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doubling条件を満たさない測度空間上のMorrey空間に関して

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1 Introduction

In this speech we discuss the property of the Morrey space with non-doubling measures.

The doubling condition has been a key condition for the Carderón-Zygmund theory. We come across the geometric observation in dealing with something on the singular integral. For example, when we use what is called $5r$-covering lemma below, we have to five times as large cubes as original cubes.

Lemma 1.1. *Let $\{Q_j\}_{j\in J}$ be a family of the cubes in $\mathbf{R}^d$. Suppose that the diameter of the cube is bounded. That is, we assume that $\sup_{j\in J}\ell(Q_j)<\infty$. Then we can select a subfamily $\{Q_j\}_{j\in J_0}$ such that $\{Q_j\}_{j\in J}$ is disjoint and that $\bigcup_{j\in J} Q_j \subset \bigcup_{j\in J_0} 5Q_j$.*

Let us see how this covering lemma is used as an example.

Theorem 1.2. *Let M be a (non-centered) Hardy-Littlewood maximal operator with respect to the Lebesgue measure $|\cdot|$:*

$$Mf(x)=\sup_{\substack{Q:cube\\ x\in Q}}\frac{1}{|Q|}\int_Q |f(y)|\,dy.$$

Then we have $\left|\{x\in\mathbf{R}^d\,:\,Mf(x)>\lambda\}\right|\le\frac{5^d}{\lambda}\int_{\mathbf{R}^d}|f(y)|\,dy.$

Proof. Let us look over the proof briefly. For the purpose of applying the lemma above, we prove the theorem for M^R instead of M, where we put M^R by the formula

$$M^Rf(x)=\sup_{\substack{Q:cube\\ x\in Q,\ell(Q)\le R}}\frac{1}{|Q|}\int_Q |f(y)|\,dy.$$

Thus what is going to be proved is reduced to showing that

$$\left|\{x \in \mathbf{R}^d : M^R f(x) > \lambda\}\right| \leq \frac{5^d}{\lambda} \int_{\mathbf{R}^d} |f(y)|\, dy$$

with the constant independent on R. If we obtain this estimate, letting $R \to \infty$, we will have the desired formula by the monotone convergence theorem. Put $E = E_\lambda^R$ by

$$E := E_\lambda^R := \{x \in \mathbf{R}^d : M^R f(x) > \lambda\}.$$

Then by the definition of E for all $x \in E$ there exists Q_x such that $\dfrac{1}{|Q_x|} \displaystyle\int_{Q_x} |f(y)|\, dy > \lambda$, $\ell(Q_x) < R$ and $x \in Q$. The authors have to apologize that they have used $5r$-covering lemma in the actual talk without verifying the assumption $\sup\limits_{x \in E} \ell(Q_x) < \infty$. Now we are restricting the sidelength of the cube less than R we are in the position of using $5r$-covering lemma. By applying $5r$-covering lemma we can find a subset $E_0 \subset E$ such that $\{Q_x\}_{x \in E_0}$ is disjoint and that $\bigcup\limits_{x \in E} Q_x \subset \bigcup\limits_{x \in E_0} 5Q_x$.

With this covering $\{Q_x\}_{x \in E_0}$, the measure of the set E can be estimated as follows.

$$|E| \leq \left|\bigcup_{x \in E} Q_x\right| \leq \left|\bigcup_{x \in E_0} 5Q_x\right| \leq \sum_{x \in E_0} |5Q_x|$$

Since we are considering the Lebesgue measure $|\cdot|$, we have $|5Q_x| = 5^d |Q_x|$. From this identity it follows that

$$|E| \leq 5^d \sum_{x \in E_0} |Q_x| \leq \frac{5^d}{\lambda} \sum_{x \in E_0} \int_{Q_x} |f(y)|\, dy \leq \frac{5^d}{\lambda} \int_{\mathbf{R}^d} |f(y)|\, dy.$$

This is the desired. □

In the proof we used the dilation property $|kQ| = k^d |Q|$. Let μ be a Radon measure and let us consider the corresponding maximal operator:

$$M' f(x) = \sup_{\substack{Q: cube \\ x \in Q}} \frac{1}{\mu(Q)} \int_Q |f(y)|\, d\mu(y)$$

What happens if μ is not the doubling? That is, if the estimate $\mu(5Q) \leq \mu(Q)$ does not hold, do we still have the weak-$(1,1)$ boundedness of $\tilde{M}$? The answer is No. If μ violates the condition $\mu(5Q) \leq C\mu(Q)$, we cannot apply the proof above. In fact there exists a Radon measure μ such that M' is not weak-$(1,1)$ bounded:

$$\sup_{\lambda > 0} \lambda \mu\{x \in \mathbf{R}^d : M' f(x) > \lambda\} = \infty$$

for some $f \in L^1(\mu)$. For this example we refer [13].

We have seen that in the proof of the weak-$(1,1)$ boundedness it is essential that we pose μ the doubling condition $\mu(5Q) \leq C\mu(Q)$ for all cubes Q centered at the support of μ. Thus it has been believed impossible to develop Carderón-Zygmund theory with non-doubling measures. Recently Nazarov, Treil and Volberg showed how to overcome this difficulty: It suffices to enlarge the denominator. They defined a modified maximal operator $\tilde{M}$.

$$\tilde{M} f(x) = \sup_{\substack{Q: cube \\ x \in Q}} \frac{1}{\mu(5Q)} \int_Q |f(y)|\, d\mu(y).$$

By using the estimate $\mu(5Q_x) \le \frac{1}{\lambda}\int_{Q_x}|f(x)|\,d\mu(x)$ instead of $|5Q_x| \le \frac{5^d}{\lambda}\int_{Q_x}|f(x)|\,dx$, we have the desired conclusion. The output we will obtain is

$$\mu\{x \in \mathbf{R}^d : \tilde{M}f(x) > \lambda\} \le \frac{1}{\lambda}\int_{\mathbf{R}^d}|f(x)|\,d\mu(x).$$

Finally let us note that interpolating the results with a trivial inequality $\|\tilde{M}f : L^\infty(\mu)\| \le \|f : L^\infty(\mu)\|$, we obtain $\|\tilde{M}f : L^p(\mu)\| \le C_p\|f : L^p(\mu)\|$ for all $1 < p \le \infty$ as a corollary of this result.

$\tilde{M}$ can be considered in the metric measure space by the analogous definition with cubes replaced by balls. Since $5r$-covering lemma holds true for any general metric space (X, d), we can consider the modified maximal operator on the metric space and the same conclusion.

But why do we have to eliminate the doubling assumption at all? There are non-doubling measures in various contexts.

Example 1.3. The following example is very similar to that in the article of Verdera [45]. Let $\mu = dx + dl$, where dx is a Lebesgue measure in $\mathbf{R}^2$ and dl is a 1-dimensional Hausdorff measure of $\{0\} \times \mathbf{R}$. Then μ is not a doubling measure. Thus the sum of the doubling measure is not always doubling.

The weighted measures can be non-doubling as the following example shows.

Example 1.4. Let dx_1dx_2 be a Lebesgue measure in $\mathbf{R}^2$. Then the weighted measure $\mu = e^{{x_1}^2+{x_2}^2}dx_1dx_2$ is not a doubling measure.

A Riemannian manifold is a typical example of the metric measure space. But when the curvature is strictly negative, the Riemannian measure is not doubling.

Example 1.5. Suppose that M is a unit disk in $\mathbf{R}^2$. Let g be a Riemannian metric defined as $g = \dfrac{4}{1-{x_1}^2-{x_2}^2}(dx_1 \otimes dx_1 + dx_2 \otimes dx_2)$. Then we have the Riemannian measure is not doubling.

In this way the non-doubling measure arises. The maximal theory which we have just seen goes very well with the aid of $5r$-covering lemma. For the Carderón-Zygmund theory with non-doubling measures, we need to introduce the assumption called the growth condition $\mu(Q) \le c_0\ell(Q)^n$. Here, c_0 and n are fixed positive constants with $0 < n \le d$.

The condition $\mu(Q) \le c_0\ell(Q)^n$ appears, for example, in the following well-known example.

Proposition 1.6. *Let μ be a measure with its support K. If the measure μ satisfies the growth condition $\mu(B(x,r)) \le c_0r^n$ with $0 < n \le d$, then the Hausdorff dimension of the set K is more than or equal to n.*

Recent researches have been showing that the doubling condition is not indispensable for the Cardeón-Zygmund theory. Nazarov, Treil and Volberg developed the theory of the singular integrals for non-doubling measures [20], [21]. Stemming from their pioneer work and X.Tolsa's Carderón-Zygmund theory, the research of this field has been developing in many ways. Originally they considered the measure with growth condition to investigate the analytic capacity on the complex plane. X. Tolsa has shown that the analytic capacity is subadditive [40] and that it is bi-Lipschitz invariant [41]. The subadditivity of the analytic capacity has been left open for a long time. Now X. Tolsa has proved it in the harmonic analysis method. The growth condition appears, for example, the following definition.

Definition 1.7. Identifying $\mathbf{R}^2$ with $\mathbf{C}$, we can consider the following maximal operator with the measure μ with $\mu(Q) \leq \ell(Q)$: $M_\mu(x) = \sup_{r>0} \frac{\mu(B(x,r))}{r}$, where $B(x,r)$ denotes a ball with center x and radius $r > 0$.

Recently the measure with growth condition has been shed light on from the other point of view because we begin to notice that the Carderón-Zygmund theory can be recovered without doubling assumption. García-Cuerva and Eduardo Gatto defined a potential operator [7]. X. Tolsa defined RBMO space and its dual $H^1(\mu)$ and the Littlewood-Paley decomposition operator for the growth measure [36], [38]. He also gave the characterization to his $H^1(\mu)$ space in terms of the grand maximal operator [37]. Chen and Sawyer have generalized the definition of RBMO to investigate the commutator of the potential operator and the RBMO function. Yang, Han and Deng have defined the Besov space and the Triebel-Lizorkin space [3], [4]. They also considered the multilinear operator [11], [12]. The authors also defined a Morrey space for non-doubling measures [27].

The first part of this report will be devoted to the survey of the theory of Morrey spaces with the underlying measure μ satisfying the growth condition.

2 Morrey spaces with non-doubling measure

In this section we will define a strong type Morrey space. We will define its norm. For $1 \leq q \leq p < \infty$ the (classical) Morrey spaces are defined as

$$\mathcal{M}_q^p(\mathbf{R}^d) := \{f \in L^q_{loc}(\mathbf{R}^d) : \|f \mid \mathcal{M}_q^p(\mathbf{R}^d)\| < \infty\},$$

where the norm $\|f \mid \mathcal{M}_q^p(\mathbf{R}^d)\|$ is given by

$$\|f \mid \mathcal{M}_q^p(\mathbf{R}^d)\| := \sup_{x \in \mathbf{R}^d, l>0} |B(x,l)|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{B(x,l)} |f|^q \, dy \right)^{\frac{1}{q}}.$$

The Morrey spaces can describe local regularity more precisely than the Lebesgue spaces $L^p(\mathbf{R}^d)$ (c.f. [10]).

Definition 2.1. Let $1 \leq q \leq p < \infty$. We define $\mathcal{M}_q^p(k,\mu)$ by a set of μ-measurable functions with the following norm finite:

$$\|f : \mathcal{M}_q^p(k,\mu)\| := \sup_{Q \in \mathcal{Q}(\mu)} \mu(kQ)^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q |f|^q \, d\mu \right)^{\frac{1}{q}}. \tag{1}$$

It is easy to see that $\mathcal{M}_q^p(k,\mu)$ is a Banach space with its norm and, if μ is doubling, then the space $\mathcal{M}_q^p(k,\mu)$ coincides the classcal Morrey space.

We remark two properties that can be seen from the definition.

Proposition 2.2. *Let $k_1, k_2 > 1$. Then we have $\mathcal{M}_q^p(k_1,\mu) \approx \mathcal{M}_q^p(k_2,\mu)$ in the sense of the equivalent norms.*

In what follows we will make a full use of this fact. For simplicity of the notation, we sometimes write $\mathcal{M}_q^p(\mu) = \mathcal{M}_q^p(2,\mu)$.

The proof is obtained by geometric observation. For the proof we refer to [27].

Proposition 2.3. *The following inclusion holds for all $1 \le q_1 \le q_2 \le p < \infty$:*

$$L^p(\mu) = \mathcal{M}^p_p(k,\mu) \subset \mathcal{M}^p_{q_2}(k,\mu) \subset \mathcal{M}^p_{q_1}(k,\mu).$$

The proof is easy by the definition of the norms and Hölder's inequality. This proposition will be recalled later when we discuss the sharp maximal inequality.

Counter example

Before proceeding further , let us see what happens if we define the Morrey norm $\mathcal{M}^p_q(1,\mu)$.

We will construct a counter example showing $\mathcal{M}^2_1(1,\mu)$ is not isomorphic to $\mathcal{M}^2_1(2,\mu)$.

Let $d=2$ and $\mathcal{H}^s$ be the s-dimensional Hausdorff measure. We denote $\mathcal{H}^s|A$ as a restriction of $\mathcal{H}^s$ to A. For $k \in \mathbf{N}$ set $S_k := \{(x,y) : \max(|x|,|y|) = 2^{-k+1}\}$, $D_k := \{(x,y) : \max(|x|,|y|) \le 2^{-k+1}\}$ and $A_k := \{(x,y) : 2^{-k} \le \max(|x|,|y|) \le 2^{-k+1}\}$.

Example 2.4. Set $\mu := \sum_{k=1}^{\infty} \frac{4^k}{(2k)!^2}\mathcal{H}^2|A_k + \sum_{k=1}^{\infty} \frac{2^k}{(2k-1)!^2}\mathcal{H}^1|S_k$.

To see that this measure μ gives a counterexample, we need the following lemma.

Lemma 2.5. *Let $Q, R \in \mathcal{Q}(dx)$ such that $\partial Q \cap R \ne \emptyset$. For such Q, R we set*

$$\alpha(Q,R) := \mathcal{H}^1(\partial Q \cap R),\ \beta(Q,R) := \mathcal{H}^2(2R \setminus Q).$$

Then

$$\alpha(Q,R) \le 8\sqrt{\beta(Q,R)}.$$

Proof. Devide equally $2R$ into 16 squares and call them $R_1, R_2, \ldots, R_{16}$. Then by assumption R_j does not meet Q for some $j = 1,2,\ldots,16$. Thus

$$\alpha(Q,R) \le 4\ell(R) = 8\ell(R_j) = 8\sqrt{\mathcal{H}^2(R_j)} \le 8\sqrt{\mathcal{H}^2(2R\setminus Q)} = 8\sqrt{\beta(Q,R)}.$$

□

Proposition 2.6. *Let μ be in Example 2.4. Then $\mathcal{M}^2_1(1,\mu)$ is not isomorphic to $\mathcal{M}^2_1(2,\mu)$.*

Proof. Let $f_k = \chi_{S_k}$ and $k \in \mathbf{N}$ be large enough. Then

$$\|f_k : \mathcal{M}^2_1(1,\mu)\| \ge \sup_{Q\in\mathcal{Q}(\mu)} \mu(Q)^{-\frac{1}{2}}\mu(S_k \cap Q) \ge \mu(D_k)^{-\frac{1}{2}}\mu(S_k) \ge c_0\mu(S_k)^{\frac{1}{2}}.$$

Here we have used $\mu(D_k) \le 2\mu(S_k)$ for large $k \in \mathbf{N}$.

Now let us estimate $\|f : \mathcal{M}^2_1(2,\mu)\|$. By the definition of norm we have

$$\|f : \mathcal{M}^2_1(2,\mu)\| := \sup_{Q\in\mathcal{Q}(\mu)} \mu(2Q)^{-\frac{1}{2}}\mu(Q \cap S_k).$$

Let $Q \in \mathcal{Q}(\mu)$ be such that Q meets S_k.

Set $\alpha := \mathcal{H}^1(Q \cap S_k)$. Then we have $f(\alpha) := \mathcal{H}^2(2Q \setminus A_k)$. By Lemma 2.5 we have $\alpha \le c_0\sqrt{f(\alpha)}$. Then

$$\mu(B \cap S_k) = \frac{\alpha}{(2k-1)!^2},\ \mu(2Q) \ge \sqrt{f(\alpha)}(2k-2)!^2.$$

Using this obsevation, we have

$$\mu(2Q)^{-\frac{1}{2}}\mu(S_k \cap B) \le \frac{\alpha}{\sqrt{f(\alpha)}} \times \frac{(2k-2)!}{(2k-1)!^2} = \frac{1}{(2k-1)!(2k-1)}\frac{\alpha}{\sqrt{f(\alpha)}} \le c_2\frac{1}{(2k-1)!(2k-1)}.$$

Hence we have

$$\|f_k : \mathcal{M}^2_1(2,\mu)\| = \sup_B \mu(2B)^{-\frac{1}{2}}\mu(S_k \cap B) \le c_3\frac{(2k-2)!}{(2k-1)!^2} = c_3\frac{1}{(2k-1)\cdot(2k-1)!}$$

and

$$\|f_k : \mathcal{M}^2_1(1,\mu)\| \ge c_0\frac{1}{(2k-1)!}.$$

Thus the isomorphism $\mathcal{M}^2_1(2,\mu) \sim \mathcal{M}^2_1(1,\mu)$ does not hold. $\square$

The next proposition shows how Proposition 2.2 can be used. The proof is a typical example which needs the geometric observation.

Theorem 2.7. *Suppose that* $1 < q \le p < \infty$. $\tilde{M}$ *is bounded from* $\mathcal{M}^p_q(\mu)$ *to itself.*

Proof. Firstly let us verify what to prove. For the proof we fix a cube Q and estimate

$$\mu(300)^{\frac{1}{p}-\frac{1}{q}}\left(\int_Q \tilde{M}f(x)^q\,\mu(x)\right)^{\frac{1}{q}}.$$

We are going to obtain $\mu(300)^{\frac{1}{p}-\frac{1}{q}}\left(\int_Q \tilde{M}f(x)^q\,\mu(x)\right)^{\frac{1}{q}} \le C\,\|f : \mathcal{M}^p_q(2,\mu)\|$.

Decompose f according to $50Q$. Set $f_1 = f\chi_{50Q}$ and $f_2 = f - f_1$. By triangle inequality we have only to estimate

$$\mu(300)^{\frac{1}{p}-\frac{1}{q}}\left(\int_Q \tilde{M}f_1(x)^q\,\mu(x)\right)^{\frac{1}{q}} \text{ and } \mu(300)^{\frac{1}{p}-\frac{1}{q}}\left(\int_Q \tilde{M}f_2(x)^q\,\mu(x)\right)^{\frac{1}{q}}$$

respectively.

For the estimate of the first term we use the result on L^p space. We will have

$$\begin{aligned}
&\mu(300)^{\frac{1}{p}-\frac{1}{q}}\left(\int_Q \tilde{M}f_1(x)^q\,\mu(x)\right)^{\frac{1}{q}}\\
&\le \mu(300)^{\frac{1}{p}-\frac{1}{q}}\left(\int_{\mathbf{R}^d} \tilde{M}f_1(x)^q\,\mu(x)\right)^{\frac{1}{q}}\\
&\le C\mu(300)^{\frac{1}{p}-\frac{1}{q}}\left(\int_{\mathbf{R}^d} |f_1(x)|^q\,\mu(x)\right)^{\frac{1}{q}}\\
&= C\mu(300)^{\frac{1}{p}-\frac{1}{q}}\left(\int_{50Q} |f_1(x)|^q\,\mu(x)\right)^{\frac{1}{q}}.
\end{aligned}$$

The last term can be bounded from above by $\|f : \mathcal{M}^p_q(2,\mu)\|$. So that the estimate of the first term is finished.

The second term requires a geometric observation. We can obtain a pointwise estimate. Let $y \in Q$. Then we have, writing down explicitly

$$\tilde{M} f_2(y) = \sup_{y \in Q} \frac{1}{\mu(5R)} \int_{R \setminus 50Q} |f(z)| \, d\mu(z).$$

In order that the integral is not 0 it is necessary that $R \cup (\mathbf{R}^d \setminus 50Q) \neq \emptyset$. If we assume that $y \in R$, it means the sidelength of R is "very large". More precisely we may limit ourselves to the cubes with $y \in R$ and with $\ell(R) \geq 20\ell(Q)$, for example, which implies that R engulfs $2Q$. Thus we have

$$\tilde{M} f_2(y) \leq \sup_{R:2Q \subset R} \frac{1}{\mu(5R)} \int_R |f(z)| \, d\mu(z).$$

Inserting the above estimate, we obtain

$$\mu(300Q)^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q \tilde{M} f_2(x)^q \, \mu(x) \right)^{\frac{1}{q}} \leq \mu(300Q)^{\frac{1}{p}-\frac{1}{q}} \mu(Q)^{\frac{1}{q}} \sup_{R:2Q \subset R} \frac{1}{\mu(5R)} \int_R |f(z)| \, d\mu(z).$$

Recall that $q \leq p$ so that the last term is less than or equal to

$$\mu(Q)^{\frac{1}{p}-\frac{1}{q}} \mu(Q)^{\frac{1}{q}} \sup_{R:2Q \subset R} \frac{1}{\mu(5R)} \int_R |f(z)| \, d\mu(z) \leq \sup_{R:2Q \subset R} \mu(5R)^{\frac{1}{p}} \frac{1}{\mu(5R)} \int_R |f(z)| \, d\mu(z).$$

This term is also bounded by $\|f : \mathcal{M}^p_q(5,\mu)\|$, hence, by $\|f : \mathcal{M}^p_q(2,\mu)\|$. □

We will summarize the result on the maximal operators. In proving the maximal inequalities we do not have to pose the growth condition on μ. For $\kappa > 1$ and $f \in L^1_{loc}(\mu)$ we use the following modified maximal operator:

$$M_\kappa f(x) := \sup_{x \in Q \in \mathcal{Q}(\mu)} \frac{1}{\mu(\kappa Q)} \int_Q |f| \, d\mu.$$

By our new notation it follows that $\tilde{M} = M_5$.

Theorem 2.8. *For all $k > 1$ there exists an integer $N = N_k$, depending only on the dimension and k, that satisfies the following condition:*

Let $\{B(x_\lambda, r_\lambda)\}_{\lambda \in \Lambda}$ be a family of balls in Euclidean space. Suppose that $\sup_{\lambda \in L} r_\lambda < \infty$.

Then we can take disjoint subfamilies

$$\{B(x_\rho, r_\rho)\}_{\rho \in L_1}, \{B(x_\rho, r_\rho)\}_{\rho \in L_2}, \ldots, \{B(x_\rho, r_\rho)\}_{\rho \in L_N}$$

such that $\displaystyle\bigcup_{\lambda \in L} B(x_\lambda, r_\lambda) \subset \bigcup_{j=1,\ldots,N} \bigcup_{\rho \in L_j} B(x_\rho, kr_\rho)$.

We use the next results of this operator in our theory. By using Theorem 2.8, which is sharper than $5r$-covering lemma for our purpose, we have the following result.

Proposition 2.9 ([24], [36]). *If $\kappa > 1$ and $1 < p \leq \infty$, then we have*

$$\|M_\kappa f : L^p(\mu)\| \leq C_{d,p,\kappa} \|f : L^p(\mu)\|.$$

We also have the inequality of Fefferman-Stein type. This type of inequality is useful when we consider the Triebel-Lizorkin space with non-doubling measure [4].

Proposition 2.10 ([24]). *If $\kappa > 1$, $1 < p < \infty$ and $1 < q \le \infty$, then we have the vector-valued maximal inequality :*

$$\left\| \left(\sum_{j \in \mathbf{N}} (M_\kappa f_j)^q \right)^{1/q} : L^p(\mu) \right\| \le C_{d,p,q,\kappa} \left\| \left(\sum_{j \in \mathbf{N}} |f_j|^q \right)^{1/q} : L^p(\mu) \right\|.$$

The modified maximal operator M_κ is weak-$(1,1)$ bounded on our Morrey space.

Theorem 2.11 ([27]). *If $k, \kappa > 1$ and $1 < q \le p < \infty$, then we have*

$$\|M_\kappa f : \mathcal{M}^p_q(k,\mu)\| \le C_{d,p,q,\kappa,k} \|f : \mathcal{M}^p_q(k,\mu)\|.$$

The corresponding vector-valued inequality is also obtained.

Theorem 2.12 ([27]). *If $k, \kappa > 1$, $1 < q \le p < \infty$ and $1 < r \le \infty$, then we have*

$$\left\| \|M_\kappa f_j : l^r\| : \mathcal{M}^p_q(k,\mu) \right\| \le C_{d,p,q,r,\kappa,k} \left\| \|f_j : l^r\| : \mathcal{M}^p_q(k,\mu) \right\|.$$

The next maximal operator is called the fractional maximal operator. To control the fractional integral operator I_α appearing in the next section, we use this maximal operator.

Definition 2.13. For $0 < \alpha < n$. We set

$$M^\alpha_\kappa f(x) := \sup_{x \in Q \in \mathcal{Q}(\mu)} \frac{1}{\mu(\kappa Q)^{1-\frac{\alpha}{n}}} \int_Q |f(y)| \, d\mu(y).$$

The fractional maximal operator $M^a_\kappa l$ is also bounded from $\mathcal{M}^p_q(\mu)$.

Theorem 2.14 ([27]). *Let $1 < q \le p < \infty$, $1 < r \le \infty$, $1 < p < 1/\alpha$ and $1/s = 1/p - \alpha$. Assume further that $1 < t \le s < \infty$ and $s/t = p/q$. Then we have*

$$\| \|M^\alpha_\kappa f_j : l^r\| : \mathcal{M}^s_t(\mu) \| \le C \left\| \|f_j : l^r\| : \mathcal{M}^p_q(\mu) \right\|.$$

3 Weak-type Morrey space

In this section we define a weak-type function space. Weak-type space is often used to describe the limit case of the strong-type space.

Definition 3.1. Let $k > 1$. Then we have

$$\|f : \mathcal{M}^p_q(k,\mu)\|_w := \sup_{\lambda > 0} \sup_{\substack{Q:cube \\ \mu(Q)>0}} \mu(kQ)^{\frac{1}{p}-\frac{1}{q}} \left(\lambda^q \mu\{x \in Q : |f(x)| > \lambda\} \right)^{\frac{1}{q}}.$$

Let w-$\mathcal{M}^p_q(k,\mu)$ be a totality of μ-measurable functions with $\|f : \mathcal{M}^p_q(k,\mu)\|_w < \infty$.

The following proposition holds, whose proof is obtained in the same manner as that of the strong-type space.

Proposition 3.2. *Let $k_1, k_2 > 1$. Then we have*

$$\|f : \mathcal{M}^p_q(k_1,\mu)\|_w \sim \|f : \mathcal{M}^p_q(k_2,\mu)\|_w$$

in the sense of the equivalent norms.

Thus in view of this proposition we omit the parameter $k > 1$ again and we will denote w-$\mathcal{M}^p_q(\mu)$ =w-$\mathcal{M}^p_q(2,\mu)$.

The maximal operator is bounded from $\mathcal{M}^p_1(\mu)$ to w-$\mathcal{M}^p_1(\mu)$.

Theorem 3.3. *Suppose that $p \geq 1$. Then we have $\tilde{M}$ is bounded from $\mathcal{M}^p_1(\mu)$ to* w-$\mathcal{M}^p_1(\mu)$ *to itself.*

Proof. The proof is similar to that of Theorem 2.11 and we omit the proof. □

4 Boundedness of the linear operators and their vector-valued extension.

In this section we consider two linear operators, the singular integral operator and fractional integral operators.

4.1 Singular integral operator

Definition 4.1. ([21] p466) The singular integral operator T is a bounded linear operator on $L^2(\mu)$ with a kernel function K that satisfies the following three properties :

(1) For some appropriate constant $C > 0$, we have

$$|K(x,y)| \leq \frac{C}{|x-y|^n} \text{ for all } x \neq y, \tag{2}$$

where n is a constant in the growth condition $\mu(B(x,r)) \leq c_0 r^n$ for all $x \in \operatorname{supp}(\mu)$.

(2) There exist constants $\varepsilon > 0$ and $C > 0$ such that

$$|K(x,y) - K(z,y)| + |K(y,x) - K(y,z)| \leq C\frac{|x-z|^\varepsilon}{|x-y|^{n+\varepsilon}} \text{ if } |x-y| > 2|x-z|. \tag{3}$$

(3) If f is a bounded measurable function with a compact support, then we have

$$Tf(x) = \int_{\mathbf{R}^d} K(x,y)f(y)\,d\mu(y) \text{ for a.e. } x \notin \operatorname{supp}(f). \tag{4}$$

Nazarov, Treil and Volberg showed the boundedness of the singular integral operator on $L^p(\mu)$ space.

Theorem 4.2. *Let $1 < p < \infty$ and T be a singular integral operator. Then we have T can be extended to a $L^p(\mu)$-bounded operator. T can be also extended to a bounded operator from $L^1(\mu)$ to w-$L^1(\mu)$.*

Our first work is to extend the domain of T.

Definition 4.3. For $f \in \mathcal{M}_q^p(\mu)$, we define

$$Tf(x) = \lim_{m \to \infty} \left(Tf_m(x) + \int_{\{|y| \geq 2m\}} K(x,y) f(y)\, d\mu(y) \right),$$

where $f_m(x) = f(x)$ if $|x| < 2m$ and $f(x) = 0$ otherwise.

The following lemma shows that the integral above converges absolutely.

Lemma 4.4. *Let $1 \leq q \leq p < \infty$. For all $f \in \mathcal{M}_q^p(\mu)$ and $x \in \mathbf{R}^d$ with $|x| < m$, we have*

$$\int_{\{|y| \geq 2m\}} |K(x,y) f(y)|\, d\mu(y) \leq C\, m^{-n/p} \| f : \mathcal{M}_q^p(\mu) \|.$$

Proof. In [27] we have proved the following lemma with $q > 1$. But the same proof holds with $q = 1$. The straightforward calculation using (2) yields this lemma. □

Now we show that the singular integral operator is bounded on our Morrey space.

Theorem 4.5. *Let $1 < q \leq p < \infty$. Then the singular integral operator T is a bounded operator from $\mathcal{M}_q^p(\mu)$ to itself.*

The weak-type function space appears in the case when $q = 1$.

Theorem 4.6. *Let $p \geq 1$. T is a bounded linear operator from $\mathcal{M}_1^p(\mu)$ to w-$\mathcal{M}_1^p(\mu)$.*

Theorem 4.5 was proved in [27]. The proof of Theorem 4.6 is proved similarly. For convenience for the readers we prove Theorem 4.6.

Proof. For this purpose we fix a cube Q with positive μ-measure. We will estimate

$$\mu(100Q)^{\frac{1}{p}-1} \left(\lambda \mu\{x \in Q : |Tf(x)| > \lambda\} \right).$$

For this purpose we decompose f according to $10Q$. Let $f_1 = f\chi_{10Q}$ and $f_2 = f - f_1$. Using this decomposition we have to estimate

$$\mu(100Q)^{\frac{1}{p}-1} \left(\lambda \mu\{x \in Q : |Tf_1(x)| > \lambda/2\} \right) \text{ and } \mu(100Q)^{\frac{1}{p}-1} \left(\lambda \mu\{x \in Q : |Tf_2(x)| > \lambda/2\} \right).$$

As we have seen, T is weak-$(1,1)$ bounded from L^1 to w-L^1, the estimate of the set near the cube is over:

$$\mu(100Q)^{\frac{1}{p}-1} \left(\lambda \mu\{x \in Q : |Tf_1(x)| > \lambda/2\} \right) \leq \mu(100Q)^{\frac{1}{p}-1} \int_{10Q} |f(y)|\, d\mu(y).$$

As for the estimate of Tf_2 we have for all $x \in Q$

$$|Tf_2(x)| \leq C \int_{\mathbf{R}^d \setminus B(z_Q, \ell(Q))} \frac{1}{|y - z_Q|^n} |f(y)|\, d\mu(y).$$

Note that $\int_0^\infty \frac{\chi_{B(z_Q,l)}(y)}{l^{n+1}}\,dl = c\,|y-z_Q|^{-n}$. Hence we have

$$\begin{aligned}
&\int_{\mathbf{R}^d\setminus B(z_Q,\ell(Q))} \frac{1}{|y-z_Q|^n}|f(y)|\,d\mu(y) \\
&= c\int_0^\infty \left(\int_{B(z_Q,l)\setminus B(z_Q,\ell(Q))} \frac{1}{l^{n+1}}|f(y)|\,d\mu(y)\right)dl \\
&\le c\,\ell(Q)^{-\frac{n}{p}}\|f : \mathcal{M}_1^p(\mu)\|.
\end{aligned}$$

Thus, assuming that $\{x\in Q : |Tf_1(x)| > \lambda/2\} \neq \emptyset$, we have $\lambda\ell(Q)^{\frac{n}{p}} \le C\|f : \mathcal{M}_1^p(\mu)\|$.

Using this estimate, we obtain

$$\mu(100Q)^{\frac{1}{p}-1}\left(\lambda\mu\{x\in Q : |Tf_2(x)| > \lambda/2\}\right) \le \mu(Q)^{\frac{1}{p}}\lambda \le C\|f : \mathcal{M}_1^p(\mu)\|.$$

So we are done. □

4.2 Fractional integral operator

Fractional integral operator was introduced by D. Adams. Fractional integral operator for the Lebesgue measure is of the form

$$I_\alpha f(x) = \int_{\mathbf{R}^d} \frac{f(y)}{|x-y|^{d-\alpha}}\,d\mu(y).$$

Note that for $0<\alpha<d$ the fractional integral operator I_α is an inverse of Laplacian $\Delta^{\alpha/2}$. If the measure μ is a growth measure, García-Cuerva and Eduardo Gatto defined a fractional integral operator for μ.

Definition 4.7 ([7]). For α with $0<\alpha<n$, we define a fractional integral operator as

$$I_\alpha f(x) := \int_{\mathbf{R}^d} \frac{f(y)}{|x-y|^{n-\alpha}}\,d\mu(y),$$

where n is a constant in the growth condition of μ.

The following result is known due to Garcia and Eduardo [7].

Proposition 4.8 ([7]). *Let $1<p<n/\alpha$ and $1/s = 1/p-\alpha/n$. Then I_α is bounded from $L^p(\mu)$ to $L^s(\mu)$.*

In this section we shall extend this result to the Morrey spaces $\mathcal{M}_q^p(\mu)$. As is the case with the classical one ([2, Theorem 2]), I_α is bounded operator on Morrey spaces. More precisely we have

Theorem 4.9 ([27]). *Suppose that the parameters satisfy*

$$1<q\le p<\infty,\ 1<t\le s<\infty,\ t/s = q/p,\ 1/s = 1/p-\alpha/n.$$

Then we have I_α is bounded from $\mathcal{M}_q^p(\mu)$ to $\mathcal{M}_t^s(\mu)$:

$$\|I_\alpha f : \mathcal{M}_t^s(k,\mu)\| \le C_{p,q,s,t,\alpha,k}\|f : \mathcal{M}_q^p(k,\mu)\|, \quad k>1.$$

We can readily extend this result to l^q valued version.

4.3 Commutators and BMO

BMO space plays a substitute role in L^∞ in the classical space. X. Tolsa, as is remarked in Introduction, defined the RBMO function space to develop Carderón-Zygmund theory. Many authors defined a function space BMO. Nazarov, Treil and Volberg defined in [21] their BMO space and obtained their $T(b)$-theorem. But their function space depends on the parameter p, while the John-Nirenberg lemma says the parameter p does not affect the definition of BMO space. In [18] Mateu, Mattila, Nicolau and Orobitg considered BMO for nondoubling measures, assuming $\mu(H) = 0$ for any hyperplane of the form $H = \{(x_1, x_2, \ldots, x_d) : x_i = a\}$, where $i = 1, 2, \ldots, d$ and $a \in \mathbf{R}$. But in their space the interpolation property does not hold. Chen and Sawyer modified the definition of RBMO defined by Tolsa to consider the commutator with RBMO functions and the potential operator [32]. Returning to the function space RBMO, we do not have the similar property to

$$H(L^\infty(\mathbf{R}^d)) + L^\infty(\mathbf{R}^d) = BMO(\mathbf{R}^d).$$

Here H is a Hilbert transform. Our future job may be to define a function space BMO to recover the all classical property. But some researchers including the authors think that it is appropriate to define a BMO space suitable for their purpose: RBMO is a nice substitute for the Calderón-Zygmund theory. Now that we are going to develop the Carderón-Zygmund theory, we believe that RBMO is the most suitable function space.

Definition 4.10. Let $Q, R \in \mathcal{Q}(\mu)$. We define

$$N_{Q,R} = \min\left\{j \in \mathbf{N}_0 \,|\, R \subset 2^j Q\right\}. \tag{5}$$

Definition 4.11. We set

$$\mathcal{Q}(\mu, 2) = \{Q \in \mathcal{Q}(\mu) \,|\, Q \text{ is a } (2, 2^{d+1})\text{-doubling cube. }\}. \tag{6}$$

Definition 4.12 ([32], [37]). Let $0 \le \alpha < n$. We put the coefficient $K^{(\alpha)}_{Q,R}$ as

$$K^{(\alpha)}_{Q,R} = 1 + \sum_{j=1}^{N_{Q,R}} \left(\frac{\mu(2^j Q)}{\ell(2^j Q)^n}\right)^{1-\frac{\alpha}{n}}.$$

For the sake of simplicity we put $K_{Q,R} = K^{(0)}_{Q,R}$.

Definition 4.13. Let $Q \in \mathcal{Q}(\mu)$. Let j_0 defined by $j_0 = \min\left\{j \in \mathbf{N}_0 \,|\, 2^j Q \in \mathcal{Q}(\mu, 2)\right\}$. We denote $Q^* = 2^{j_0} Q$. Note that the minimum always exists from the reduction-to-absurdity argument.

Remark 4.14. By growth condition, for any cube Q there is a doubling cube R of the form $R = 2^j Q$. By geometrical measure theory for any $x \in \operatorname{supp}(\mu)$ and $r > 0$ we also have a doubling cube S centered at x and $\operatorname{diam}(S) < r$.

Definition 4.15. We say that a locally integrable function is an element of RBMO if it satisfies

$$\|a\|_* = \sup_{Q \in \mathcal{Q}(\mu)} \frac{1}{\mu\left(\frac{3}{2}Q\right)} \int_Q |a(x) - m_{Q^*}(a)| d\mu(x) + \sup_{\substack{Q \subset R \\ Q, R\, \in \mathcal{Q}(\mu, 2)}} \frac{|m_Q(a) - m_R(a)|}{K_{Q,R}} < \infty,$$

where $m_Q(a)$ is a mean of a over Q.

X. Tolsa showed the following result.

Theorem 4.16 ([38]). *T is a bounded operator from $L^2(\mu) \cap L^\infty(\mu)$ to* RBMO(μ). *More precisely we have $\|Tf\|_* \le C\|f : L^\infty(\mu)\|$ for all $f \in L^2(\mu) \cap L^\infty(\mu)$ with constant C independent on f.*

As for I_α, García-Cuerva and Eduardo Gatto proved the following theorem.

Theorem 4.17. *Let $0 < a < n$. Then I_α can be extended to a bounded operator from $L^q(\mu) \cap L^{n/\alpha}(\mu)$ to $RBMO(\mu)$. More precisely we have $\|Tf\|_* \le C\|f : L^{n/\alpha}(\mu)\|$ for all $f \in L^{n/\alpha}(\mu) \cap L^2(\mu)$ with constant C independent on f.*

•

The following theorem is a supplement for the limiting case. The result is somehow weaker.

Theorem 4.18. *Let $f \in \mathcal{M}_q^{n/\alpha}(\mu) \cap L^r(\mu)$. Then there exists a constants C_1, C_2 such that*

$$\frac{1}{\mu(2Q)} \int_Q |f(y) - m_{Q^*}(f)| \, d\mu(y) \le C_1 \|f\|_{\mathcal{M}_q^{n/\alpha}(\mu)}$$

and that

$$|m_Q(I_\alpha f) - m_R(I_\alpha f)| \le C_2 \, K_{Q,R}^{(\alpha)} \|f\|_{\mathcal{M}_q^{n/\alpha}(\mu)}$$

for all $f \in \mathcal{M}_q^{n/\alpha}(\mu) \cap L^r(\mu)$.

Proof. First we will treat $I := \dfrac{1}{\mu(2Q)} \displaystyle\int_Q |I_\alpha f(x) - m_{Q^*}(I_\alpha f)| \, d\mu(x)$. Decompose $f = f_1 + f_2 + f_3$, where $f_1 = f\chi_{\frac{3}{2}Q}$ and $f_3 = f\chi_{\mathbf{R}^d \setminus \frac{3}{2}Q^*}$. Using this decomposition, we can decompose I as

$$\begin{aligned} I \ &\le \ \frac{1}{\mu(2Q)} \int_Q |I_\alpha f_3(x) - m_{Q^*}(I_\alpha f_3)| \, d\mu(x) + \frac{1}{\mu(2Q)} \int_Q |I_\alpha f_1(x)| \, d\mu(x) \\ &+ \ \frac{1}{\mu(2Q)} \int_Q |I_\alpha f_2(x)| \, d\mu(x) + \frac{1}{\mu(2Q)} \int_Q |m_{Q^*} I_\alpha f_1 + f_2)| \, d\mu(x) =: I_1 + I_2 + I_3 + I_4. \end{aligned}$$

We write down I_1 explicitly and estimate by using the mean-value theorem.

$$\begin{aligned} &I_1 \\ &\le \ \frac{1}{\mu(2Q)\mu(Q^*)} \int_{Q \times Q^*} d\mu(x) d\mu(y) \left| \int_{\mathbf{R}^d \setminus \frac{3}{2}Q^*} \frac{f(z)}{|x - z|^{n-\alpha}} - \frac{f(z)}{|y - z|^{n-\alpha}} \, d\mu(z) \right| \\ &\le \ \frac{C}{\mu(2Q)\mu(Q^*)} \int_Q d\mu(x) \int_{Q^*} d\mu(y) \int_{\mathbf{R}^d \setminus \frac{3}{2}Q^*} \frac{|x - y||f(z)|}{|z_Q - z|^{n-\alpha+1}} \, d\mu(z) \\ &\le \ C\ell(Q^*) \int_{\mathbf{R}^d \setminus \frac{3}{2}Q^*} \left(\int_0^\infty \frac{\chi_{B(z_Q,l)}(z)}{l^{n-\alpha+2}} \, dl \right) |f(z)| \, d\mu(z) \\ &= \ C\ell(Q^*) \int_0^\infty \frac{dl}{l^{n-\alpha+2}} \int_{B(z_Q,l) \setminus \frac{3}{2}Q^*} |f(z)| \, d\mu(z) \\ &\le \ C\ell(Q^*) \int_{\ell(Q^*)}^\infty \left(l^{-2} \, \mu(B(z_Q, 2l))^{\frac{\alpha}{n} - \frac{1}{q}} \left(\int_{B(z_Q,l)} |f(z)|^q \, d\mu(z) \right)^{\frac{1}{q}} \right) dl \\ &= \ C\|f : \mathcal{M}_q^{n/\alpha}(\mu)\|. \end{aligned}$$

The treatment of I_2 is simpler. We may assume that $q < n/\alpha$ because we have a monotonicity in the space M_q^p for parameter q. And fix an auxiliary constant u such that $\dfrac{1}{u} = \dfrac{1}{q} - \dfrac{\alpha}{n}$. Then we have

$$\begin{aligned}
I_2 &\le \left(\frac{1}{\mu(2Q)}\int_Q |I_\alpha f_1(x)|^u \, d\mu(x)\right)^{\frac{1}{u}} \\
&\le C\mu(2Q)^{\frac{\alpha}{n}-\frac{1}{q}}\left(\int_Q |f_1(x)|^q \, d\mu(x)\right)^{\frac{1}{q}} \\
&\le C\|f : \mathcal{M}_q^{n/\alpha}\|.
\end{aligned}$$

The treatment of I_4 is quite similar. It remains to estimate I_3. We proceed as follows:

$$\begin{aligned}
I_3 &\le C\int_{\frac{3}{2}Q^*\setminus\frac{3}{2}Q} \frac{|f(y)|}{|y-z_Q|^{n-\alpha}} \, d\mu(y) \\
&\le C\int_{\frac{3}{2}Q^*\setminus\frac{3}{2}Q} \left(\int_0^\infty \frac{\chi_{B(z_Q,l)}(y)}{l^{n-\alpha+1}}\right)|f(y)| \, d\mu(y) \\
&\le C\int_{\ell(Q)}^\infty \left(\frac{1}{l^{n-\alpha+1}}\int_{\frac{3}{2}Q^*\cap B(z_Q,l)} |f(y)| \, d\mu(y)\right) dl \\
&\le C\int_{\ell(Q)}^{\ell(Q^*)} \frac{dl}{l^{n-\alpha+1}}\int_{B(z_Q,l)} |f(y)| \, d\mu(y) + C\int_{\ell(Q^*)}^\infty \frac{dl}{l^{n-\alpha+1}}\int_{2Q^*} |f(y)| \, d\mu(y) \\
&\le C\int_{\ell(Q)}^{\ell(Q^*)} \frac{dl}{l^{n+1}}\|f\|_{\|f : \mathcal{M}_q^{n/\alpha}(\mu)\|} + C\|f\|_{\mathcal{M}_1^{n/\alpha}} \le CK_{Q,Q^*}\|f : \mathcal{M}_q^{n/\alpha}(\mu)\| \\
&\le C\|f : \mathcal{M}_q^{n/\alpha}(\mu)\|.
\end{aligned}$$

Next we will treat $|m_Q(I_\alpha f) - m_R(I_\alpha f)|$, where $Q \subset R$ and Q, R are doubling cubes. But the estimates are almost the same using the technique used in the previous estimates but the one of

$$\left|m_Q\left(I_\alpha\left(f\chi_{2^{N_{Q,R}}Q\setminus\frac{3}{2}Q}\right)\right)\right| \le C K_{Q,R}^{(\alpha)}\|f\|_{\mathcal{M}_q^{n/\alpha}(\mu)}.$$

So we prove this only. Writing down the left-hand-side explicitly, we have

$$\begin{aligned}
&\left|I_\alpha\left(f\chi_{2^{N_{Q,R}}Q\setminus\frac{3}{2}Q}\right)(x)\right| \\
&\le C\int_{2^{N_{Q,R}}Q\setminus Q} \frac{|f(y)|}{|y-z_Q|^{n-\alpha}} \, d\mu(y) \\
&\le C\sum_{j=0}^{N_{Q,R}} \frac{1}{\ell(2^jQ)^{n-\alpha}}\int_{2^jQ} |f(y)| \, d\mu(y) \le C\sum_{j=0}^{N_{Q,R}} \frac{\mu(2^jQ)^{1-\frac{1}{q}}}{\ell(2^jQ)^{n-\alpha}}\left(\int_{2^jQ} |f(y)|^q \, d\mu(y)\right)^{\frac{1}{q}} \\
&\le C\sum_{j=0}^{N_{Q,R}} \frac{\mu(2^{j+1}Q)^{1-\frac{\alpha}{n}}}{\ell(2^jQ)^{n-\alpha}}\mu(2^{j+1}Q)^{\frac{\alpha}{n}-\frac{1}{q}}\left(\int_{2^jQ} |f(y)|^q \, d\mu(y)\right)^{\frac{1}{q}} \\
&\le C\left(1+\sum_{j=0}^{N_{Q,R}}\left(\frac{\mu(2^jQ)}{\ell(2^jQ)^n}\right)^{1-\frac{\alpha}{n}}\right)\|f : \mathcal{M}_q^{n/\alpha}(\mu)\| = CK_{Q,R}^{(\alpha)}\|f : \mathcal{M}_q^{n/\alpha}\|.
\end{aligned}$$

This is the desired. □

Remark 4.19. The condition $f \in L^r(\mu)$ in the assumption of the theorem is added to avoid the technical modification of I_α. If we modify I_α trivially, we can remove the assumption $f \in L^r(\mu)$. We do not go into the detail.

5 Sharp-maximal inequality and its applications

In this section we consider the sharp-maximal operators.

5.1 Definition of the sharp maximal operator

In this section we state the main results. Before going into details, we recall the definition of RBMO which recovers classical results such as John-Nirenberg's property.

Definition 5.1 ([37]). Let $0 \le \alpha < n$. Then we define a sharp-maximal operator:

$$\begin{aligned} T^\sharp_\alpha f(x) &:= \sup_{x \in Q \in \mathcal{Q}(\mu)} \frac{1}{\mu\left(\frac{3}{2}Q\right)} \int_Q |f(x) - m_{Q^*}(f)| d\mu(x) \\ &+ \sup_{x \in Q \subset R:\ Q,R \in \mathcal{Q}(\mu,2)} \frac{|m_Q(f) - m_R(f)|}{K^{(\alpha)}_{Q,R}}. \end{aligned} \tag{7}$$

For the sake of simplicity we put $T^\sharp := T^\sharp_0$.

We shall distinguish the sharp-maximal operators $T^\sharp$ and $M^\sharp$. To describe sharp maximal inequalities we introduce one more maximal operator.

Definition 5.2 ([37]). Define $Nf(x)$ as

$$Nf(x) = \sup_{x \in Q \in \mathcal{Q}(\mu,2)} \frac{1}{\mu(Q)} \int_Q |f(x)| d\mu(x).$$

Example 5.3. If $\mu = dx$, then we have $\displaystyle\sup_{x \in Q,R} \frac{|m_Q(f) - m_R(f)|}{K_{Q,R}} \le CM^\sharp f(x)$ and if $Q \subset R$ are concentric $K_{Q,R} \le C \log_2 \left(2 + \dfrac{l(R)}{l(Q)}\right)$.

As for a L^p result, Tolsa obtained the following.

Proposition 5.4. [37] *Suppose that $f \in L^p(\mu)$ with $1 < p < \infty$. Then there exists a constant $C > 0$ such that for almost all $x \in \operatorname{supp} \mu$, we have $|f(x)| \le Nf(x)$.*

Assume further that $\int f = 0$ if $\mu(\mathbf{R}^d) < \infty$. Furthermore if $\min(1, Nf) \in L^p(\mu)$, then we have

$$\|Nf \mid L^p(\mu)\| \sim \|T^\sharp f \mid L^p(\mu)\|.$$

Now it is time to state sharp-maximal inequalities.

Theorem 5.5 (Sharp-maximal inequality A). *Suppose that $1 < q \le p < \infty$. For any locally integrable function f we have*

$$\|Nf \mid \mathcal{M}^p_q(\mu)\| \sim \|T^\sharp_\alpha f \mid \mathcal{M}^p_q(\mu)\| + \|f \mid \mathcal{M}^p_1(\mu)\|. \tag{8}$$

We would like to emphasize that this inequality is admissible for any locally integrable functions. Furthermore assumption that $\mu(\mathbf{R}^d) < \infty$ is also unnecessary.

Next we shall obtain the inequality without the second term of the right-hand-side of Theorem 5.5 by assuming even weaker integrability condition. As for this kind of approach, Fujii obtained a result with μ doubling via CZ-decomposition of $M^\sharp f$. We shall prove our results by a good λ-inequality.

Proposition 5.6. [6] *Suppose that μ is a doubling Radon measure. If there is a cube I such that*

$$\lim_{k\to\infty}\frac{1}{\mu(kI)}\int_{kI} f(x)d\mu(x)=0, \tag{9}$$

then we have

$$\|f\,|\,L^p(\mu)\|\sim\|Mf\,|\,L^p(\mu)\|\sim\|M^\sharp f\,|\,L^p(\mu)\|.$$

Here $M^\sharp$ is a usual sharp maximal operator.

$$M^\sharp f(x)=\sup_{x\in Q\in\mathcal{Q}(\mu)}\frac{1}{\mu(Q)}\int_Q|f(x)-m_Q(f)|\,d\mu(x).$$

As a corollary of Theorem 5.5, we obtain another sharp-maximal inequality.

Theorem 5.7 (Sharp-maximal inequality B). *Let $0\le\alpha<n$. Suppose that there are concentric doubling cubes $Q_1,Q_2,\dots,Q_k,\dots\in\mathcal{Q}(\mu,2)$ with $\displaystyle\lim_{k\to\infty}\frac{1}{\mu(Q_k)}\int_{Q_k}f(x)d\mu(x)=0$, and $Q_j\uparrow\mathbf{R}^d$. Then we have*

$$\|f\,|\,\mathcal{M}^p_q(\mu)\|\sim\|Nf\,|\,\mathcal{M}^p_q(\mu)\|\sim\|T^\sharp_\alpha f\,|\,\mathcal{M}^p_q(\mu)\|$$

independently on f.

As a selfimprovement of this theorem, we obtain one more sharp-maximal inequality.

Theorem 5.8 (Sharp-maximal inequality C). *Let $0\le\alpha<n$. If $\mu(\mathbf{R}^d)<\infty$, then for all μ-measurable function f we have the norm equivalence*

$$\|f\,|\,\mathcal{M}^p_q(\mu)\|\sim\|T^\sharp_\alpha f\,|\,\mathcal{M}^p_q(\mu)\|+\|f\,|\,L^1(\mu)\|.$$

5.2 Outline of the proof

In this subsection we will explain the outline of Theorem 5.5.

We prove this lemma by using a good λ-inequality. We state our good λ-inequality for Morrey space. We have denoted $\mathcal{Q}(\mu,2)$ as a totality of doubling cube. (If the measure is non-doubling, a cube Q is said to be doubling if it is $(2,2^{d+1})$-doubling.) For the proof we put

$$\Lambda_Q(f)=\sup_{R\in\mathcal{Q}(Q,bad)}m_R(|f|), \tag{10}$$

where for a cube $Q\in\mathcal{Q}(\mu)$, we have put

$$\mathcal{Q}(Q,bad)=\{R\in\mathcal{Q}(\mu,2)\,|\,R\cap Q\ne\emptyset,\ R\text{ is not contained in }3Q\}.$$

We intend to say that a cube $R\in\mathcal{Q}(2,Q,bad)$ is difficult to control.

Theorem 5.9 ([28]). *Let $\epsilon>0$ and $\eta>0$. There exists sufficiently small $\delta>0$ such that*

$$\mu\{x\in Q\,|\,Nf(x)>(1+\epsilon)\lambda,\ T^\sharp f(x)\le\delta\lambda\}\le\eta\mu\{x\in 3Q\,|\,Nf(x)>\lambda\}$$

for all $\lambda\ge\Lambda_Q(f)$.

If we consider a doubling measure, then the following is a substitute for good-λ inequality for the doubling measure.

Theorem 5.10 ([29]). *Suppose that μ satisfies the doubling condition. For all $\delta > 0$ and for all $\lambda \geq \Lambda_Q(f)$ we have*

$$\mu\{x \in Q \,|\, Mf(x) > 2C_0{}^3\lambda,\ M^\sharp f(x) \leq \delta\lambda\} \leq C\delta\mu\{x \in 3Q \,|\, Mf(x) > \lambda\}, \tag{11}$$

$M^\sharp$ is given by (12) *not by* (7).

$$M^\sharp f(x) = \sup_{x \in Q} \frac{1}{\mu(Q)} \int_Q |f(y) - m_Q(f)| \, d\mu(y). \tag{12}$$

In general we will obtain good-λ inequality for all $\lambda > 0$. Now we obtain our good-λ inequality for $\lambda \geq \Lambda$. Thus the information for $\lambda < \Lambda$ is missing now. To fill the gap in our situation we use the following lemma.

Lemma 5.11. *Under the same assumption in Theorem 5.9, we have*

$$\mu(Q)^{\frac{1}{p}} \Lambda_Q(f) \leq C\|f \,:\, \mathcal{M}^p_q(\mu)\|.$$

Using this lemma, we can estimate

$$\mu(64Q)^{\frac{1}{p}-\frac{1}{q}} \left(\int_0^\Lambda q\lambda^{q-1} \mu\{x \in Q \,:\, Nf(x) > \lambda\} \, d\lambda \right)^{\frac{1}{q}}$$

from above by

$$\mu(64Q)^{\frac{1}{p}-\frac{1}{q}} \mu(Q)^{\frac{1}{q}} \Lambda \leq C\|f \,:\, \mathcal{M}^p_q(\mu)\|.$$

By the technique of the weight we can extend all the results in this section to the vector-valued versions.

Theorem 5.12. *Suppose that $1 \leq q \leq p < \infty$, $1 < r < \infty$, $\kappa > 1$ and $0 \leq \alpha < n$. Let $f_j \in \mathcal{M}^p_q(\mu)$ with $j = 1, 2, \ldots$.*

(1) *Assume that $\mu(\mathbf{R}^d) = \infty$. Then we have*

$$\left\| \left(\sum_{j=1}^\infty N f_j{}^r \right)^{\frac{1}{r}} \,:\, \mathcal{M}^p_q(\mu) \right\| \leq C \left\| \left(\sum_{j=1}^\infty M^{\sharp,\alpha} f_j{}^r \right)^{\frac{1}{r}} \,:\, \mathcal{M}^p_q(\mu) \right\|. \tag{13}$$

(2-a) *Assume that $\mu(\mathbf{R}^d) < \infty$. If $m_{\mathbf{R}^d}(f_j) = 0$ for all $j = 1, 2, \ldots$, then we have* (13).

(2-b) *Assume that $\mu(\mathbf{R}^d) < \infty$. Then we have for all $\{f_j\}_{j=1}^\infty \subset \mathcal{M}^p_q(\mu)$*

$$\begin{aligned} &\left\| \left(\sum_{j=1}^\infty N f_j{}^r \right)^{\frac{1}{r}} \,:\, \mathcal{M}^p_q(\mu) \right\| \\ &\leq C \left\| \left(\sum_{j=1}^\infty M^{\sharp,\alpha} f_j{}^r \right)^{\frac{1}{r}} \,:\, \mathcal{M}^p_q(\mu) \right\| + C \left\{ \sum_{j=1}^\infty \left(\int_{\mathbf{R}^d} |f_j(x)| \, d\mu(x) \right)^r \right\}^{\frac{1}{r}}. \end{aligned} \tag{14}$$

6 Prospect for the future research

6.1 More general measures on $\mathbb{R}^d$

In this report we have assumed that the measure satisfies the growth condition $\mu(B(x,r)) \leq c_0 r^n$. But recently there are many attempts to remove this condition. In fact some of the results involving the maximal operator can be obtained without the growth condition. For details see [14], [15], [16], [18], [24].

6.2 Metric measure space

In $\mathbf{R}^d$ there are good covering lemmas. But generally the metric space (X, d) does not have covering lemmas as good as those in $\mathbf{R}^d$. Our problem is to apply our theory to the metric measure space. For details we refer [21], [24], [35].

6.3 BMO function for non-doubling measure

As is referred in Subsection 4.3, given a Radon measure μ with growth condition, we have to define a nice BMO space. Probably we have to define BMO function space for each problem one is considering.

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