e\right)+4(g-\alpha(f))\left(\gamma-\frac{g+2-\alpha(f)}{2}e\right)$$
$$-\sum_{i=0}^{k-1}2(m_i-1)^2-\sum_{i=0}^{k-1}\frac{4n_i(n_i-3)}{3}-k+\varepsilon.$$

where ε is the number of (-1)-curves blown down by $\widetilde{\tau}$.

Then

$$12\chi_f - K_f^2 = (4\alpha(f)-4)\left(\beta-\frac{\alpha(f)}{2}e\right)+4(2g+3-2\alpha(f))\left(\gamma-\frac{g+2-\alpha(f)}{2}e\right)$$
$$+3k-2\sum_{i=0}^{k-1}m_i(2m_i-1)-2\sum_{i=0}^{k-1}n_i(n_i-1)-\varepsilon.$$

$$12\chi_f - K_f^2-\mu\chi_f = \left(4\alpha(f)-4-\left(\frac{5\alpha(f)}{9}-1\right)\mu\right)\left(\beta-\frac{\alpha(f)}{2}e\right) \tag{3.1}$$
$$+(8g+12-8\alpha(f)-(g+1-\alpha(f))\mu)\left(\gamma-\frac{g+2-\alpha(f)}{2}e\right)$$
$$+\left[3k-2\sum_{i=0}^{k-1}m_i(2m_i-1)-2\sum_{i=0}^{k-1}n_i(n_i-1)-\varepsilon\right.$$
$$\left.+\left(\sum_{i=0}^{k-1}\frac{m_i(m_i-1)}{2}+\sum_{i=0}^{k-1}\frac{n_i(5n_i-9)}{18}\right)\mu\right]$$

Let

$$h_p = \sum_i (3-2m_i(2m_i-1)-2n_i(n_i-1))-\varepsilon_p, \tag{3.2}$$

$$\delta_p = \sum_i\left(\frac{m_i(m_i-1)}{2}+\frac{n_i(5n_i-9)}{18}\right). \tag{3.3}$$

From these 2 invariants, we will define a slope function

$$s_p(\mu) = h_p + \delta_p\mu,$$

Our goal is to find the lower bound of the slope function, especially when μ is sufficiently small.

Let D be a horizontal effective divisor in the ruled surface $\varphi_0 : P_0 \longrightarrow C$. Afterwards, we always denote a fibre of the minimal ruled surface by F. Then the relative ramification index of D is defined as

$$r_D = D(D+K_{P_0/C}) \geq 0.$$

If after k blow-ups σ_i the strict transform $\widetilde{D}$ of D becomes smooth (it may be composed of several disjoint nonsingular curves), then

$$\widetilde{D} = (\sigma_1\cdots\sigma_k)^*D - \sum_{i=1}^{k} m_i\mathcal{E}_i,$$

where $\mathcal{E}_i$ is the total transform of the exceptional divisor of σ_i and m_i is the multiplicity of the strict transform of D at the center of blow-up σ_i. In fact, we have

$$r_D = \sum_{i=1}^{k} m_i(m_i - 1) + (\text{ramification index of the finite morphism } \widetilde{D} \longrightarrow C).$$

Denote the contribution to r_D of each singular point p of D by r_p, then

$$r_D = \sum_p r_p.$$

It is obvious that $r_p = \sum_j m_j(m_j - 1) +$ the contribution of the inverse image of p to the ramification index of $\widetilde{D} \longrightarrow C$.

Afterwards we use the following notation:

$$r_1 = r_{D_1}, \qquad r_2 = r_{D_2},$$

$$r_{1,p} = r_{D_1,p}, \qquad r_{2,p} = r_{D_2,p}.$$

By analysing the singularities on the branch locus, we can obtain the following key lemmas:

LEMMA 3.7. *Let $\varphi_0 : P_0 \longrightarrow C$ be a normalized (Cf. Lemma 3.5) ruled surface with triple cover data $(s, t, \mathcal{L})$ such that the obtained generically triple cover fibration $f : S \longrightarrow C$ is semistable and of maximal slope. It is also assumed that $R = R_h$ and D_1 and D_2 are composed of sections. Then for any singular point p in R, one has*

$$s_p(\mu) \geq M_{1,\min}(\mu) r_{1,p} + M_{2,\min}(\mu) r_{2,p} + M_{3,\min}(\mu)(D_1 D_2)_p, \tag{3.4}$$

where

$$M_{1,\min}(\mu) = \frac{\mu}{9} - 1, \qquad \text{if } \mu \leq 2.25,$$

$$M_{2,min}(\mu) = \begin{cases} \dfrac{12(-3g^2 - 6g + 2) + g(5g + 2)\mu}{18g(g + 2)} & \text{if } g \text{ is even, } \alpha(f) \geq \frac{g}{2} + 1 \text{ and } \mu \leq \frac{6}{g-2}, \\ \dfrac{12(-3g^2 + 5) + (5g^2 - 8g + 3)\mu}{18(g^2 - 1)} & \text{if } g \text{ is odd, } \alpha(f) \geq \frac{g+1}{2} \text{ and } \mu \leq \frac{6}{g-3}, \\ \dfrac{6(-6\alpha(f)^2 + 6\alpha(f) + 1) + (\alpha(f) - 1)(5\alpha(f) - 4)\mu}{18\alpha(f)(\alpha(f) - 1)} & \text{if } 2 \leq \alpha(f) \leq \frac{g}{2} \text{ and } \mu \leq \frac{1}{\alpha(f)-1}. \end{cases}$$

$$M_{3,\min}(\mu) = \begin{cases} \dfrac{4(3 - g\mu)}{9g(g + 2)} & \text{if } g \text{ is even, } \alpha(f) \geq \frac{g}{2} + 1 \text{ and } \mu \leq \frac{6}{g-2}, \\ \dfrac{12 - 4(g - 1)\mu}{9(g^2 - 1)}. & \text{if } g \text{ is odd, } \alpha(f) \geq \frac{g+1}{2} \text{ and } \mu \leq \frac{6}{g-3}, \\ \dfrac{-\mu}{9\alpha(f)} & \text{if } 2 \leq \alpha(f) \leq \frac{g}{2} \text{ and } \mu \leq \frac{1}{\alpha(f)-1}. \end{cases}$$

LEMMA 3.8. *Let $\varphi_0 : P_0 \longrightarrow C$ be a normalized ruled surface with triple cover data $(s, t, \mathcal{L})$ such that the obtained generically triple cover fibration $f : S \longrightarrow C$ is semistable and of maximal slope. It is also assumed that $R = R_h$ and D_1 and D_2 are composed of sections. If D_2 is composed of disjoint sections and that $\alpha(f) < (g + 5)/2$, then for any singular point p in R one has*

$$s_p(\mu) \geq M_{1,\min}(\mu) r_{1,p} + M_{4,\min}(\mu)(D_1 D_2)_p, \tag{3.5}$$

where

$$M_{1,\min}(\mu) = \frac{\mu}{9} - 1, \qquad \text{if } \mu \le 2.25,$$

$$M_{4,\min}(\mu) =$$

$$\begin{cases} \dfrac{24 + g(g-10)\mu}{72g}, & \text{if } g \text{ even, } \alpha(f) \le \frac{g}{2} + 2,\ \mu \le \frac{24}{g(g-2)}, \\ \dfrac{24 + (g-1)(g-11)\mu}{72(g-1)}, & \text{if } g \text{ odd, } \alpha(f) \le \frac{g+5}{2},\ \mu \le \frac{24}{(g-1)(g-3)}. \end{cases}$$

By these lemmas, we can prove the theorem 3.2. We take the simplest case $\alpha = g + 2$ as an example. Then $D_1 = 0$, $\gamma = 0$ and $r_2 = D_1 D_2 = 0$.

By formula (3.1), (3.2), (3.3), we have

$$12\chi_f - K_f^2 - \mu\chi_f = \left(4g + 4 - \frac{5g+1}{9}\mu\right)\left(\beta - \frac{g+2}{2}e\right) + \sum_p (h_p + \delta_p \mu).$$

By Lemma 3.7,

$$\sum_p (h_p + \delta_p\mu) \ge \sum_p (M_{1,\min}(\mu) r_{1,p} + M_{2,\min}(\mu) r_{2,p} + M_{3,\min}(\mu)(D_1 D_2)_p) = M_{2,\min}(\mu) r_2.$$

Here

$$r_2 = D_2(D_2 + K_f) = 2(g+1)\left(\beta - \frac{g+2}{2}e\right) \ge 0.$$

If g is even and $\mu \le \frac{6}{g-2}$, then

$$12\chi_f - K_f^2 - \mu\chi_f \ge \frac{4(6(g+1) - g^2\mu)}{9g(g+2)}\left(\beta - \frac{g+2}{2}e\right).$$

Take $\mu = \frac{6(g+1)}{g^2} < \frac{6}{g-2}$, then

$$12\chi_f - K_f^2 - \frac{6(g+1)}{g^2}\chi_f \ge 0,$$

That is

$$\lambda_f \le 12 - \frac{6(g+1)}{g^2}.$$

If $\alpha(f) = g + 2$, g is odd and $\mu \le \frac{6}{g-3}$, then

$$12\chi_f - K_f^2 - \mu\chi_f \ge \frac{4(g+1)(6 - (g-1)\mu)}{9(g^2-1)}\left(\beta - \frac{g+2}{2}e\right).$$

Take $\mu = \frac{6}{g-1} < \frac{6}{g-3}$, then

$$\lambda_f \le 12 - \frac{6}{g-1}.$$

4. Examples of smooth hyperelliptic central fibre

In this section we will give some examples to show how to construct local fibration by triple cover such that its central fibre is a smooth hyperelliptic curve of genus 3. Let $P = \mathbb{P}^1_{\mathbb{C}[[t]]} = \mathbb{P}^1_{\mathbb{C}} \times_{\mathbb{C}} \mathrm{Spec}(\mathbb{C}[[t]])$. Then $\varphi : P \longrightarrow \mathrm{Spec}(\mathbb{C}[[t]])$ is a local $\mathbb{P}^1$-bundle whose central fibre is $F_0 = \varphi^{-1}(0) \cong \mathbb{P}^1$. Let y denote the affine coordinate in $\mathbb{P}^1_{\mathbb{C}}$. Let $P = U \cup V$ be an affine open cover of P where $P \setminus U$ is the line at infinity $\infty \times_{\mathbb{C}} \mathrm{Spec}(\mathbb{C}[[t]])$, $P \setminus V = Z(y)$. Let $U_y = U - Z(y) = \mathrm{Spec}(\mathbb{C}[[t]][y, y^{-1}])$.

EXAMPLE 4.1. Let

$$\mathbf{s} = (-9t^3 + 9t^2 - 3)y^4 + 12ty^2 - 3t^2 \in \Gamma(P, \mathcal{O}_P(4)),$$
$$\mathbf{t} = (9t^3 - 9t^2 + 2)y^6 + (9t^4 + 18t^3 - 12t)y^4 + 15t^2y^2 + 2t^3 \in \Gamma(P, \mathcal{O}_P(6)).$$

and $\mathcal{L} = \mathcal{O}_P(2)$. By using the following polynomial equation in $\mathcal{L}^3$

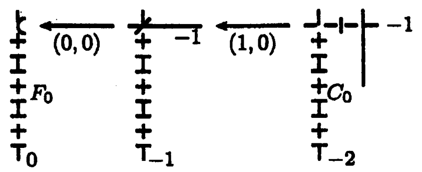
$$p(z) = z^3 + \mathbf{s}z + \mathbf{t},$$

we can define the triple cover $f : Y \to P$ determined by the triple cover data $(\mathbf{s}, \mathbf{t}, \mathcal{L})$.

Then we have

$$a_0 = \mathbf{s} = (-9t^3 + 9t^2 - 3)y^4 + 12ty^2 - 3t^2,$$
$$b_0 = \mathbf{t} = (9t^3 - 9t^2 + 2)y^6 + (9t^4 + 18t^3 - 12t)y^4 + 15t^2y^2 + 2t^3,$$
$$a_1 = a_2 = b_1 = 1, \qquad c_0 = 27t^2y,$$
$$c_1 = (-4t^5 + 12t^4 - 12t^3 + 3t^2 + 2t - 1)y^{10} + (22t^3 - 26t^2 + 4t + 4)y^8$$
$$+ (-t^4 + 20t^3 + 8t^2 - 22t - 2)y^6 + (22t^2 + 8t)y^4 + (4t^2 - 1)y^2 + 4t.$$

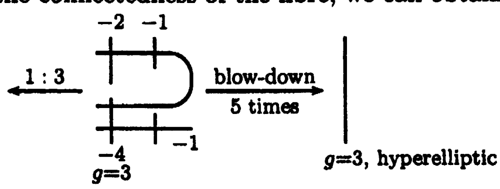
The discriminant of c_1 is a polynomial in t, hence it has 10 simple roots in an infinitely small neighborhood of $t = 0$. When $t = 0$, c_1 has a double root $y = 0$ and 8 simple roots. Thus this triple cover has only double ramification. The following diagram shows the resolution of the singular points of the branch locus.



Note that U_y is invariant during the resolution, $F_0 \cap U_y \cong C_0 \cap U_y$. Since F_0 is contained in the zero set of c_0, Y is not normal over $f^{-1}(F_0)$ (cf. [26]). But the restriction of the defining polynomial $p(z)$ to $F \cap U_y$ is

$$p(z) \equiv z^3 - 3y^4z + 2y^6 = (z + 2y^2)(z - y^2)^2 \pmod t$$

So $p(z)$ is reducible in $\mathbb{C}[[t]][y, y^{-1}]$. This implies that after the normalization $\tilde{Y} \longrightarrow Y$, the triple cover of C_0 has 2 components. By the connectedness of the fibre, we can obtain the smooth fibre bundle.



PROPOSITION 4.2. *Let F_0 be a fibre of a minimal ruled surface $\varphi : P \longrightarrow C$, and let $f : S \to C$ be a relatively minimal fibration obtained by a triple cover of P. If the fibre of f over F_0 is a smooth hyperelliptic fibre, then*

(1) *$\alpha = D_2F \leq 1$;*
(2) *There is only one singular point $p \in F_0$ of branch locus. If $D_2F = 0$, then $\mu_p(D_1) \leq 3$. If $D_2F = 1$, then $\mu_p(D_1) = \mu_p(D_2) = 1$. Hence the other intersecting points of branch locus with F_0 are all of double ramification.*

The examples above imply that smooth hyperelliptic fibres may exist when $\alpha = D_2F \leq 1$. As we know the Kodaira fibration do exist when $g \geq 3$, so the slope may reach the upper bound 12 when $\alpha \leq 1$. At last we will investigate the behavior of the branch locus if f is Kodaira fibration.

COROLLARY 4.3. *If f is a Kodaira fibration, then the branch locus must satisfy the following conditions:*

(1) *$D_2F = 0$: A singular point p of the branch locus (good cusp is excluded) must be one of following type. If a fibre has a singular point as follows, it can have neither second singular point nor good cusps.*
  (a) *Double point not tangent to the fibre;*

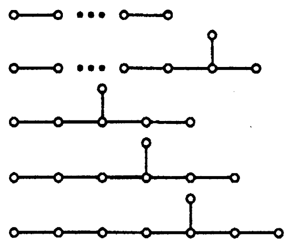
(b) *Triple point not tangent to the fibre;*
(c) *Smooth point tangent to the fibre with order* 2.

(2) $D_2F = 1$: *A singular point p of the branch locus (good cusp is excluded) must be of following type. If a fibre has a singular point as follows, it can have neither second singular points nor good cusps.*

(a) $\mu_p(D_1) = \mu_p(D_2) = 1$ *and the intersection number* $(D_1D_2)_p$ *is even.* D_1, D_2 *are not tangent to the fibre.*

5. Cubic equations of rational triple points of a surface

Rational double points of dimension two were studied first by Du Val ([11]) in 1934. There are 5 types of rational double points and each type has one standard quadratic defining equation. These equations are very useful in the classification of algebraic surfaces.



A_n : $z^2 + x^2 + y^{n+1} = 0, \quad (n \geq 1)$

D_n : $z^2 + y(x^2 + y^{n-2}) = 0, \quad (n \geq 4)$

E_6 : $z^2 + x^3 + y^4 = 0$

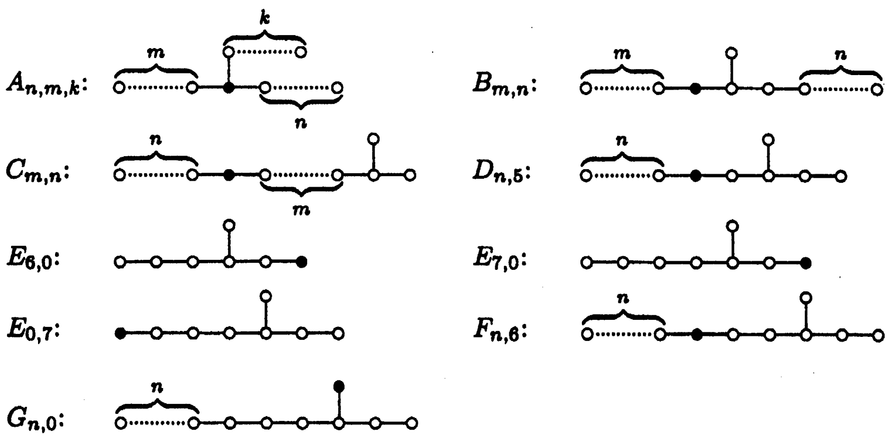
E_7 : $z^2 + x(x^2 + y^3) = 0$

E_8 : $z^2 + x^3 + y^5 = 0$

From the quadratic equations, we can resolve the surface singularity by using a canonical method for double covers (see [4], p.107).

A rational point of multiplicity higher than 2 is not a hypersurface singularity, so it is impossible to define the singularity itself by one equation (cf. [1]). On the other hand, a surface singularity is isomorphic to the normalization of a local hypersurface $f(x, y, z) = 0$ in $\mathbb{C}^3$. Sometimes, it is very convenient if we know f, especially when we know the processes of normalization and resolution directly from f. A typical example is the Hirzebruch-Jung singularity defined by the normalization of $z^n = xy^{n-q}$. We do not need to find the defining equations of the normalized singularity. In fact, the singularity is determined by n and q.

In 1966, M. Artin [1] classified the dual graphs of rational triple points of dimension 2 into 9 classes, and he proved that each rational triple point can be embedded into $\mathbb{C}^4$. In 1968, Tyurina [28] gave explicitly 3 defining equations for each singularity. Tyurina [29] proved also that a rational triple point is determined uniquely by its dual graph. So isomorphically, there are 9 rational triple points.



Where $\circ$ is a (-2)-curve, $\bullet$ is a (-3)-curve.

On the other hand, the singularities coming from the normalization of a local surface defined by a cubic equation $z^3 + s(x, y)z + t(x, y) = 0$ can be resolved by canonical resolution. Theoretically, a rational triple point might be defined by one cubic equation (up to normalization). So it is interesting to find out the equations similar to rational double points. By using triple cover theory, the local cubic equations of all the rational cubic points are obtained(Cf. [7]).

$A_{n,m,k}$: $(n \geq m \geq k)$

$$\begin{cases} z^3 + (x + y^{k+1})z^2 + x^2y^{n-k}z + x^2y^{m+2n-3k}(x + y^{k+1}) = 0, & (m > k); \\ z^3 + x(x + y^{k+1})z + x(x + y^{k+1})^2y^p = 0, & (n = 3p + k, m = k); \\ z^3 + x(x + y^{k+1})z + (x + y^{k+1})^2y^{n+1} = 0, & (n \not\equiv k \pmod 3,\ m = k). \end{cases}$$

$B_{m,n}$:

$$\begin{cases} z^3 + (-x + y^{p+1})z^2 + y^{2m+3}z + xy^{2m+3} = 0, & n = 2p\,; \\ z^3 - xz^2 + y^{m+3}(y^m + y^p)z + xy^{2m+3} = 0, & n = 2p + 1\,. \end{cases}$$

$C_{m,n}$:

$$\begin{cases} z^3 + x(y^2 + x^{m+2})z + x(y^2 + x^{m+2})^2y^p = 0, & n = 3p + 1\,; \\ z^3 + x(y^2 + x^{m+2})z + (y^2 + x^{m+2})^2y^{n+1} = 0, & n \not\equiv 1 \pmod 3). \end{cases}$$

$D_{n,5}$:

$$z^3 + xz^2 + y^{n+3}z + x^2y^{2n+2} = 0\,.$$

$E_{6,0}$:

$$z^3 + y^3z + x^2y^2 = 0\,.$$

$E_{7,0}$:

$$z^3 + x^2yz + y^4 = 0\,.$$

$E_{0,7}$:

$$z^3 + y^2(x^2 + y^3) = 0\,.$$

$F_{n,6}$:

$$\begin{cases} z^3 + x(x^2 + y^3)z + x(x^2 + y^3)^2y^p = 0, & n = 3p + 2\,; \\ z^3 + x(x^2 + y^3)z + (x^2 + y^3)^2y^{n+1} = 0, & n \not\equiv 2 \pmod 3)\,. \end{cases}$$

$G_{n,0}$:

$$\begin{cases} z^3 + x^{p+2}yz + xy^3 = 0, & n = 3p\,; \\ z^3 + x^{p+2}yz + x^2y^3 = 0, & n = 3p + 1\,; \\ z^3 + xy^2(y + x^{p+2}) = 0, & n = 3p + 2\,. \end{cases}$$

Here is an example to show the canonical resolution. We use the following notations:

——— -3 : a rational curve with self-intersection number -3 which is a component of the totally ramified branch locus D_2;

——— -4 : a rational curve with self-intersection number -4 which is a component of the simply ramified branch locus D_1, note that the self-intersection number -2 will not be marked;

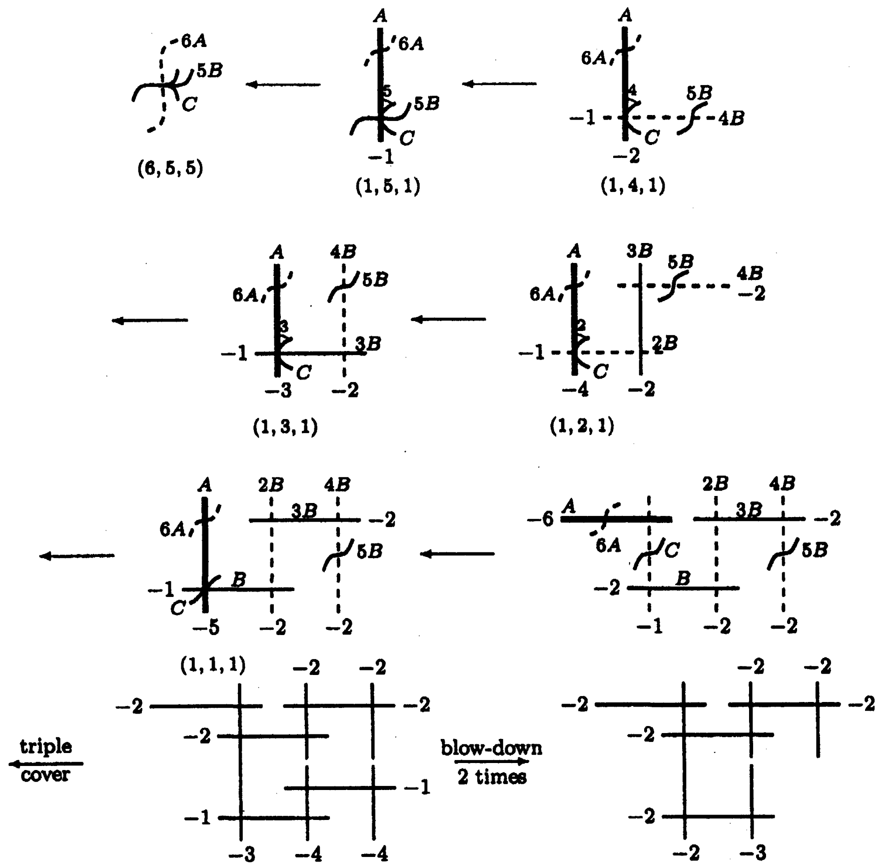
-------- -1 : a rational curve with self-intersection number -1 which is not a component of the branch locus, note that the self-intersection number -2 will not be marked;

------: a simply ramified point on a rational curve;

------: a totally ramified point on a rational curve;

EXAMPLE 5.1. $z^3 + x^2yz + y^4 = 0$, $p_0 = (0,0)$

(0) $s = x^2y$, $t = y^4$
(1) $a = 4x^6$, $b = 27y^5$, $c = 4x^6 + 27y^5$. (Step 0: eliminate $y^3 = \gcd(s^3, t^2)$)
(2) Multiplicities of (a, b, c) at p_0 are $(6, 5, 5)$.
(3) Pullback of $a + b = c$: $e^6\bar{a} + e^5\bar{b} = e^5\bar{c} \implies e\bar{a} + \bar{b} = \bar{c}$. (Eliminate e^5)
(4) New data: $a' = e\bar{a}$, $b' = \bar{b}$, $c' = \bar{c}$. So e is in a_1.
(5) Multiplicities of (a', b', c') at p_1 are $(1, 5, 1)$.
...



This is a rational triple point of type $E_{7,0}$.

6. Further remarks

The final purpose of this study is to get the computation formulas for the global invariants of a trigonal fibration $f : X \to C$ from the local data of the special fibers. If the genus g of a generic fiber is 2, G. Xiao [31] got nice formulas:

$$\begin{aligned} \chi_f &= \frac{1}{12}s_2(f) + \frac{1}{5}s_3(f), \\ K_f^2 &= \frac{1}{5}s_2(f) + \frac{7}{5}s_3(f), \\ e_f &= s_2(f) + s_3(f), \end{aligned}$$

where $s_2(f) = \sum_F s_2(F)$ and $s_3(f) = \sum_F s_3(F)$ are two nonnegative indices of the singular fibers. When F is a semistable fiber, $s_2(F)$ (resp. $s_3(F)$) is the number of inseparable (resp. separable) double points of F. A double point p of F is called inseparable if the partial normalization of F at p is still connected. Otherwise, p is called separable.

Based on the local analysis of the singularities, Jun Lu and the two authors of the present paper get similar formulas for non-hyperelliptic fibrations of genus $g \geq 3$. When $g = 3$, we have

$$\begin{aligned} \chi_f &= \frac{1}{9}a_1 + \frac{1}{3}a_2 + \frac{4}{9}a_3 + \frac{1}{3}a_4 + \frac{4}{9}a_5 + \frac{1}{9}a_6, \\ K_f^2 &= \frac{1}{3}a_1 + 3a_2 + \frac{7}{3}a_3 + 2a_4 + \frac{13}{3}a_5 + \frac{4}{3}a_6, \\ e_f &= 2a_2 + a_3 + a_4 + 3a_5 + a_6, \end{aligned}$$

For non-hyperelliptic fibartions of genus 3, M. Reid has some conjectural formulas [19]. We will compare them with our formulas later.

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