Title: Rudin's Dowker space is base-normal: a direct proof

General and Geometric Topology and Geometric Group Theory

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Citation: 数理解析研究所講究録 (2006), 1492: 94-99

Issue Date: 2006-05

URL: http://hdl.handle.net/2433/58266

Type: Departmental Bulletin Paper

Textversion: publisher

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Rudin’s Dowker space is base-normal  
— a direct proof —

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The theorem ‘Rudin’s Dowker space is base-normal’ was proved in [7] by using some results of K. P. Hart in [3]. In this report, we give a direct proof to this theorem.

Throughout this paper, all spaces are assumed to be $T_1$ topological spaces. The symbol $\mathbb{N}$ denotes the set of all natural numbers. As usual, a cardinal is the initial ordinal and an ordinal is the set of smaller ordinals. The cardinality of a set $X$ is denoted by $|X|$. For a space $X$, $w(X)$ stands for the weight of $X$. For a space $X$ and a subspace $A$ of $X$, the closure of $A$ in $X$ is denoted by $\overline{A}$.

Motivated by base-paracompactness of J. E. Porter [4], we introduced in [6] the notion of base-normality. Recall that a space $X$ is said to be base-normal if there is a base $\mathcal{B}$ for $X$ with $|\mathcal{B}| = w(X)$ satisfying that every pair of disjoint closed subsets $F_0, F_1$ of $X$ admits a locally finite cover $\mathcal{B}'$ of $X$ by members of $\mathcal{B}$ such that, for every $B \in \mathcal{B}'$, either $\overline{B} \cap F_0 = \emptyset$ or $\overline{B} \cap F_1 = \emptyset$ holds. A space $X$ is said to be base-collectionwise normal if there is a base $\mathcal{B}$ for $X$ with $|\mathcal{B}| = w(X)$ satisfying that every discrete closed collection $\{F_\alpha : \alpha \in \Omega\}$ of $X$ admits a locally finite cover $\mathcal{B}'$ of $X$ by members of $\mathcal{B}$ such that, for every $B \in \mathcal{B}'$, $|\{\alpha \in \Omega : \overline{B} \cap F_\alpha \neq \emptyset\}| \leq 1$. Note that every base-normal space is normal, and G. Gruenhage constructed in [2] a ZFC example of a countably compact zero-dimensional LOTS which is not base-normal.

Recall that a Dowker space is a normal space $X$ for which $X \times [0, 1]$ is not normal. In [6] we pointed out that a base-normal Dowker space can be constructed by using a technique of Porter in [4]. Indeed, let $Y$ be any Dowker space. Then, the direct sum $Y \oplus (\kappa + 1)$, where $\kappa$ is the cardinality of all open subsets of $Y$ and $\kappa + 1$ has the usual order topology, is a base-normal Dowker space (although $Y$ itself is not necessarily assumed to be base-normal) ([6]). Thus, it seems to be an interesting problem to find base-normal spaces among Dowker spaces which have been obtained so far. In fact, on the 3rd Japan-Mexico Joint Meeting on Topology and its Applications held in December, 2004, a participant asked a question if Rudin’s Dowker space is base-normal.
or not, and in [7] this question is affirmatively answered.

Let us first recall the construction of Rudin’s Dowker space in [5]. The symbol \( cf(\lambda) \) stands for the cofinality of \( \lambda \). Let

\[
F = \left\{ f : \mathbb{N} \to \omega_\omega : f(n) \leq \omega_n \text{ for all } n \in \mathbb{N} \right\}
\]

and

\[
X = \left\{ f \in F : \exists i \in \mathbb{N} \text{ such that } \omega < cf(f(n)) < \omega_i \text{ for all } n \in \mathbb{N} \right\}.
\]

Let \( f, g \in F \). Then, we define \( f < g \) if \( f(n) < g(n) \) for every \( n \in \mathbb{N} \), and define \( f \leq g \) if \( f(n) \leq g(n) \) for every \( n \in \mathbb{N} \). Moreover, define

\[
U_{f,g} = \{ h \in X : f < h \leq g \}.
\]

The set \( \{U_{f,g} : f, g \in F\} \) is a base for a topology of \( X \). The space \( X \) is Rudin’s Dowker space. We set \( B = \{U_{f,g} : f, g \in F\} \). Note that \( w(X) = \omega_\omega^\omega = |B| \).

For \( U \subset F \), define a map \( t_U \in F \) by \( t_U(n) = \sup\{f(n) : f \in U\} \) for each \( n \in \mathbb{N} \). For undefined terminology, see [1].

To prove base-normality of Rudin’s Dowker space, we give a more strict result as follows.

**Theorem.** Let \( X \) be Rudin’s Dowker space, and \( B \) the base for \( X \) defined as above. For every discrete closed collection \( \{F_\alpha : \alpha \in \Omega\} \) of \( X \), there is a disjoint cover \( B' \) of \( X \) by members of \( B \) satisfying that, for every \( B \in B' \), \(|\{\alpha \in \Omega : B \cap F_\alpha \neq \emptyset\}| \leq 1\).

This theorem was proved in [7, Theorem 3.4] by using results in [3]. As was announced in the introduction, we directly prove this.

**Proof of Theorem.** First show the following statements are valid.

(i) \( X \in B \).

(ii) If \( U(1), U(2) \in B \), then \( U(1) \cap U(2) \in B \).

(iii) If \( U(i) \in B \), \( i \in \mathbb{N} \), then \( \bigcap_{i \in \mathbb{N}} U(i) \in B \).

Indeed, (i) is easy to see and (ii) follows from (i) and (iii), so we only give a proof of (iii). To prove (iii), let \( U(i) \in B \), \( i \in \mathbb{N} \). Then, each \( U(i) \) is expressed as \( U(i) = U_{f_i, g_i} \) for some \( f_i, g_i \in F \). Define \( f, g \in F \) by \( f(n) = \sup_{i \in \mathbb{N}} f_i(n) \), \( n \in \mathbb{N} \), and \( g(n) = \min_{i \in \mathbb{N}} g_i(n) \), \( n \in \mathbb{N} \). Notice that \( f \notin X \). Hence, we have \( \bigcap_{i \in \mathbb{N}} U_{f_i, g_i} = U_{f,g} \). Thus, \( \bigcap_{i \in \mathbb{N}} U(i) \in B \).

Next, we show the following:
Claim. For every disjoint closed subsets $F_0, F_1$ of $X$, there is a disjoint cover $B'$ of $X$ by members of $B$ such that, for every $B \in B'$, either $B \cap F_0 = \emptyset$ or $B \cap F_1 = \emptyset$ holds.

To show this, let $F_0$ and $F_1$ be disjoint closed subsets of $X$. The proof in [5] makes for each countable ordinal $\alpha$ a disjoint open collection $J_\alpha$ of $X$ which covers $F_0 \cup F_1$. We modify the proof in [5] so as to make disjoint open covers $J_\alpha$ of $X$ (consisting of members of $B$).

Inductively, we construct disjoint open covers $J_\alpha$ of $X$, $0 \leq \alpha < \omega_1$, with $J_\alpha \subset B$ having the following property:

For every $\beta < \alpha$ and every $V \in J_\alpha$, there exists $U \in J_\beta$ such that

1. $V \subset U$,
2. if $V \cap F_0 \neq \emptyset \neq V \cap F_1$, then $t_V \neq t_U$,
3. if $U \cap F_0 = \emptyset$ or $U \cap F_1 = \emptyset$, then $U = V$.

First, set $J_0 = \{X\}$. By (i), it follows that $X \in B$, hence $J_0 \subset B$.

Next, assume that $J_\beta$ has been constructed for every $\beta < \alpha$.

Case 1. $\alpha$ is limit. For every $f \in X$ and every $\beta < \alpha$, choose a unique $U(f)_\beta$ such that $f \in U(f)_\beta \in J_\beta$. Define

$$U_f = \bigcap_{\beta<\alpha} U(f)_\beta \quad \text{for every } f \in X, \quad \text{and } J_\alpha = \{U_f : f \in X\}.$$

Then, by (iii), it follows that $J_\alpha \subset B$. Moreover, $J_\alpha$ is a disjoint cover of $X$ because each $J_\beta$ is a disjoint cover of $X$. Fix $\beta < \alpha$. We shall show that $U_f$ and $U(f)_\beta$ satisfying conditions (1), (2) and (3) above. Since $U_f \subset U(f)_\beta$, (1) holds. To show (2), assume $U_f \cap F_0 \neq \emptyset \neq U_f \cap F_1$. Then, $U(f)_{\beta+1} \cap F_0 \neq \emptyset \neq U(f)_{\beta+1} \cap F_1$. Hence, it follows from the assumption of induction that $t_{U(f)_{\beta+1}} \neq t_{U(f)_\beta}$. Since $t_{U_f} \leq t_{U(f)_{\beta+1}} \leq t_{U(f)_\beta}$, we have $t_{U_f} < t_{U(f)_\beta}$, so (2) holds. To show (3), assume either $U(f)_\beta \cap F_0 = \emptyset$ or $U(f)_\beta \cap F_1 = \emptyset$ holds. Then, since $U(f)_\beta = U(f)_{\beta'}$ for every $\beta'$ with $\beta < \beta' < \alpha$, we have $U(f)_{\beta} = U(f)_{\beta'}$. It follows that $U_f = U(f)_\beta$. So, (3) holds.

Case 2. $\alpha = \beta + 1$. Fix $U \in J_\beta$. We shall construct a disjoint cover $J(U)$ of $U$ with $J(U) \subset B$ so as to have the following property:

For every $V \in J(U)$,

1. if $V \cap F_0 \neq \emptyset \neq V \cap F_1$, then $t_V \neq t_U$,
2. if $U \cap F_0 = \emptyset$ or $U \cap F_1 = \emptyset$, then $U = V$.

Case A. $U \cap F_0 = \emptyset$ or $U \cap F_1 = \emptyset$. Define

$$J(U) = \{U\}.$$

Then, $J(U) \subset B$, and $U$ satisfies conditions $2'$ and $3'$.
Case B. $U \cap F_0 \neq \emptyset \neq U \cap F_1$, and there exists $i \in \mathbb{N}$ such that $cf(t_U(i)) \leq \omega$. Then, we select $i_U$ so as to satisfy $cf(t_U(i_U)) \leq \omega$. Then, as in [5], we can show that $cf(t_U(i_U)) = \omega$. Choose an increasing sequence $\{\lambda_U(n) : n \in \mathbb{N}\}$ of terms of $t_U(i_U)$ cofinal with $t_U(i_U)$. Set

$V(U, n) = \{f \in U : \lambda_U(n-1) < f(i_U) \leq \lambda_U(n)\}$
for each $n \in \mathbb{N}$. Define

$\mathcal{J}(U) = \{V(U, n) : n \in \mathbb{N}\}$.

Note that $V(U, n) = U_{f,g} \cap U$, where $f, g \in F$ is defined by $f(i_U) = \lambda_U(n-1)$ and $f(n) = 0$ if $n \neq i_U$, and $g(i_U) = \lambda_U(n)$ and $g(n) = \omega_n$ if $n \neq i_U$. Since $U_{f,g}, U \in B$, it follows from (ii) that $V(U, n) \in B$. Thus, $\mathcal{J}(U) \subset B$. For every $M \subset \mathbb{N}$, set

$V(U, M, f_U) = \{h \in U : \text{for every } h(n) \text{ for every } n \in M \text{, and } h(n) > f_U(n) \text{ for every } n \in \mathbb{N} - M\}$

Define

$\mathcal{J}(U) = \{V(U, M, f_U) : M \subset \mathbb{N}\}$.

Likewise the proof of Case B, by (ii), we can show that $V(U, M, f_U) \in B$ for each $M \subset \mathbb{N}$. Thus, $\mathcal{J}(U) \subset B$. Also, we can show that $\mathcal{J}(U)$ is a disjoint cover of $U$. Finally, it is not difficult to show $V(U, M, f_U)$ and $U$ satisfy conditions (2)' and (3)'.

Case C. $U \cap F_0 \neq \emptyset \neq U \cap F_1$, and $cf(t_U(n)) > \omega$ for every $n \in \mathbb{N}$. By the quite similar proof to those of [5, Lemmas 5 and 6], we can select $f_U \in F$ such that $f_U < t_U$ and such that either $\{h \in U : f_U < h\} \cap F_0 = \emptyset$ or $\{h \in U : f_U < h\} \cap F_1 = \emptyset$ holds. For every $M \subset \mathbb{N}$, set

$V(U, M, f_U) = \{h \in U : h(n) \leq f_U(n) \text{ for every } n \in M, \text{ and } h(n) > f_U(n) \text{ for every } n \in \mathbb{N} - M\}$

Define

$\mathcal{J}(U) = \{V(U, M, f_U) : M \subset \mathbb{N}\}$.

Likewise the proof of Case B, by (ii), we can show that $V(U, M, f_U) \in B$ for each $M \subset \mathbb{N}$. Thus, $\mathcal{J}(U) \subset B$. Also, we can show that $\mathcal{J}(U)$ is a disjoint cover of $U$. Finally, it is not difficult to show $V(U, M, f_U)$ and $U$ satisfy conditions (2)' and (3)'.

Set

$\mathcal{J}_\alpha = \bigcup_{U \in \mathcal{J}_\beta} \mathcal{J}(U)$.

By using conditions (2)' and (3)' above and the assumption of induction, we can show that $\mathcal{J}_\alpha, 0 \leq \alpha < \omega_1$, have the required property.

For every $f \in X$ and every $\alpha$ with $0 \leq \alpha < \omega_1$, there exists a unique $U(f)_\alpha \in \mathcal{J}_\alpha$ such that $f \in U(f)_\alpha$. Let $\beta$ and $\alpha$ with $\beta < \alpha < \omega_1$. Then, we have $U(f)_\alpha \subset U(f)_\beta$, hence $t_{U(f)_\alpha} \leq t_{U(f)_\beta}$. If $U(f)_\alpha \cap F_0 \neq \emptyset \neq U(f)_\alpha \cap F_1,$
then \( t_{U(f)}(n) < t_{U(f)}(n) \) for some \( n \in \mathbb{N} \). As in [5], for every \( n \in \mathbb{N} \) one can move backward in \( \omega_n \) only finitely many steps. Hence, there exists \( \alpha(f) < \omega_1 \) such that

\[
U(f)_{\alpha(f)} \cap F_0 = \emptyset \quad \text{or} \quad U(f)_{\alpha(f)} \cap F_1 = \emptyset.
\]

By (3), if \( \alpha(f) < \beta < \omega_1 \) then \( U(f)_{\beta} = U(f)_{\alpha(f)} \). Clearly, \( \{U(f)_{\alpha(f)} : f \in X\} \) is a cover of \( X \) consisting of elements of \( B \). To prove \( \{U(f)_{\alpha(f)} : f \in X\} \) is pairwise disjoint, assume \( U(f)_{\alpha(f)} \cap U(g)_{\alpha(g)} \neq \emptyset \). Take \( \beta < \omega_1 \) so as to satisfy \( \alpha(f) < \beta \) and \( \alpha(g) < \beta \). It follows from \( U(f)_{\beta} = U(f)_{\alpha(f)} \) and \( U(g)_{\beta} = U(g)_{\alpha(g)} \) that \( U(f)_{\beta} \cap U(g)_{\beta} \neq \emptyset \). Since \( \mathcal{J}_\beta \) is pairwise disjoint, we have \( U(f)_{\beta} = U(g)_{\beta} \), hence \( U(f)_{\alpha(f)} = U(g)_{\alpha(g)} \). This shows that \( \{U(f)_{\alpha(f)} : f \in X\} \) is pairwise disjoint, and this is the required \( B' \) in Claim.

Finally, to complete the proof, let \( \{F_\alpha : \alpha \in \Omega\} \) be a discrete closed collection of \( X \). Since \( X \) is collectionwise normal, there is a discrete open collection \( \{U_\alpha : \alpha \in \Omega\} \) of \( X \) such that \( F_\alpha \subset U_\alpha \) for each \( \alpha \in \Omega \). Due to the fact shown above, for every \( \alpha \in \Omega \), there is a disjoint cover \( B_\alpha \) of \( X \) by members of \( B \) such that, for every \( B \in B_\alpha \), either \( B \cap F_\alpha = \emptyset \) or \( B \subset U_\alpha \) holds. For every \( \alpha \in \Omega \), define

\[
B^*_\alpha = \{B \in B_\alpha : B \subset U_\alpha\}.
\]

Note that \( F_\alpha \subset \bigcup B^*_\alpha \subset U_\alpha \) for every \( \alpha \in \Omega \). Set

\[
B^* = \bigcup_{\alpha \in \mathbb{N}} B^*_\alpha.
\]

Since \( \bigcup B^*_\alpha \) is clopen for each \( \alpha \in \Omega \), and \( \{\bigcup B^*_\alpha : \alpha \in \Omega\} \) is discrete in \( X \), it follows that \( B^* \) is clopen in \( X \). Hence, by the fact shown in the above, there is a disjoint cover \( C \) of \( X \) by members of \( B \) such that, for each \( C \in \mathcal{C} \), either \( C \cap B^* = \emptyset \) or \( C \subset B^* \) holds. Then, \( \{C \in \mathcal{C} : C \cap B^* = \emptyset\} \cup \mathcal{C} \) is the required disjoint cover of \( X \) by members of \( B \). This completes the proof.  

The notion of base-normality is motivated by the well-known fact that \( X \) is normal if and only if every pair of disjoint closed subsets \( F_0, F_1 \) of \( X \) admits a locally finite open cover \( \mathcal{U} \) of \( X \) such that, for every \( U \in \mathcal{U} \), either \( U \cap F_0 = \emptyset \) or \( U \cap F_1 = \emptyset \) holds. On the other hand, it is easy to see that “locally finite” in the above fact can be replaced by “star-finite”; a collection \( \{U_\alpha : \alpha \in \Omega\} \) of subsets of \( X \) is said to be star-finite if \( |\{\beta \in \Omega : U_\beta \cap U_\alpha \neq \emptyset\}| < \omega \) holds for every \( \alpha \in \Omega \). In order to consider a base version of this fact, we define a space \( X \) to be strongly base-normal if there is a base \( B \) for \( X \) with \( |B| = w(X) \) satisfying that every pair of disjoint closed subsets \( F_0, F_1 \) of \( X \) admits a star-finite cover \( B' \) of \( X \) by members of \( B \) such that, for every
$B \in B'$ either $\overline{B} \cap F_0 = \emptyset$ or $\overline{B} \cap F_1 = \emptyset$ holds. The Theorem in the above shows that Rudin's Dowker space possesses this property. Also, note that there is a base-normal space (in fact, a metric space) which is not strongly base-normal ([7]). Related results on strongly base-normal spaces, see [7].

References


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