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# Besov Spaces of Self-affine Lattice Tilings and Pointwise regularity

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## 1 Introduction

There are many ways to characterize Besov spaces. Among them in the discrete version are regular wavelet expansion, Littlewood-Paley decomposition, polynomial approximation, spline approximation, mean oscillation, and difference operator (See [9], [13] and [15]). We give these characterizations in context of self-affine lattice tilings of $\mathbb{R}^n$ and we apply to study these pointwise versions. In particular we see to give most of these characterizations in a framework of multiresolution approximation on self-affine lattice tilings of $\mathbb{R}^n$. We also give conditions of finitely many functions which generate the Besov spaces of self-affine lattice tilings of $\mathbb{R}^n$ in a view of multiresolution approximation scheme (cf. [6]). This result is a generalization of characterizations of Besov spaces given by regular wavelet functions and by spline functions.(See [3] , [12] and [15]) Moreover we apply to give descriptions of scaling exponents by characterizations of the Besov space, and we also consider a pointwise Hölder exponent of oscillatory functions given by a multiresolution approximation series in self-affine lattice tilings of $\mathbb{R}^n$.

In the second section we introduce self-affine lattice tilings of $\mathbb{R}^n$ which arise in many contexts, particularly, in fractal geometry and in construction of wavelet bases. See [14] for a survey on related topics. We define Besov spaces of self-affine lattice tilings, and give its characterizations and its pointwise versions.

In the third section we consider a multiresolution analysis $\{V_l\}$ generated by finitely many functions associated with a self-affine lattice tiling. We give properties of Besov space norms defined by approximation errors associated with $\{V_l\}$.

In the fourth section we give some conditions of finitely many functions which characterize the Besov space by multiresolution approximation on self-affine lattice tilings of $\mathbb{R}^n$. We apply this result to give a generalization of characterizations of Besov spaces given by regular wavelet functions and by spline functions, and we also give characterizations of the pointwise Hölder space by multiresolution approximation.

In the fifth section we give descriptions of scaling exponents of global and poitwise regularity by characterizations of the Besov space. We give properties of a pointwise Hölder exponent for a multiresolution approximation series in self-affine lattice tilings and apply to compute a pointwise Hölder exponent of several oscillatory functions.

We use $C$ to denote a positive constant different in each occasion. But it will depend on the parameter appearing in each problem. The same notations $C$ are not necessarily the same on any two occurrences.

## 2 Self-affine lattice tilings and Besov spaces

Let $\Gamma$ be a lattice in $\mathbb{R}^n$, that is, $\Gamma$ is an image of the integer lattice $\mathbb{Z}^n$ under some nonsingular linear transformation and let $M$ be a dilation matrix, that is, all eigenvalues of $M$ have absolute values greater than one and $M$ preserves the lattice $\Gamma$: $M\Gamma \subset \Gamma$. This implies that $|\det M| = m$ is a positive integer greater than one and $m$ is the order of the quotient space $\Gamma/M\Gamma$. We say that a compact set $T$ generates a self-affine tiling $\{T+\gamma\}_{\gamma\in\Gamma}$ if

$$\begin{aligned} \cup_{\gamma\in\Gamma}(T+\gamma) &= \mathbb{R}^n \ \text{ disjoint a.e.} \\ \cup_{\gamma\in\Gamma_0}(T+\gamma) &= MT \ \text{ disjoint a.e.} \end{aligned} \tag{1}$$

where $\Gamma_0$ is a finite subset of $\Gamma$ consisting of representatives for disjoint cosets in $\Gamma/M\Gamma$. The set $\Gamma_0$ is called a set of digits and the compact set $T$ is called a self-affine tile. The self-affine tile $T$ has nonempty interior $T^o$. We suppose that $\Gamma = \mathbb{Z}^n$. In this case the dilation matrix $M$ has integer entries.

For $1 \le p \le \infty$, let $\mathcal{L}^p = \mathcal{L}^p(\mathbb{R}^n)$ be the linear space of all functions $\phi$ for which

$$|\phi|_p = (\int_T (\sum_{\nu\in\mathbb{Z}^n} |\phi(x-\nu)|)^p dx)^{1/p} < \infty. \tag{2}$$

with the usual modification for $p=\infty$. Clearly, $\mathcal{L}^p \subset L^p(\mathbb{R}^n)$ and $\mathcal{L}^\infty \subset \mathcal{L}^p \subset \mathcal{L}^q \subset \mathcal{L}^1 = L^1(\mathbb{R}^n)$ for $1 \le q \le p \le \infty$. If $\phi \in L^p(\mathbb{R}^n)$ $(1 \le p \le \infty)$ is compactly supported, then $\phi \in \mathcal{L}^p$. Furthermore, we observe that if there are constants $C > 0$ and $\delta > 0$ such that $|\phi(x)| \le C(1+|x|)^{-n-\delta}$ for all $x \in \mathbb{R}^n$ then $\phi \in \mathcal{L}^\infty$.

A finite subset $\Phi = \{\phi_1, \dots, \phi_N\}$ of $\mathcal{L}^\infty$ is said to have $L^p$-stable shifts $(1 \le p \le \infty)$, if there are constants $C_1 > 0$ and $C_2 > 0$ such that for any sequences $c_j \in l^p(\mathbb{Z}^n)$ $(j = 1, \dots, N)$,

$$C_1 \sum_{j=1}^N ||c_j||_{l^p} \le ||\sum_{j=1}^N \sum_{\nu\in\mathbb{Z}^n} c_j(\nu)\phi_j(x-\nu)||_p \le C_2 \sum_{j=1}^N ||c_j||_{l^p}.$$

From now those equivalences shall be described as

$$\sum_{j=1}^N ||c_j||_{l^p} \sim ||\sum_{j=1}^N \sum_{\nu\in\mathbb{Z}^n} c_j(\nu)\phi_j(x-\nu)||_p.$$

**Theorem A** ([6]). *For a finite subset $\Phi = \{\phi_1, \dots, \phi_N\}$ of $\mathcal{L}^\infty$, we have following equivalent conditions:*
(i) *$\Phi$ has $L^2$-stable shifts,*
(ii) *$\Phi$ has $L^p$-stable shifts for $1 \le p \le \infty$,*

(iii) *there is a set of functions* $\tilde{\Phi} = \{\tilde{\phi}_1, \dots, \tilde{\phi}_N\}$ *in* $\mathcal{L}^\infty$, *dual to* $\Phi$ *in the sense that*

$$\int \phi_j(x-\mu)\bar{\tilde{\phi}}_k(x-\nu)dx = \delta_{\mu\nu}\delta_{jk}, \quad j,k = 1,\dots,N, \quad \mu,\nu \in \mathbb{Z}^n,$$

*where* $\delta$ *is the Kronecker's symbol.*

Let $\Pi = \{T+\nu\}_{\nu\in\mathbb{Z}^n}$ be a self-affine lattice tiling of $\mathbb{R}^n$ with a dilation matrix $M$. For a nonnegative integer $k$, we denote the function $p_\alpha$ with $|\alpha| \le k, \alpha \in \mathbb{Z}_+^n$, where $\mathbb{Z}_+$ is the set of all nonnegative integers, given by

$$\begin{aligned} p_\alpha(x) &= x^\alpha, \quad x \in T^o \\ p_\alpha(x) &= 0 \quad \text{otherwise.} \end{aligned} \tag{3}$$

Since $\Phi = \{p_\alpha\}_{|\alpha|\le k}$ of $\mathcal{L}^\infty$ has $L^2$-stable shifts, there is a set of functions $\tilde{\Phi} = \{\tilde{p}_\alpha\}_{|\alpha|\le k}$ dual to $\Phi$.

Let $Q_0$ be a translate of the tile $T$ containing the origin as an interior point and let $p'_\alpha$, $\tilde{p}'_\alpha$ be corresponding translates of $p_\alpha$, $\tilde{p}_\alpha$ respectively. For $Q_l(x_0) = M^{-l}Q_0 + x_0$, we write

$$p_\alpha^{Q_l(x_0)}(x) = m^{l/2}p'_\alpha(M^l(x-x_0)), \quad \tilde{p}_\alpha^{Q_l(x_0)}(x) = m^{l/2}\tilde{p}'_\alpha(M^l(x-x_0))$$

$$P_{Q_l(x_0)}f(x) = \sum_{|\alpha|\le k} \langle f, \tilde{p}_\alpha^{Q_l(x_0)}\rangle p_\alpha^{Q_l(x_0)}(x). \tag{4}$$

We define

$$\operatorname{osc}_p^k f(x,l) = \inf_{P\in\mathbb{P}^k}\Big(\frac{1}{|Q_l(x)|}\int_{Q_l(x)} |f(y) - P(y)|^p dy\Big)^{1/p} \tag{5}$$

and

$$\Omega_p^k f(x,l) = \Big(\frac{1}{|Q_l(x)|}\int_{Q_l(x)} |f(y) - P_{Q_l(x)}f(y)|^p dy\Big)^{1/p}$$

where $Q_l(x) = M^{-l}Q_0 + x$ and $P_{Q_l(x)}f$ is given in (4), and $|Q_l(x)|$ is the volume element of $Q_l(x)$, and $\mathbb{P}^k$ is the linear space of all polynomials of degree no greater than $k$ on $\mathbb{R}^n$.

**Definition**. Let $\lambda_0$ be the least value of absolute values of eigenvalues of the dilation matrix $M$. Given $s > 0$, $k$ a nonnegative integer with $k+1 > s$ and $1 \le p,q \le \infty$. A function $f$ is said to belong to the Besov space $B_{pq}^s(M)$ if

$$||f||_{B_{pq}^s(M)} = ||f||_p + \Big(\sum_{l=0}^{\infty}(\lambda_0^{ls}||\operatorname{osc}_p^k f(\cdot,l)||_p)^q\Big)^{1/q} < \infty. \tag{6}$$

with the usual modification for $q = \infty$. We note that the above definition is independent of the choice of nonnegative integers $k$ with $k+1 > s$ and $\operatorname{osc}_p^k$ in the definition can be replaced by $\operatorname{osc}_1^k$. We can see $W_{k+1}^p(\mathbb{R}^n) \subset B_{pq}^s(M)$ if $s < k+1$. When the dilation matrix $M$ is $\lambda_0$-times of the identy $Id$ with $\lambda_0 > 1$, the above Besov space coincides the usual Besov space on $\mathbb{R}^n$.

**Remark 1.** We have the embedding theorem : $B^{\beta}_{p\xi}(M) \subset B^{\alpha}_{p\eta}(M)$ for $\beta > \alpha > 0$, $1 \le \xi, \eta \le \infty$ and $1 \le p \le \infty$, and $B^{\alpha}_{p\xi}(M) \subset B^{\alpha}_{p\eta}(M)$ for $\alpha > 0$, $1 \le \xi \le \eta \le \infty$ and $1 \le p \le \infty$.

Let $\triangle_u f$ denote the difference operator $\triangle_u f(x) = f(x+u) - f(x)$. Let us choose positive constants $r$ and $d$ such that

$$\{u \in \mathbb{R}^n : |u| < r\} \subset Q_0 \subset \{u \in \mathbb{R}^n : |u| < dr\}. \tag{7}$$

We obtain a following equivalent statment

**Theorem 1** . *Given $s > 0$ , a nonnegative integer $k$ with $k+1 > s$ and $1 \le p, q \le \infty$, we have equivalent ones of the Besov space norm given in* (6), *if one of them exists, with the usual modification for $q = \infty$,*

$$\begin{aligned} &||f||_{B^s_{pq}(M)} \\ &\sim\ ||f||_p + \Big(\sum_{l=0}^{\infty}(\lambda_0^{ls}||\Omega_p^k f(\cdot,l)||_p)^q\Big)^{1/q} \equiv |||f|||_1, \\ &\sim\ ||f||_p + \Big(\sum_{l=0}^{\infty}(\lambda_0^{ls} \sup_{(k+1)|M^l u|<r/2} ||\triangle_u^{k+1} f||_p)^q\Big)^{1/q} \equiv |||f|||_2. \end{aligned}$$

If $0 < s < k+1$ for a nonnegative integer $k$ and $1 \le p, q \le \infty$, then for $x \in \mathbb{R}^n$, a function $f \in T^s_{pq}(x)$ means that

$$\Big(\sum_{l=0}^{\infty}(\lambda_0^{ls}\operatorname{osc}_p^k f(x,l))^q\Big)^{1/q} < \infty$$

with the usual modification for $q = \infty$. We note that the definition is independent of the choice of $k$ with $k+1 > s$.

**Remark 2.** We have the embedding theorem : $T^{\beta}_{p\xi}(x) \subset T^{\alpha}_{p\eta}(x)$ for $\beta > \alpha > 0$, $1 \le \xi, \eta \le \infty$ and $1 \le p \le \infty$, and $T^{\alpha}_{p\eta}(x) \subset T^{\alpha}_{p\xi}(x)$, $T^{\alpha}_{\xi q}(x) \subset T^{\alpha}_{\eta q}(x)$ for $\alpha > 0$, $1 \le \eta \le \xi \le \infty$ and $1 \le p, q \le \infty$.

We have a poinwise version of Theorem 1, which is proved by the same way as the proof of Theorem 1.

**Corollary.** *Given $s > 0$ , a nonnegative integer $k$ with $k+1 > s$ and $1 \le p, q \le \infty$. Then for $x \in \mathbb{R}^n$ following properties of a bounded function $f$ are equivalent, with the usual modification for $q = \infty$,*

(i) $f \in T^s_{pq}(x)$,

(ii) $\Big(\sum_{l=0}^{\infty}(\lambda_0^{ls}\Omega_p^k f(x,l))^q\Big)^{1/q} < \infty$,

(iii) $\Big(\sum_{l=0}^{\infty}(\lambda_0^{ls} \sup_{(k+1)|M^l u|<r/2} (\frac{1}{|Q_l(x)|}\int_{Q_l(x)} |\triangle_u^{k+1} f(y)|^p dy)^{1/p})^q\Big)^{1/q} < \infty.$

We will define the Littlewood-Paley decomposition. Let us $\lambda_0 > 1$ and $\varphi$ a function in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ with the following properties: supp $\hat{\varphi} \subset \{\xi \in \mathbb{R}^n : |\xi| \leq 1\}$ and $\hat{\varphi}(\xi) = 1$ on $\{\xi \in \mathbb{R}^n : |\xi| \leq \lambda_0^{-1}\}$. Let $\psi(x) = \lambda_0^n \varphi(\lambda_0 x) - \varphi(x)$. Let $\varphi_l(x) = \lambda_0^{ln}\varphi(\lambda_0^l x)$, $S_l f = f * \varphi_l$, $\psi_l(x) = \lambda_0^{ln}\psi(\lambda_0^l x)$ and $f_l = f * \psi_l$ for $l = 0,\ 1,\ 2,\ \ldots$. Then for $f \in S'$ we have Littlewood-Paley decomposition:

$$f = \varphi * f + \sum_{l=0}^{\infty} \psi_l * f \equiv S_0 f + \sum_{l=0}^{\infty} f_l. \tag{8}$$

**Theorem B** ([13]). *Suppose that a dilation matrix is of the form $M = \lambda_0 Id$ with $\lambda_0 > 1$. Let $1 \leq p, q \leq \infty$ and $s > 0$. Then we have equivalence of norms if one of them exit, for Littlewood-Paley decomposition given in* (8), *with the usual modification $q = \infty$:*

$$\begin{aligned} &\text{(i)} && ||f||_{B^s_{pq}(M)}, \\ \sim\ &\text{(ii)} && ||f||_p + \Big(\sum_{l=0}^{\infty}(\lambda_0^{ls}||f - S_l f||_p)^q\Big)^{1/q}, \\ \sim\ &\text{(iii)} && ||S_0 f||_p + \Big(\sum_{l=0}^{\infty}(\lambda_0^{ls}||f_l||_p)^q\Big)^{1/q}. \end{aligned}$$

We write $T^s_{\infty\infty}(x) = C^s(x)$. The following statement is a pointwise version of Theorem B and can be proved by the corollary of Theorem 1 using the same way as in [1].

**Proposition 1** . *Suppose that a dilation matrix is of the form $M = \lambda_0 Id$ with $\lambda_0 > 1$. Let $s > 0$. Then for $x \in \mathbb{R}^n$, following properties of a bounded function $f$ for Littlewood-Paley decomposition given in* (8) *are equivalent:*

$$\begin{aligned} &\text{(i)} \quad f \in C^s(x), \\ &\text{(ii)} \quad |f(y) - S_l f(y)| \leq C(\lambda_0^{-l} + |x-y|)^s \ \text{ for all } l \geq 0. \end{aligned}$$

**Corollary**. *Suppose that a dilation matrix $M = \lambda_0 Id$. Let $f$ be a bounded function. If $f \in C^s(x)$, then it holds*

$$\text{(iii)} \qquad |f_l(y)| \leq C(\lambda_0^{-l} + |x-y|)^s \quad \textit{for all } l \ \geq\ 0.$$

*Conversely, if it holds for $s > s' > 0$,*

$$\text{(iii)}' \qquad |f_l(y)| \leq C\lambda_0^{-ls}(1 + \lambda_0^l|x-y|)^{s'} \quad \textit{for all } l \ \geq\ 0,$$

*then $f \in C^s(x)$.*

# 3 Multiresolution approximation

Let $\Pi$ denote a self-affine lattice tiling $\{T+\nu\}_{\nu\in\mathbb{Z}^n}$ with a dilation matrix $M$. For an integer $l$ and a finite subset $\Phi=\{\phi_1,\ldots,\phi_N\}$ of $\mathcal{L}^\infty$ with $L^2$-stable shifts, we define operators $P_lf$ given by

$$P_lf(x)=\sum_{j=1}^{N}\sum_{\nu\in\mathbb{Z}^n} m^l\langle f,\tilde{\phi}_j(M^l\cdot-\nu)\rangle\phi_j(M^lx-\nu) \tag{9}$$

where $\langle f,\tilde{\phi}_j(M^l\cdot-\nu)\rangle=\int f(y)\bar{\tilde{\phi}}_j(M^ly-\nu)\,dy$ and $\tilde{\Phi}=\{\tilde{\phi}_1,\ldots,\tilde{\phi}_N\}$ is dual to $\Phi$ in Theorem A.

Let $V_0^p=\{\sum_{j=1}^N\sum_{\nu\in\mathbb{Z}^n}a_j(\nu)\phi_j(x-\nu):\ a_j\in l^p(\mathbb{Z}^n)\}$ and let $V_l^p=\{f(M^lx):f\in V_0^p\}$. Then for $1\le p\le\infty$, the operator $P_l$ is a bounded projection operator of $L^p(\mathbb{R}^n)$ onto $V_l^p$ $(1\le p\le\infty)$ in the sense that $P_lf=f$ for any $f\in V_l^p$. We say $\Phi=\{\phi_1,\ldots,\phi_N\}$ of $\mathcal{L}^\infty$ is $M$-refinable if there exist sequences $c_{jk}\in l^1(\mathbb{Z}^n)$ $(1\le j,k\le N)$ such that

$$\phi_j(x)=\sum_{k=1}^{N}\sum_{\nu\in\mathbb{Z}^n}c_{jk}(\nu)\phi_k(Mx-\nu),\quad x\in\mathbb{R}^n,\quad j=1,\ldots,N.$$

A following theorem implies that $\{V_l^p\}$ is a multiresolution analysis in $L^p(\mathbb{R}^n)$ for $1\le p<\infty$.

**Theorem C** ( [7] and [16]). *If a finite subset $\Phi$ of $\mathcal{L}^\infty$ is $M$-refinable and has $L^2$-stable shifts, then the sequence of sets $\{V_l^p\}$ $(1\le p\le\infty)$ satisfies following properties:*
(i) $f\in V_0^p\Leftrightarrow f(x-\nu)\in V_0^p$ *for all* $\nu\in\mathbb{Z}^n$ ,
(ii) $f\in V_l^p\Leftrightarrow f(Mx)\in V_{l+1}^p$,
(iii) $\cdots\subset V_l^p\subset V_{l+1}^p\subset\cdots$ ,
(iv) $\cap_{l\in\mathbb{Z}}V_l^p=\{0\}$ $(1\le p<\infty)$,
(v) $\cup_{l=0}^\infty V_l^p$ *is dense in* $L^p(\mathbb{R}^n)$ $(1\le p<\infty)$.

Given a function $f$ in $L^p(\mathbb{R}^n)$ $(1\le p\le\infty)$, $\sigma_l^p(f)$ denotes the error of $L^p$-approximation from $V_l^p$ in $L^p(\mathbb{R}^n)$:

$$\sigma_l^p(f)=\inf\{||f-S||_p:S\in V_l^p\}. \tag{10}$$

Clearly we have the following equivalence:

$$\sigma_l^p(f)\sim||f-P_lf||_p,\quad f\in L^p(\mathbb{R}^n)\ \ (1\le p\le\infty).$$

Given $s>0$, $\lambda>1$ and $1\le p,q\le\infty$. A function $f$ is said to belong to $B_{pq}^{s,\lambda}(\Phi)$ if

$$||f||_{B_{pq}^{s,\lambda}(\Phi)}=||f||_p+\Big(\sum_{l=0}^{\infty}(\lambda^{ls}\sigma_l^p(f))^q\Big)^{1/q}<\infty \tag{11}$$

with the usual modification when $q=\infty$.

Let

$$R_lf=P_{l+1}f-P_lf,\quad l=0,1,\ldots. \tag{12}$$

We put

$$P_0f(x)=\sum_{j=1}^{N}\sum_{\nu\in\mathbf{Z}^n}a_{j0}(\nu)\phi_j(x-\nu),\quad R_lf(x)=\sum_{j=1}^{N}\sum_{\nu\in\mathbf{Z}^n}a_{j(l+1)}(\nu)\phi_j(M^{l+1}x-\nu). \tag{13}$$

Since $\Phi$ has stable shifts, we have

$$||P_0f||_p\sim\sum_{j=1}^{N}||a_{j0}||_{l^p},\quad ||R_lf||_p\sim m^{-(l+1)/p}\sum_{j=1}^{N}||a_{j(l+1)}||_{l^p},\quad l=0,1,\dots. \tag{14}$$

Then for $f\in B_{pq}^{s,\lambda}(\Phi)$ we have

$$f(x)=P_0f(x)+\sum_{l=0}^{\infty}R_lf(x)\equiv\sum_{j=1}^{N}\sum_{l=0}^{\infty}\sum_{\nu\in\mathbf{Z}^n}a_{jl}(\nu)\phi_j(M^lx-\nu).$$

Moreover from [15, Theorem 5.10] there exists an associated set of wavelets $\{\psi_j^\epsilon\}_{j=1,\dots,N}^{\epsilon=1,\dots,m-1}$, that is, $\{\psi_j^\epsilon(x-\nu)\}_{j=1,\dots,N,\nu\in\mathbf{Z}^n}^{\epsilon=1,\dots,m-1}$ is an orthonormal basis in $W_0=V_1^2\ominus V_0^2$ in $L^2(\mathbb{R}^n)$ , whose wavelet expansion of a function $f\in L^2(\mathbb{R}^n)$ is given by

$$f(x)=\sum_{j=1}^{N}\sum_{\nu\in\mathbf{Z}^n}a_{j0}(\nu)\phi_j(x-\nu)+\sum_{j=1}^{N}\sum_{\epsilon=1}^{m-1}\sum_{l=0}^{\infty}\sum_{\nu\in\mathbf{Z}^n}b_{jl}^\epsilon(\nu)m^{l/2}\psi_j^\epsilon(M^lx-\nu) \tag{15}$$

where

$$a_{j0}(\nu)=\langle f(y),\tilde{\phi}_j(y-\nu)\rangle,\quad b_{jl}^\epsilon(\nu)=\langle f(y),m^{l/2}\psi_j^\epsilon(M^ly-\nu)\rangle. \tag{16}$$

Then we have

$$P_0f(x)=\sum_{j=1}^{N}\sum_{\nu\in\mathbf{Z}^n}a_{j0}(\nu)\phi_j(x-\nu),$$

$$R_lf(x)=\sum_{j=1}^{N}\sum_{\epsilon=1}^{m-1}\sum_{\nu\in\mathbf{Z}^n}b_{jl}^\epsilon(\nu)m^{l/2}\psi_j^\epsilon(M^lx-\nu),l=0,1,\dots.$$

When $m>(n+1)/2$, there exist $\psi_j^\epsilon\in\mathcal{L}^\infty$ and

$$||R_lf||_p\sim m^{l(1/2-1/p)}\sum_{j=1}^{N}\sum_{\epsilon=1}^{m-1}||b_{jl}^\epsilon||_{l^p}\quad(1\le p\le\infty).$$

A following result can be proved from easy routine using Hardy's inequalty.

**Theorem 2** . *Assume that a finite subset $\Phi=\{\phi_1,\dots,\phi_N\}$ of $\mathcal{L}^\infty$ is $M$-refinable and has $L^2$-stable shifts. Given $\lambda>1$ and $\alpha>0$, there are equivalences of the norm $||f||_{B_{pq}^{\alpha,\lambda}(\Phi)}$ given in* (11), *if one of them exits, for any $1\le p,q\le\infty$, with the usual modification for $q=\infty$:*
(i) $||f||_p+(\sum_{l=0}^\infty(\lambda^{l\alpha}||f-P_lf||_p)^q)^{1/q}$,
(ii) $||P_0f||_p+(\sum_{l=0}^\infty(\lambda^{l\alpha}||R_lf||_p)^q)^{1/q}$,
(iii) $(\sum_{l=0}^\infty(\lambda^{l\alpha}m^{-l/p}\sum_{j=1}^N||a_{jl}||_{l^p})^q)^{1/q}$,
*where $\{a_{jl}\}$ are given in* (13).

(iv) $\inf(\sum_{l=0}^{\infty}(\lambda^{l\alpha}m^{-l/p}\sum_{j=1}^{N}||c_{jl}||_{l^p})^q)^{1/q}$ *where the infimum is taken over all admissible $L^p$-convergent representations*

$$f(x)=\sum_{j=1}^{N}\sum_{l=0}^{\infty}\sum_{\nu\in\mathbb{Z}^n}c_{jl}(\nu)\phi_j(M^lx-\nu),$$

(v) $\sum_{j=1}^{N}||a_{j0}||_{l^p}+(\sum_{l=0}^{\infty}(\lambda^{l\alpha}m^{l(1/2-1/p)}\sum_{j=1}^{N}\sum_{\epsilon=1}^{m-1}||b_{jl}^{\epsilon}||_{l^p})^q)^{1/q}$ when $m>(n+1)/2$, *where $\{a_{j0}\}$ and $\{b_{jl}^{\epsilon}\}$ are given in* (16).

**Proposition 2** . *Given $k+1>s>0$. Assume that $\Phi=\{\phi_1,\dots,\phi_N\}$ of $\mathcal{L}^{\infty}$ is M-refinable and has $L^2$-stable shifts. Then we have for any $1\le p,q\le\infty$,*

$$B_{pq}^{s,\lambda_0}(\Phi)\subset B_{pq}^{s}(M)$$

*provided that there exists a positive number $s_0$ with $s_0>s$ such that $\sup_{l\ge 0}\lambda^{ls_0}|\mathrm{osc}_p^k\phi_j(\cdot,l)|_p<\infty$ for all $j=1,\dots,N$, where the norm $|\cdot|_p$ and $\mathrm{osc}_p^k$ are given in* (2) *and* (5) *respectively, and $\lambda_0$ is the least value of absolute values of eigenvalues of $M$.*

**Sketch of Proof.** We shall prove for any $f\in B_{pq}^{s,\lambda_0}(\Phi)$,

$$(\sum_{l=0}^{\infty}(\lambda_0^{ls}\tilde{\sigma}_l^p(f))^q)^{1/q}\le C(||f||_p+(\sum_{l=0}^{\infty}(\lambda_0^{ls}\sigma_l^p(f))^q)^{1/q})$$

where $\sigma_l^p$ is the errors of $L^p$-approximation given in (10) associated with $\Phi$ and $\tilde{\sigma}_l^p(f)=||\mathrm{osc}_p^kf(\cdot,l)||_p$ . Since $\sigma_l^p(f)\to 0$ as $l\to\infty$ $(1\le p\le\infty)$, we have an $L^p$-convergent series

$$f(x)=P_0f(x)+\sum_{l=0}^{\infty}R_lf(x)\equiv\sum_{j=1}^{N}\sum_{l=0}^{\infty}\sum_{\nu\in\mathbb{Z}^n}a_{jl}(\nu)\phi_j(M^lx-\nu)$$

where $P_0f(x)=\sum_{j=1}^{N}\sum_{\nu\in\mathbb{Z}^n}a_{j0}(\nu)\phi_j(x-\nu)$ and $R_lf(x)=\sum_{j=1}^{N}\sum_{\nu\in\mathbb{Z}^n}a_{j(l+1)}(\nu)\phi_j(M^{l+1}x-\nu)$ are given in (13).

Then we have

$$\begin{aligned}\tilde{\sigma}_{l_0}^p(f)&=\tilde{\sigma}_{l_0}^p(P_0f+\sum_{l=0}^{\infty}R_lf)\\&\le\ \tilde{\sigma}_{l_0}^p(P_0f)+\sum_{l=0}^{\infty}\tilde{\sigma}_{l_0}^p(R_lf)\equiv I_0+\sum_{l=0}^{\infty}I_l'.\end{aligned}$$

We shall give an estimate of $I_0$. By (14) we have

$$\begin{aligned}I_0&\le C\sum_{j=1}^{N}||\sum_{\nu\in\mathbb{Z}^n}|a_{j0}(\nu)|\mathrm{osc}_p^k\phi_j(x-\nu,l_0)||_p\\&\le\ C\sum_{j=1}^{N}||a_{j0}||_{l^p}|\mathrm{osc}_p^k\phi_j(\cdot,l_0)|_p\le C||P_0f||_p\sup_j|\mathrm{osc}_p^k\phi_j(\cdot,l_0)|_p.\end{aligned}$$

If $l < l_0$, then we see by (14) that

$$I'_l \le Cm^{-(l+1)/p} \sum_{j=1}^{N} || \sum_{\nu} |a_{j(l+1)}(\nu)| \mathrm{osc}_p^k \phi_j(x-\nu, l_0-l-1)||_p$$
$$\le \; C\sum_{j=1}^{N} m^{-(l+1)/p} ||a_{j(l+1)}||_{l^p} |\mathrm{osc}_p^k \phi_j(\cdot, l_0-l-1)|_p \le C||R_l f||_p \sup_j |\mathrm{osc}_p^k \phi_j(\cdot, l_0-l-1)|_p.$$

If $l \ge l_0$, then we have by the definition,

$$I'_l \le ||R_l f||_p.$$

From Hardy's inequality and Theorem 2, these complete the proof of Proposition 2.

A following corollary can be proved by the same way in the proof of Proposition 2.

**Corollary.** *Given $\lambda > 1$ and $s > 0$. Assume that $\Phi = \{\phi_1, \dots, \phi_N\}$ and $\Phi' = \{\phi'_1, \dots, \phi'_L\}$ of $\mathcal{L}^\infty$ are M-refinable and have $L^2$-stable shifts. Then we have for any $1 \le p, q \le \infty$,*

$$B_{pq}^{s,\lambda}(\Phi') \subset B_{pq}^{s,\lambda}(\Phi)$$

*provided that there exists a positive number $s_0$ with $s_0 > s$ such that $\sup_{l\ge 0} \lambda^{ls_0} |\phi'_j - P_l \phi'_j|_p < \infty$ for all $j = 1, \dots, L$, where the operator $P_l$ is given in* (9) *associated with $\Phi$.*

For a positive integer $k$ and $1 \le p \le \infty$, $\mathcal{L}_k^p = \mathcal{L}_k^p(\mathbb{R}^n)$ is denoted to be the space of all functions $f$ such that $f(x)(1+|x|)^k \in \mathcal{L}^p$. If $\phi \in L^p(\mathbb{R}^n)$ $(1 \le p \le \infty)$ is compactly supported, then $\phi \in \mathcal{L}_k^p$. Furthermore, we observe that if there are constants $C > 0$ and $\delta > k$ such that $|\phi(x)| \le C(1+|x|)^{-n-\delta}$ for all $x \in \mathbb{R}^n$ then $\phi \in \mathcal{L}_k^\infty$.

For a finite subset $\Phi$ of $\mathcal{L}_k^\infty$, the domain of the operator $P_l$ given in (9), can be extended to include the linear space $\mathbb{P}^k$ of all polynomials of degree no greater than $k$ on $\mathbb{R}^n$. For a finite subset $\Phi$ of $\mathcal{L}_k^1$, we say that $\Phi$ satisfies the Strang-Fix condition of order $k$ if there is a finite linear combination $\phi$ of the functions of $\Phi$ and their shifts such that $\hat{\phi}(0) \ne 0$ and $\partial^\alpha \hat{\phi}(2\pi\nu) = 0$, $|\alpha| \le k-1$, $\nu \in \mathbb{Z}^n$ with $\nu \ne 0$.

**Lemma 1** . *Let $\Phi$ be a finite subset of $\mathcal{L}_k^\infty$ that has $L^2$- stable shifts. Then $\Phi$ satisfies the Strang-Fix condition of order $k$ if and only if $P_0 q = q$ for any $q \in \mathbb{P}^{k-1}$.*

*Moreover, if this is the case, then we have $||P_l f - f||_p \le C\lambda_0^{-lk} \sum_{|\alpha|=k} ||\partial^\alpha f||_p$ for any $f$ in the Sobolev space $W_k^p(\mathbb{R}^n)$ $(1 \le p \le \infty)$, with a constant $C$ independent of $f, p$ and $l$ where $\lambda_0$ is the least value of absolute values of eigenvalues of the dilation matrix $M$, that is, $W_k^p(\mathbb{R}^n) \subset B_{pq}^{s,\lambda_0}(\Phi)$ if $0 < s < k$ and $1 \le q \le \infty$.*

**Proof.** We can prove by the same way of [8, Theorem 5.2]. We will omit its details.

# 4 Characterization of Besov spaces

Let $\Pi$ be a self-affine lattice tiling $\{T+\nu\}_{\nu\in\mathbb{Z}^n}$ and $\Pi_l$ denote the subdivision $\{M^{-l}(T+\nu)\}_{\nu\in\mathbb{Z}^n}$ of $\mathbb{R}^n$ for a nonnegative integer $l$. Let $\Phi = \{\phi_1, \dots, \phi_N\}$ be a finite subset of $\mathcal{L}^\infty$ and $\lambda_0$ the least value of absolute values of eigenvalues of the dilation matrix $M$.

**Proposition 3** . *Given* $1 \le p,\ q \le \infty$ *and* $k > s > 0$. *Assume that a finite subset* $\Phi = \{\phi_1, \dots, \phi_N\}$ *of* $\mathcal{L}_k^\infty$ *satisfies*

(a) $\Phi$ *has* $L^2$*-stable shifts,*

(b) $\Phi$ *is* $M$*-refinable,*

(c) $\Phi$ *satisfies the Strang-Fix condition of order* $k$.

*Then we have* $B^s_{pq}(M) \subset B^{s,\lambda_0}_{pq}(\Phi)$.

**Proof.** We shall prove for any $f \in B^s_{pq}(M)$

$$(\sum_{l=0}^{\infty}(\lambda_0^{ls}\sigma_l^p(f))^q)^{1/q} \le C||f||_{B^s_{pq}(M)}$$

where $\sigma_l^p$ is given in (10) associated with $\Phi$. We choose a function $\chi$ in $C_c^\infty(\mathbb{R}^n)$ such that $\int |\chi(u)|du = 1$ and supp $\chi \subset \{u \in \mathbb{R}^n : |u| < r/2k\}$ where $r$ is the positive number given in (7). We write $\chi_l(u) = m^l\chi(M^lu)$, $h_l(x) = \int(f(x) - \Delta_u^k f(x))\chi_l(u)du$ and $g_l = P_l h_l - h_l$ where $P_l$ is given in (9) associated with $\Phi$. Then we have for $1 \le p \le \infty$,

$$||f - P_l f||_p \le ||f - h_l||_p + ||g_l||_p + ||P_l h_l - P_l f||_p \le C||f - h_l||_p + ||g_l||_p \equiv CI_1 + I_2.$$

Obviously we have :

$$I_1 \le C \sup_{k|M^lu|<r/2} ||\Delta_u^k f||_p.$$

We shall give an estimate of $I_2$ by (1):

$$I_2 = (\sum_{Q\in\Pi_l}\int_Q |g_l(x)|^p dx)^{1/p} = (\sum_{\nu\in\mathbf{Z}^n}\int_{M^{-l}T}|g_l(x - M^{-l}\nu)|^p dx)^{1/p}. \tag{17}$$

Let $q_z$ be the $(k-1)$-th Taylor polynomial of $h_l$ about $z \in \mathbb{R}^n$ and let $r_z$ be the corresponding remainder. Since $\Phi$ satisfies the Strang-Fix condition of order $k$, we see from Lemma 1

$$g_l(x - M^{-l}\nu) = P_l r_{x-M^{-l}\nu}(x - M^{-l}\nu) = m^l\int K(M^lx, M^ly)r_{x-M^{-l}\nu}(y - M^{-l}\nu)dy$$

where $K(x,y) = \sum_{j=1}^N \sum_{\nu\in\mathbf{Z}^n}\phi_j(x-\nu)\bar{\tilde{\phi}}_j(y-\nu)$.

To estimate $I_2$, we use

$$r_{x-M^{-l}\nu}(y - M^{-l}\nu) = \int_0^1 \sum_{|\beta|=k}\frac{k}{\beta!}\partial^\beta h_l(x + t(y-x) - M^{-l}\nu)(1-t)^{k-1}(y-x)^\beta dt,$$

and

$$|\partial^\beta h_l(x)| \le C\sum_{e=1}^k(\int_{|u|<r/2k}|f(x - eM^{-l}u)|^p du)^{1/p}$$

$$\le C\sum_{e=1}^k(m^l\int_{|M^lu|<re/2k}|f(x-u)|^p du)^{1/p} \le Cm^{l/p}(\int_{|M^lu|<r/2}|f(x-u)|^p du)^{1/p}.$$

Hence we get an estimate:

$$
\begin{aligned}
&(\sum_{\nu\in\mathbb{Z}^n} |r_{x-M^{-l}\nu}(y-M^{-l}\nu)|^p)^{1/p} \\
\le\ & C\int_0^1 \sum_{|\beta|=k} (\sum_{\nu} |\partial^\beta h_l(x+t(y-x)-M^{-l}\nu)|^p)^{1/p}(1-t)^{k-1}|x-y|^k dt \\
\le\ & C\int_0^1 \sum_{|\beta|=k} (\sum_{\nu} m^l \int_{|M^l u|<r/2} |f(x+t(y-x)-M^{-l}\nu-u)|^p du)^{1/p}(1-t)^{k-1}|x-y|^k dt \\
\le\ & C\int_0^1 m^{l/p}(\sum_{\nu} \int_{M^{-l}(T+\nu)} |f(x+t(y-x)+u)|^p du)^{1/p}(1-t)^{k-1}|x-y|^k dt \\
\le\ & C\int_0^1 m^{l/p}||f||_p(1-t)^{k-1}|x-y|^k dt \le C|x-y|^k m^{l/p}||f||_p.
\end{aligned}
$$

Hence, since $\Phi\subset\mathcal{L}_k^\infty$, we get an estimate of $I_2$ in (17):

$$
\begin{aligned}
I_2 \le\ & Cm^l(\int_{M^{-l}T}\sum_{\nu}(\int |K(M^l x, M^l y)||r_{x-M^{-l}\nu}(y-M^{-l}\nu)|dy)^p dx)^{1/p} \\
\le\ & Cm^l(\int_{M^{-l}T}(\int |K(M^l x, M^l y)|(\sum_{\nu} |r_{x-M^{-l}\nu}(y-M^{-l}\nu)|^p)^{1/p} dy)^p dx)^{1/p} \\
\le\ & Cm^{l+l/p}||f||_p(\int_{M^{-l}T}(\int |K(M^l x, M^l y)||x-y|^k dy)^p dx)^{1/p} \\
\le\ & C||f||_p(\int_T(\int |K(x,y)||M^{-l}(x-y)|^k dy)^p dx)^{1/p} \\
\le\ & C||f||_p\lambda_0^{-lk}(\int_T(\int |K(x,y)||x-y|^k dy)^p dx)^{1/p} \le C||f||_p\lambda_0^{-lk}.
\end{aligned}
$$

Now we combine the estimates of $I_1$ and $I_2$ to write

$$
||f-P_l f||_p \le CI_1 + I_2 \le C(\sup_{k|M^l u|<r/2} ||\Delta_u^k f||_p + \lambda_0^{-lk}||f||_p).
$$

This implies that

$$
(\sum_{l=0}^{\infty}(\lambda_0^{ls}\sigma_l^p(f))^q)^{1/q} \le C||f||_{B^s_{pq}(M)}.
$$

This completes the proof of Proposition 3.

A following theorem is an immediate consequence of Proposition 2 and Proposition 3. This theorem is a generalization of results in [3], [4] and [12].

**Theorem 3** . *Given $1\le p,\ q\le\infty$ and $k>s>0$. Assume that a finite subset $\Phi=\{\phi_1,\dots,\phi_N\}$ of $\mathcal{L}_k^\infty$ satisfies*

(a) *$\Phi$ has $L^2$-stable shifts,*

(b) *$\Phi$ is $M$-refinable,*

(c) *there exists a positive number $s_0$ with $s_0>s$ such that $\sup_{l\ge0}\lambda_0^{ls_0}|\mathrm{osc}_p^{k-1}\phi_j(\cdot,l)|_p<\infty$ for all $j=1,\dots,N$,*

(d) *$\Phi$ satisfies the Strang-Fix condition of order $k$.*

*Then we have* $B^s_{pq}(M) = B^{s,\lambda_0}_{pq}(\Phi)$ *with equivalent norms*

$$||f||_{B^s_{pq}(M)} \sim ||f||_{B^{s,\lambda_0}_{pq}(\Phi)}$$

*where the norms* $||f||_{B^s_{pq}(M)}$ *and* $||f||_{B^{s,\lambda_0}_{pq}(\Phi)}$ *are given in* (6) *and* (11) *respectively, and* $\lambda_0$ *is the least value of absolute values of eigenvalues of the dilation matrix* $M$.

**Remark 3**. When $\{\phi_j\}_{j=1}^N$ have compact supports, we see that the condition (c) in Theorem 3 can be rephrased as :

(c)' There exists a positive number $s_0 > s$ such that $\sup_{l\geq 0} \lambda_0^{ls_0}||\mathrm{osc}_p^{k-1}\phi_j(\cdot, l)||_p < \infty$, (that is, $\phi_j \in B^{s_0}_{p\infty}(M)$ if $s_0 < k$) for all $j = 1, \dots, N$.

We say that a function on $\mathbb{R}^n$ is $k$-regular if it is of class $C^k$ and rapidly decreasing in the sense that $|\partial^\alpha f(x)| \leq C_N(1 + |x|)^{-N}$ for all $N = 0,\ 1,\ 2,\ \dots$ and all $|\alpha| \leq k$. Any $k$-regular function belongs to $\mathcal{L}_N^\infty$ for any $N \geq 0$ and any $k$-regular function $f$ satisfies the condition (c) in Theorem 3 : $\sup_{l\geq 0} \lambda_0^{lk}|\mathrm{osc}_p^{k-1} f(\cdot, l)|_p < \infty$.

**Corollary 1** . *Suppose that a dilation matrix is of the form* $M = \lambda_0 Id$ *with* $\lambda_0 > 1$. *Let* $1 \leq p,\ q \leq \infty$ *and* $k > s > 0$. *Assume that a finite subset* $\Phi = \{\phi_1, \dots, \phi_N\}$ *of* $k$*-regular functions on* $\mathbb{R}^n$ *satisfies:*

(a) $\Phi$ *has* $L^2$*-stable shifts,*

(b) $\Phi$ *is* $M$*-refinable.*

*Then there exits a set* $\{\psi_j^\epsilon\}_{j=1,\dots,N}^{\epsilon=1,\dots,m-1}$ *of* $k$*-regular wavelets associated with* $\Phi$*, and we have equivalence of norms, if one of them exit, for wavelet expansion given in* (15) *with the usual modification for* $q = \infty$*:*

$$\begin{aligned} &\text{(i)} \quad ||f||_{B^s_{pq}(M)}, \\ \sim\ &\text{(ii)} \quad ||f||_{B^{s,\lambda_0}_{pq}(\Phi)}, \\ \sim\ &\text{(iii)} \quad \sum_{j=1}^N ||a_{j0}||_{l^p} + (\sum_{l=0}^\infty (\lambda_0^{l(s+n/2-n/p)} \sum_{j=1}^N \sum_{\epsilon=1}^{m-1} ||b^\epsilon_{jl}||_{l^p})^q)^{1/q}. \end{aligned}$$

**Proof.** From [15, Theorem 5.15], for a finite subset $\Phi$ of $k$- regular functions there exists an associated set of $k$-regular wavelets for a general dilation matrix $M$ if $m > (n+1)/2$. Since a finite subset of $k$-regular functions satisfies the Strang-Fix condition of order $k+1$ in the case $M = \lambda_0 Id$ (See [9, Theorem 4 in 2.6] and Lemma 1), we have the equivalence of (i) and (ii) from Theorem 3. The equivalence of (ii) and (iii) can be proved by Theorem 2.

We define the tensor product B-spline by $\mathcal{M}_k = \prod_{i=1}^n M_k(x_i),\quad x = (x_1, \dots, x_n) \in \mathbb{R}^n,\quad k = 1, 2, \dots$. where $M_k(t)$ is the $k$-th order central B-spline, that is, $\hat{M}_k(t) = (\dfrac{\sin(t/2)}{t/2})^k$. Let us denote by $\{e^i\}_{i=1}^n$ the set of unit vectors in $\mathbb{R}^n$. We put $e^{n+1} = \sum_{i=1}^n e^i$, and $X = \{x^1, \dots, x^{d_0}\}$ with $x^1 = e^1, \dots, x^{d_1} = e^1, x^{d_1+1} = e^2, \dots, x^{d_1+d_2} = e^2, \dots, x^{d_1+\cdots+d_n+1} = e^{n+1}, \dots, x^{d_0} = e^{n+1}$ where $d_0 = d_1 + \cdots + d_{n+1}$. We denote the box spline $B(x, X)$ corresponding to $X$ given by $\hat{B}(x, X) = (2\pi)^{-n/2}\prod_{j=1}^{d_0}\dfrac{1-e^{ix^j\cdot x}}{ix^j\cdot x}$. In the case that the self-affine lattice tiling is the net of closed cubes generated by $T = [0,1]^n$ and the dilation matrix is $2Id$, the $k$-th order tensor product B-spline $\mathcal{M}_k$ satisfies the conditions of Theorem 3, particularly, $\mathcal{M}_k \in B_{p\infty}^{k-1+1/p}(\mathbb{R}^n)$ and $\mathcal{M}_k$ satisfies the Strang-Fix condition of order $k$. The above box spline $B(x, X)$ also satisfies the conditions of Theorem 3 replacing the above $k$ by $k = \min\{d_i + d_j : \ i, j = 1, \dots, n+1, \ i \neq j\}$. Hence we get results of [3] and [12].

**Corollary 2** . *Suppose that the self-affine lattice tiling is the net $\Pi = \{T+\nu\}_{\nu\in\mathbb{Z}^n}$ of closed cubes generated by $T = [0,1]^n$ and the dilation matrix is $2Id$. Then Theorem 3 remains true for the tensor product B-spline $\Phi = \{\mathcal{M}_k\}$ or the box spline $\Phi = \{B(x, X)\}$.*

A following proposition is a pointwise version of Corollary 1 in Theorem 3.

**Proposition 4** . *Suppose that a dilation matrix is of the form $M = \lambda_0 Id$ with $\lambda_0 > 1$ and $k > s > 0$. Assume that a finite subset $\Phi = \{\phi_1, \dots, \phi_N\}$ of $k$-regular functions on $\mathbb{R}^n$ satisfies:*

(a) *$\Phi$ has $L^2$-stable shifts,*

(b) *$\Phi$ is $M$-refinable.*

*Then for $x \in \mathbb{R}^n$ and a bounded function $f$ on $\mathbb{R}^n$ , following properties are equivalent:*

$$\text{(i)} \quad f \in C^s(x),$$

$$\text{(ii)} \quad |f(y) - P_l f(y)| < C(\lambda_0^{-l} + |x-y|)^s \quad l \geq 0$$

*where $P_l f$ is given in* (9).

**Proof**. This can be proved by the same way as in Proposition 1. See [1, Theorem 3].

**Corollary**. *Suppose that the conditions in Proposition 4 are satisfied. Let $s > s' > 0$.*

(a) *If $f \in C^s(x)$, we have*

$$|R_l f(y)| \leq C(\lambda_0^{-l} + |x-y|)^s \quad l = 0, 1, 2, 3, \dots$$

*where $R_l f$ is given in* (12).

*If it holds*

$$|R_l f(y)| \leq C\lambda_0^{-sl}(1 + \lambda_0^l |x-y|)^{s'} \quad l = 0, 1, 2, 3, \dots,$$

*then $f \in C^s(x)$.*

(b) *If $f \in C^s(x)$, we have*

$$|b_{jl}^\epsilon(\nu)| \leq C\lambda_0^{-(s+\frac{n}{2})l}(1 + |\lambda_0^l x - \nu|)^s$$

*for $j = 1, \dots, N, l = 1, 2, 3, \dots, \epsilon = 1, \dots, m-1$ and any $\nu \in \mathbb{Z}^n$ where $b_{jl}^\epsilon(\nu)$ is given in* (16).

*If it holds*

$|b_{jl}^\epsilon(\nu)| \leq C\lambda_0^{-(s+\frac{n}{2})l}(1+|\lambda_0^l x-\nu|)^{s'}$ for $j = 1, \dots, N, \ l = 1, 2, 3, \dots$ and $\epsilon = 1, \dots, m-1$ *and any $\nu \in \mathbb{Z}^n$, then $f \in C^s(x)$.*

(c) *For $\{a_{jl}(\nu)\}$ given in* (13), *if it holds*

$$|a_{jl}(\nu)| \leq C\lambda_0^{-sl}(1 + |\lambda_0^l x - \nu|)^{s'} \quad j = 1, \dots, N, \ l > 0 \text{ and } \nu \in \mathbb{Z}^n,$$

*then $f \in C^s(x)$.*

# 5 Scaling exponents

For $1 \leq p, q \leq \infty$ we define $\alpha_{pq}(f) = \sup\{s \geq 0 : f \in B^s_{pq}(M)\}$ for functions $f \in L^p(\mathbb{R}^n)$. If there is not a positive number $s$ with $f \in B^s_{pq}(M)$, then we define $\alpha_{pq}(f) = 0$. We remark that $\alpha_{pq}(f) > 0$ for any $f \in L^p(\mathbb{R}^n)$ in the case $1 \leq p < \infty$. In the same manner we define $\alpha_{pq}(f, x) = \sup\{s \geq 0 : f \in T^s_{pq}(x)\}$ for $x \in \mathbb{R}^n$ and bounded functions $f$ on $\mathbb{R}^n$. We put $\alpha_p(f) = \alpha_{p\infty}(f)$, $\alpha(f) = \alpha_\infty(f)$, $\alpha_p(f, x) = \alpha_{p\infty}(f, x)$ and $\alpha(f, x) = \alpha_\infty(f, x)$.

We can prove a following proposition by the embedding theorem ( See [11]).

**Proposition 5** .
(i) $\alpha_p(f) = \alpha_{p\eta}(f)$ *for* $1 \leq p, \eta \leq \infty$,
(ii) $\alpha(f) > \alpha_p(f) - \frac{n}{p} \geq \alpha_q(f) - \frac{n}{q}$ *for* $1 \leq q \leq p < \infty$ *when* $M = \lambda_0 Id$,
(iii) $\alpha_p(f, x) = \alpha_{p\eta}(f, x)$ *for* $1 \leq p, \eta \leq \infty$,
(iv) $\alpha(f) \leq \alpha(f, x) \leq \alpha_p(f, x) \leq \alpha_q(f, x)$ *for* $1 \leq q \leq p < \infty$.

For $1 \leq p \leq \infty$ we have by Theorem 1 and Theorem B

$$\alpha_p(f) = -\frac{\log A_p(f)}{\log \lambda_0}$$

if the right hand side of the above equality is less than $k + 1$ where

$$A_p(f) = \limsup_{l\to\infty} ||\mathrm{osc}^k_p f(\cdot, l)||_p^{1/l} = \limsup_{l\to\infty} \sup_{(k+1)|M^l u|<r/2} ||\Delta^{k+1}_u f||_p^{1/l}$$

and furthermore when $M = \lambda_0 Id$ with $\lambda_0 > 1$

$$A_p(f) = \limsup_{l\to\infty} ||f - S_l f||_p^{1/l} = \limsup_{l\to\infty} ||f_l||_p^{1/l}.$$

For $1 \leq p \leq \infty$ we have by the corollary of Theorem 1

$$\alpha_p(f, x) = -\frac{\log A_p(f, x)}{\log \lambda_0}$$

if the right hand side of the above equality is less than $k + 1$ where

$$A_p(f, x) = \limsup_{l\to\infty} \mathrm{osc}^k_p f(x, l)^{1/l} = \limsup_{l\to\infty} \sup_{(k+1)|M^l u|<r/2} \left(\frac{1}{|Q_l(x)|} \int_{Q_l(x)} |\Delta^{k+1}_u f(y)|^p dy\right)^{1/pl}.$$

Furthermore when $M = \lambda_0 Id$ with $\lambda_0 > 1$, we have by Proposition 1 and its corollary

$$\alpha(f, x) = \liminf_{\lambda_0^{-l}+|x-y|\to 0} \frac{\log|f(y) - S_l f(y)|}{\log(\lambda_0^{-l} + |x - y|)}$$

and, if $\alpha(f) > 0$

$$\alpha(f, x) = \liminf_{\lambda_0^{-l}+|x-y|\to 0} \frac{\log|f_l(y)|}{\log(\lambda_0^{-l} + |x - y|)}$$

where $S_l f$ and $f_l$ are given for Littlewood-Paley decompostion in (8).

We can prove a following proposition by Theorem 2, Theorem 3, Proposition 4 and its corollary.

**Proposition 6** . (i). *Assume that a finite subset* $\Phi = \{\phi_1, \ldots, \phi_N\}$ *of* $\mathcal{L}_k^\infty$ *satisfies the conditions* (a), (b), (c) *and* (d) *of Theorem 3.*

*Then for* $f \in L^p(\mathbb{R}^n)$ $(1 \le p \le \infty)$ *we have*

$$\alpha_p(f) = -\frac{\log A_p(f)}{\log \lambda_0} = \frac{\log m}{p \log \lambda_0} - \frac{\log \rho_p(f)}{\log \lambda_0}$$

*if the second and third parts of the above equality are less than* $\min(k, s_0)$ *where*

$$A_p(f) = \limsup_{l \to \infty} \sigma_l^p(f)^{1/l} = \limsup_{l \to \infty} ||R_l(f)||_p^{1/l}$$

*and*

$$\rho_p(f) = \limsup_{l \to \infty} \sum_{j=1}^{N} ||a_{jl}||_{l^p}^{1/l} = \inf \limsup_{l \to \infty} \sum_{j=1}^{N} ||c_{jl}||_{l^p}^{1/l}$$

*and* $\{a_{jl}\}$ *is given by* (13) *and* inf *is taken over all admissible representations* $f(x) = \sum_{j=1}^N \sum_{l=0}^\infty \sum_{\nu \in \mathbf{Z}^n} c_{jl}(\nu)\phi_j(M^l x - \nu)$ *as in Theorem 2.*

(ii). *Furthermore when* $m > (n+1)/2$, *we have*

$$\alpha_p(f) = (1/p - 1/2)\frac{\log m}{\log \lambda_0} - \frac{\log \rho_p'(f)}{\log \lambda_0}$$

*if the right hand side of the above equality is less than* $\min(k, s_0)$ *where*

$$\rho_p'(f) = \limsup_{l \to \infty} \sum_{j=1}^{N} \sum_{\epsilon=1}^{m-1} ||b_{jl}^\epsilon||_{l^p}^{1/l}$$

*and* $\{b_{jl}^\epsilon\}$ *is given in* (16) *for the wavelet expansion* (15) *associated with* $\Phi$.

(iii). *Suppose that conditions in Proposition 4 hold for a bounded function* $f$. *Then we have*

$$\alpha(f, x) = \liminf_{\lambda_0^{-l} + |x-y| \to 0} \frac{\log |f(y) - P_l f(y)|}{\log(\lambda_0^{-l} + |x-y|)}$$

*if the right hand side of the above equality is less than* $k$ *and,*

$$\alpha(f, x) = \liminf_{\lambda_0^{-l} + |x-y| \to 0} \frac{\log |R_l f(y)|}{\log(\lambda_0^{-l} + |x-y|)}$$

$$= \liminf_{\lambda_0^{-l} + |x - \lambda_0^{-l}\nu| \to 0} \inf_j \frac{\log \lambda_0^{\frac{n}{2}l} |b_{jl}^\epsilon(\nu)|}{\log(\lambda_0^{-l} + |x - \lambda_0^{-l}\nu|)} \le \liminf_{\lambda_0^{-l} + |x - \lambda_0^{-l}\nu| \to 0} \inf_j \frac{\log |a_{jl}(\nu)|}{\log(\lambda_0^{-l} + |x - \lambda_0^{-l}\nu|)}$$

*if* $\alpha(f) > 0$ *and the right hand side of the above inequality is less than* $k$ *where* $P_l f$, $R_l f$ *and* $\{a_{jl}\}$ *are given in* (9), (12) *and* (13) *respectively.*

Let $\Pi = \{T + \nu\}_{\nu \in \mathbf{Z}^n}$ be a self-affine lattice tiling with a dilation matrix $M$ and a set $\Gamma_0$ of digits, and $\Pi_l$ denote the subdivision $\{M^{-l}(T + \nu)\}_{\nu \in \mathbf{Z}^n}$ of $\mathbb{R}^n$ for a nonnegative integer $l$. We write $Q = M^{-l}(T + \nu_Q)$ for $Q \in \Pi_l$. Let $\Pi_l(T) = \{Q \in \Pi_l : \ Q \subset T\}$ and $\Pi(T) = \cup_{l=0}^\infty \Pi_l(T)$. We put $\Gamma_0 = \{\gamma_1, \cdots, \gamma_m\}$. Then from (1) for $Q \in \Pi_l(T)$, $\nu_Q$ is of a form $\nu_Q = M^{l-1}\gamma_{i_1} + \cdots + \gamma_{i_l}$, $\gamma_{i_1}, \cdots, \gamma_{i_l} \in \Gamma_0$ and let $M_Q y = M^l y - \nu_Q$

and $\mu_Q = \mu_{i_1} \cdots \mu_{i_l}$ for $l > 0$ where $\mu_1, \mu_2, \ldots, \mu_m$ are real or complex numbers with $0 < |\mu_i| < 1$ , $i = 1, \ldots, m$. For $l = 0$ we put $M_T = Id$ and $\mu_T = 1$.

From now we suppose that a dilation matrix $M$ is of a form $M = \lambda_0 Id$ with $\lambda_0 > 1$ and we consider a bounded function $f$ which is given by a series

$$f(y) = \sum_{Q \in \Pi(T)} \mu_Q \phi(M_Q y), \quad y \in \mathbb{R}^n \tag{18}$$

where a function $\phi$ is bounded and zero outside $T^o$. We remark that $\alpha(f) \leq \alpha(\phi)$. Let

$$\tau_0(x) \equiv \liminf_{l \to \infty} \inf_{K_l(x) \ni Q} \frac{\log |\mu_Q|}{\log(\lambda_0^{-l} + |x - \lambda_0^{-l} \nu_Q|)} = \liminf_{l \to \infty} \inf_{K_l(x) \ni Q} \frac{\log |\mu_Q|}{\log \lambda_0^{-l}}$$

where $K_l(x) \equiv \{Q \in \Pi_l(T) : \ B(x, \lambda_0^{-l}) \cap Q \neq \emptyset\}$ and $B(x, \lambda_0^{-l})$ is a ball centered at $x$ with a radius $\lambda_0^{-l}$. When $x \in \Omega \equiv \cap_{l=0}^{\infty} \cup_{Q \in \Pi_l(T)} Q^o$(the interior of $Q$) there exits a unique sequence $\{Q_{l,x}\}_{l \geq 0}$ such that $Q_{l,x} \in \Pi_l(T)$ and $x \in Q_{l,x}^o$. Then we have for $x \in \Omega$

$$\tau_0(x) = \liminf_{l \to \infty} \frac{\log \mu_{Q_{l,x}}}{\log \lambda_0^{-l}}.$$

Let for $x \in \Omega$

$$\tau_1(x) \equiv \liminf_{l \to \infty} \frac{\log |\mu_{Q_{l,x}}|}{\log \Delta_l(x)}$$

where $\Delta_l(x) = \mathrm{dist}(x, \partial Q_{l,x})$ is the distance from $x$ to the boundary $\partial Q_{l,x}$ of $Q_{l,x}$. We remark for $x \in \Omega$, $\tau_0(x) = \tau_1(x)$ if $\sup_{l \geq 0} \dfrac{\Delta_l(x)}{\Delta_{l+1}(x)} < \infty$.

A following theorem may be proved by the same way as in [11].

**Theorem 4** . *Let $f$ and $\phi$ be bounded functions given in* (18). *Then we have*

(i) $\alpha(f, x) \geq \min(\alpha(\phi), \tau_0(x))$ *for* $x \in T$,

(ii) $\alpha(f, x) \geq \min_i(\alpha(\phi, \Omega_i), \tau_1(x))$ *for* $x \in \Omega$ *with* $\sup_{l \geq 0} \dfrac{\Delta_l(x)}{\Delta_{l+1}(x)} < \infty$

*where* $\Omega_i \equiv M^{-1}(T^o + \gamma_i)$, $\gamma_i \in \Gamma_0$, $i = 1, \ldots, m$ *and* $\alpha(\phi, \Omega_i) = \sup\{s \geq 0 : \phi \in C^s(\Omega_i)\}$ *and* $C^s(\Omega_i)$ *is defined as the Besov space* $B^s_{\infty\infty}(\Omega_i)$ *on* $\Omega_i$.

(iii) *Suppose that* $\phi \in C^\infty(\Omega_i)$, $i = 1, \ldots, m$ *and there exit a positive number* $s_0$ *and* $y_0 \in T^o$ *such that*

$$\sup_{l \geq 0} \sup_{y} \frac{|f_l(y)|}{(\lambda_0^{-l} + |y - y_0|)^{s_0}} = \infty.$$

*Then* $\tau_0(x) \geq \alpha(f, x)$ *for* $x \in T$.

**Corollary**. *Let $\phi$ be a bounded function on $\mathbb{R}^n$ such that $\phi \in C^\infty(\Omega_j), j = 1, \ldots, m$ and $\phi = 0$ outside $T^o$. Consider a bounded function $f$ given by* (18) *satisfying the condition* (iii) *in Theorem 4. Then we have*

(i) $\quad \tau_0(x) \geq \alpha(f, x) \geq \min(\alpha(\phi), \tau_0(x)), \quad x \in T,$

(ii) *for $x$ in $\Omega$ with* $\sup_{l \geq 0} \dfrac{\Delta_l(x)}{\Delta_{l+1}(x)} < \infty, \quad \alpha(f, x) = \tau_0(x) = \tau_1(x)$.

**Examples**. We consider a self-affine tiling $\Pi = \{T + \nu\}_{\nu \in \mathbb{Z}}$ such that a tile $T = [0, 1]$ and a dilation $M = 2Id$ on $\mathbb{R}$.

(a) We consider the Takagi function such that

$$f(x) = \sum_{l=0}^{\infty} \sum_{Q \in \Pi_l(T)} \mu^l \phi(M_Q x), \quad \forall x \in \mathbb{R}$$

where $0 < \mu < 1$ and $\phi$ is a bounded function such that $\phi(x) = x$ $(0 < x \leq \frac{1}{2})$, $\phi(x) = 1 - x$ $(\frac{1}{2} \leq x < 1)$, $\phi(x) = 0$ (otherwise). Let $\tau = \dfrac{\log \mu}{\log 2^{-1}}$. Then from the corollary of Theorem 4, if $\tau \leq 1$, $\tau = \alpha(f, x)$ for each $x \in T$.
(b) We consider the Weierstrass function $f(x) = \sum_{l=0}^{\infty} \mu^l \phi(2^l x)$ with $0 < \mu < 1$ and $\phi(x) = \sin 2\pi x$ $(x \in \mathbb{R})$. The proof of Theorem 4 can be also applied to this function case. Then we have

$$\tau = \alpha(f, x), \quad \forall x \in \mathbb{R}.$$

where the constant $\tau = \dfrac{\log \mu}{\log 2^{-1}}$ is given in the part (a) above.

(c) We consider Lèvy's function

$$f(x) = \sum_{l=0}^{\infty} \sum_{Q \in \Pi_l(T)} 2^{-l} \phi(M_Q x), \quad \forall x \in \mathbb{R}$$

where $\phi(x) = x - \dfrac{1}{2}$ $(0 < x < 1)$, $\phi(x) = 0$ (otherwise). Then we can see that $1 = \tau_1(x) = \alpha(f, x)$ for a point $x$ in $\Omega$ with $\sup_{l \geq 0} \dfrac{\Delta_l(x)}{\Delta_{l+1}(x)} < \infty$.

# References


[1] P. Anderssom, Characterization of pointwise Hölder regularity, *Appl. Comput. Harm. Anal.*, **4**(1997), 429-443.

[2] R. A. DeVore and R. Sharpley, Maximal functions measuring smoothness, *Mem. Amer. Math. Soc.*, **293**(1984), 1-115.

[3] R.A. DeVore and V. Popov, Interpolation of Besov spaces, *Trans. Amer. Math. Soc.*, **305**(1988), 397-414.

[4] R.A. DeVore, B. Jarwerth and V. Popov, Compression of wavelet decompositions, *Amer. J. Math.*, **114**(1992), 737-785.

[5] J. R. Dorronsoro, Mean oscillation and Besov spaces, *Canad. Math. Bull.*, **28**(1985), 474-480.

[6] G. Garrigós and A. Tabacco, Wavelet decompositions of anisotropic Besov spaces, *Math. Nachr.* , **239-240**(2002), 80-102.

[7] R.-Q. Jia and C.A. Micchelli, Using the refinement equations for construction of pre-wavelets II, in *Curves and surfaces*, P.J. Laurent et al.,eds., Academic Press, Boston, 1991, 209-246.

[8] J. Lei, R.-Q. Jia and E.W. Cheney, Approximation for shift-invariant spaces by integral operators, *SIAM J. Math. Anal.*, **28**(1997), 481-498.

[9] Y. Meyer, *Wavelets and operators*, Cambridge Univ. Press, Cambridge, 1992.

[10] Y. Meyer, *Wavelets, vibrations and scalings*, CRM Monograghs Series 9, Amer. Math. Soc., Providence, R. I., 1998.

[11] K. Saka, Scaling exponents of self-similar functions and wavelet analysis, *Proc. Amer. Math. Soc.*, **133**(2005), 1035-1045.

[12] W. Sickel, Spline representations of functions in Besov-Tribel-Lizorkin spaces on $\mathbb{R}^n$, *Form Math.*, **2**(1990), 451-475.

[13] H. Triebel, *Theory of Function Spaces*, Birkhäuser, Leipzig, 1983.

[14] Y. Wang, Self-affine Tiles, in *Adv. in Wavelets*, K.-S. Lau ed., Springer, Singapore, 1998, 261-282.

[15] P. Wojtaszczyk, *A mathematical introduction to wavelets*, Cambridge Univ. Prss, 1997.

[16] K. Zhao, Density of dilates of a principal shift-invariant subspace, *J. Math. Anal. Appl.*, **184**(1994), 517-532.