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Weak topologies, and determining covers

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We assume that spaces are regular $T_1$, and maps are continuous and onto.

For a cover $\mathcal{P}$ of a space $X$, $X$ is determined by $\mathcal{P}$ [6], if $X$ has the weak topology with respect to $\mathcal{P}$ [3]; that is, $G \subset X$ is open in $X$ if $G \cap P$ is open in $P$ for each $P \in \mathcal{P}$. Here, we can replace “open” by “closed”. We call such a cover $\mathcal{P}$ a determining cover in [20].

We recall that a space $X$ is respectively a sequential space [4]; $k$-space; quasi-$k$-space [11] if $X$ has a determining cover by (compact) metric subsets; compact subsets; countably compact subsets. Sequential spaces are $k$-spaces, and $k$-spaces are quasi-$k$-spaces.

As is well-known, every sequential space; $k$-space; quasi-$k$-space is respectively characterized as a quotient space of a (locally compact) metric space; locally compact (paracompact) space; $M$-space.

Let $\mathcal{P}$ be a collection of subsets of a space $X$. Then, $\mathcal{P}$ is closure-preserving (abbreviated by CP), if for any subfamily $\mathcal{P}'$ of $\mathcal{P}$, $cl(\bigcup\{P : P \in \mathcal{P}'\}) = \bigcup\{clP : P \in \mathcal{P}'\}$. Also, $\mathcal{P}$ is hereditarily closure-preserving (abbreviated by HCP), if for any subcollection $\mathcal{P}' = \{P_\alpha : \alpha\}$ of $\mathcal{P}$, and any $\{A_\alpha : \alpha\}$ such that $A_\alpha \subset P_\alpha$, the collection $\{A_\alpha : \alpha\}$ is CP.

For a closed cover $\mathcal{F}$ of a space $X$, $X$ is dominated by $\mathcal{F}$ [7] if $\mathcal{F}$ is a CP cover, and any $\mathcal{P} \subset \mathcal{F}$ is a determining cover of the union of $\mathcal{P}$. (Sometimes, we also say that $X$ has the Whitehead weak topology; Morita weak topology (in the sense of [9]); or hereditarily weak topology, with respect to $\mathcal{F}$). We call such a closed cover $\mathcal{F}$ a dominating cover in [20].

A space $X$ having an increasing determining cover $\{X_n : n \in N\}$ is called the inductive limit of $\{X_n : n \in N\}$. When the $X_n$ are closed in $X$, $\{X_n : n \in N\}$ is a dominating cover of $X$. Also, every CW-complex has a dominating cover by compact metric subsets.

Open covers $\Rightarrow$ Determining covers $\Leftarrow$ Dominating covers $\Leftarrow$ HCP closed covers $\Leftarrow$ Locally finite closed covers.

Every space having a determining cover by sequential spaces (resp. $k$-spaces; quasi-$k$-spaces) is a sequential space (resp. $k$-space; quasi-$k$-space).
While, every space having a dominating cover by paracompact spaces (resp. normal spaces) is paracompact (resp. normal); see [7] or [10].

Concerning “preservations” of weak topologies, we have the following natural questions (Q1), (Q2) and (Q3), and the same questions which are replaced “determining” by “dominating”.

(Q1) Let $f : X \to Y$ be a map, and let $\mathcal{P}$ be a determining cover of $X$ (resp. $Y$). Under what conditions, is $f(\mathcal{P}) = \{f(P) : P \in \mathcal{P}\}$ (resp. $f^{-1}(\mathcal{P}) = \{f^{-1}(P) : P \in \mathcal{P}\}$) a determining cover of $Y$ (resp. $X$)?

(Q2) Let $\mathcal{P}$ be a determining cover of a space $X$. For a (or any) subset $S \subseteq X$, under what conditions, is $\mathcal{P}|S = \{P \cap S : P \in \mathcal{P}\}$ a determining cover of $S$?

(Q3): For each $i = 1, 2$, let $\mathcal{P}_i$ be a determining cover of a space $X_i$. Under what conditions, is $\mathcal{P}_1 \times \mathcal{P}_2 = \{P_1 \times P_2 : P_i \in \mathcal{P}_i\}$ a determining cover of $X_1 \times X_2$?

In [20], we gave some related answers to the question (Q3) (containing countable products of weak topologies), and their applications to products of paracompact spaces. For products of weak topologies (determining covers), see [19]. In this paper, let us give some related answers to the questions (Q1) and (Q2) in Section 1 and 2, respectively. Related to (Q3), we also give some results on countable products of spaces having certain determining covers in Section 3, containing additional matters to [20].

We recall some elementary facts which will be used in this paper. For basic matters on weak topologies, see [17] or [18], for example.

**Fact A:** (1) Let $C$ be a determining cover of $X$. Let $\mathcal{P}$ be a cover of $X$. If each element of $C$ is contained in some element of $\mathcal{P}$, then $\mathcal{P}$ is a determining cover of $X$.

(2) Let $\{P_\alpha : \alpha\}$ be a determining cover of $X$. If each $P_\alpha$ has a determining cover $\mathcal{P}_\alpha$, then $\bigcup \{\mathcal{P}_\alpha : \alpha\}$ is a determining cover of $X$.

(3) Let $\mathcal{P}$ be a determining cover of $X$. If $S$ is a closed or open subset of $X$, then $\mathcal{P}|S$ is a determining cover of $S$.

(4) For a determining cover $\mathcal{P}$ of a space $X^\omega$, $\mathcal{P}_1 \times \mathcal{P}_2 \times \cdots$ is a determining cover of $X^\omega$, where $\mathcal{P}_i = P_i(\mathcal{P})$ for the projection $P_i$ from $X^\omega$ onto the $i$-th coordinate space $X$.

A cover $\mathcal{P}$ of $X$ is **point-countable** if every $x \in X$ is in at most countably many $P \in \mathcal{P}$. A decreasing sequence $(A_n)$ of non-empty subsets of $X$ is a $q$-
sequence (resp. \textit{k-sequence}) [8], if $C = \bigcap \{A_n : n \in \mathbb{N}\}$ is countably compact (resp. compact) in $X$, and each open subset $U$ with $C \subseteq U$ contains some $A_n$ (equivalently, for any $x_n \in A_n$, \{x_n : n \in \mathbb{N}\} has an accumulation point in $C$).

\textbf{Fact B:} (1) Let $\mathcal{P}$ be a point-countable determining cover of $X$. Then, for each $q$-sequence $(A_n)$ in $X$, some $A_n$ is contained in a finite union of elements of $\mathcal{P}$ ([14, Lemma 6]).

(2) Let $\mathcal{F} = \{X_\alpha : \alpha \leq \gamma\}$ be a dominating cover of $X$. For each $\alpha \leq \gamma$, let $L_\alpha = X_\alpha - \bigcup \{X_\beta : \beta < \alpha\}$. Then $\{clL_\alpha : \alpha \leq \gamma\}$ is a determining cover of $X$ such that, for each $q$-sequence $(A_n)$ in $X$, some $A_n$ meets only finitely many $L_\alpha$ (cf. [16, Lemma 2.5]).

1. Maps

\textbf{Example 1.1.} (1) An open map $f : X \to Y$ with each $f^{-1}(y)$ at most two points, and $X$ has a discrete, closed and open cover $\mathcal{F}$ by compact subsets, but $f(\mathcal{F})$ is not a CP cover (hence, not a dominating cover).

(2) An open map $g : X \to Y$ with each $g^{-1}(y)$ at most two points, and $Y$ has a countable determining cover $\mathcal{F}$ by convergent sequences (or, a dominating cover by compact metric subsets), but $g^{-1}(\mathcal{F})$ is not a determining cover of $X$.

\textbf{Theorem 1.2.} (1) Let $f : X \to Y$ be a quotient map. If $\mathcal{P}$ is a determining cover of $X$, as is well-known, $f(\mathcal{P})$ is a determining cover of $Y$.

(2) Let $f : X \to Y$ be a closed map. Then the following hold.

(a) If $\mathcal{F}$ is a dominating cover of $X$, $f(\mathcal{F})$ is a dominating cover of $Y$.

(b) If $\mathcal{P}$ is a determining (resp. dominating) cover of $Y$, $f^{-1}(\mathcal{P})$ is a determining (resp. dominating) cover of $X$ ([13, Lemma 1.2]).

\textbf{Corollary 1.3.} Let $f : X \to Y$ be a closed map such that each $f^{-1}(y)$ is compact (resp. countably compact; first countable). Then $X$ is a k-space ([1]) (resp. quasi-k-space; sequential space ([13])) if (and only if) $Y$ is so respectively.

\textbf{Corollary 1.4.} Let $f : X \to Y$ be a map. Then the following hold.

(1) Let $X$ be a k-space. If $\mathcal{P}$ is a determining cover of $Y$, then $f^{-1}(\mathcal{P})$ is a determining cover of $X$.

(2) Let $X$ be a quasi-k-space. If $\mathcal{F}$ is a dominating (resp. point-countable closed) cover of $Y$, then $f^{-1}(\mathcal{F})$ is a dominating (resp. point-countable closed) cover of $X$. 

The author doesn't know whether the above (1) remains true under $X$ being a quasi-$k$-space. This is affirmative if any countably compact subset of $Y$ is closed (as $Y$ is a sequential space, or a space whose points are $G_{4}$-sets, for example).

### 2. Subsets

For an open (resp. HCP closed) cover $\mathcal{P}$ of $X$, $\mathcal{P}|S$ is a determining (resp. dominating) cover of $S$ for any $S \subset X$. But, we have the following example. Here, the Arens' space $S_{2}$ is the space obtained from the disjoint union $\Sigma\{L_{n} : n = 0, 1, \cdots\}$ of the convergent sequence $\{1/n : n \in N\} \cup \{0\}$ by identifying each $1/n \in L_{0}$ with $0 \in L_{n}$ ($n \geq 1$). The quotient space $S_{2}/L_{0}$ is called the *sequential fan* which is denoted by $S_{\omega}$.

**Example 2.1.** The Arens' space $S_{2}$ has the obvious increasing and dominating countable cover $\mathcal{F}$ by compact metric subsets, but for $S = S_{2} - \{1/n \in L_{0} : n \in N\}$, $\mathcal{F}|S$ is not a determining cover of $S$.

A space is *Fréchet*, if whenever $a \in clA$, then there exists a sequence in $A$ converging to the point $a$. We recall that a space $X$ is a *$k'$-space* [1], if whenever $a \in clA$, then there exists a compact subset $K$ of $X$ such that $a \in cl(A \cap K)$. Let us recall other related spaces due to [8]. A space $X$ is a *countably bi-quasi-$k$-space* if, whenever $x \in clA_{n}$ with $A_{n+1} \subset A_{n}$ ($n \in N$), there exists a $q$-sequence $(B_{n})$ such that $x \in cl(A_{n} \cap B_{n})$ for all $n \in N$. If the $A_{n}$ are all the same set, then such a space is a *singly bi-quasi-$k$-space*. Fréchet spaces and locally compact spaces are $k'$-spaces. $k'$-spaces, and $M$-spaces (generally, countably bi-quasi-$k$-spaces) are singly bi-quasi-$k$-spaces. Singly bi-quasi-$k$-spaces are quasi-$k$-spaces. For properties related to dominating or point-countable determining covers among singly bi-$k$-spaces (or, singly bi-quasi-$k$-spaces), see [15] or [21].

**Theorem 2.2.** (1) Let $\mathcal{P}$ be a determining cover of $X$. For $S \subset X$, $\mathcal{P}|S$ is a determining cover of $S$ if $S$ has a determining cover by open or closed sets in $X$, in particular, $S$ is a $k$-space. When $\mathcal{P}$ is point-countable and closed, the same result holds if $S$ is a quasi-$k$-space.

(2) Let $\mathcal{F}$ be a dominating cover of $X$. For $S \subset X$, $\mathcal{F}|S$ is a dominating cover if $S$ has a determining cover by open or closed subsets in $X$, or $S$ is a quasi-$k$-space.

**Corollary 2.3.** Let $\mathcal{F}$ be a dominating cover of $X$ by Fréchet spaces. For $S \subset X$, the following are equivalent.

(a) $S$ has a dominating cover $\mathcal{F}|S$.

(b) $S$ has a determining cover $\mathcal{F}|S$. 
(c) $S$ is a quasi-$k$-space.
(d) $S$ is a sequential space.

Theorem 2.4. (1) Let $\mathcal{P}$ be a cover of $X$. Then the following are equivalent.
   (a) For any $S \subset X$, $\mathcal{P}|S$ is a determining cover of $S$.
   (b) For any $A \subset X$ and any $a \in clA$, there exists $P \in \mathcal{P}$ such that $a \in cl_P(A \cap P)$.

(2) Let $\mathcal{F}$ be a closed cover of $X$. Then the following are equivalent.
   (a) For any $S \subset X$, $\mathcal{F}|S$ is a dominating cover of $S$.
   (b) For any $S \subset X$, $\mathcal{F}|S$ is CP in $X$.

Corollary 2.5. Let $X$ be a singly bi-quasi-$k$-space, and let $\mathcal{F}$ be a dominating (resp. point-countable determining closed) cover of $X$. Then, for any $S \subset X$, $\mathcal{F}|S$ is a dominating (resp. determining) cover of $S$.

Corollary 2.6. (1) For a space $X$, the following are equivalent.
   (a) $X$ is Fréchet.
   (b) $X$ has a determining cover $\mathcal{P}$ by compact metric subsets such that for any $S \subset X$, $\mathcal{P}|S$ is a determining cover of $S$.
   (c) $X$ is a sequential space, and for any determining cover $\mathcal{P}$ of $X$ and any $S \subset X$, $\mathcal{P}|S$ is a determining cover of $S$.

(2) For a space $X$, the following are equivalent.
   (a) $X$ is a $k'$-space.
   (b) $X$ has a determining cover $\mathcal{P}$ by compact subsets such that for any $S \subset X$, $\mathcal{P}|S$ is a determining cover of $S$.

Corollary 2.7. Let $X$ be a sequential space. If any subset of $X$ is a quasi-$k$-space, then $X$ is Fréchet (cf. [5]).

Corollary 2.8. (1) Let $X$ have a determining cover $\mathcal{P}$ by Fréchet spaces. Then the following are equivalent.
   (a) $X$ is Fréchet.
   (b) For any $S \subset X$, $\mathcal{P}|S$ is a determining cover of $S$.

(2) Let $X$ have a dominating (or point-countable determining closed) cover $\mathcal{F}$ by $k'$-spaces. Then the following are equivalent.
   (a) $X$ is a $k'$-space.
   (b) For any $S \subset X$, $\mathcal{F}|S$ is a determining cover of $S$.

Remark 2.9. Not every compact sequential space is Fréchet (the space $\Psi^*$ in [5, Example 7.1], for example). Thus, in (c) $\Rightarrow$ (a) of Corollary 2.6(1), we can't replace "determining" by "dominating". While, under $X$ being a $k$-space, (c) implies $X$ is a $k'$-space, but the converse need not hold even if
$X$ is compact sequential. Also, in (a) $\Rightarrow$ (b) of Corollary 2.8(2), we can't replace "dominating" by "determining".

**Question 2.10.** (1) Let $\mathcal{P}$ be a determining cover of $X$. Let $S \subset X$, and $S$ be a quasi-$k$-space. Is $\mathcal{P}|S$ a determining cover of $S$?

(2) Let $\mathcal{F}$ be a dominating cover of $X$. For any $S \subset X$, let $\mathcal{F}|S$ be a determining cover of $S$. Is $\mathcal{F}|S$ a dominating cover of $S$?

(3) Let $X$ be a $k$-space. For any determining cover $\mathcal{P}$ of $X$, and any $S \subset X$, let $\mathcal{P}|S$ be a determining cover of $S$. Is $X$ Fréchet?

3. **Countable products**

In this section, we consider countable products of weak topologies, as additional matters to Section 4 in [20]. For finite products of weak topologies in terms of Question 3, see [19] or [20]. First, let us give the following notations.

For a cover $\mathcal{P}$ of a space, let $[\mathcal{P}] = \{A : A$ is a finite union of elements of $\mathcal{P}\}$, $\mathcal{P}^* = \{P \cup F : P \in \mathcal{P}, F$ is finite\}, and let $\mathcal{P}^\omega = \{intP : P \in \mathcal{P}\}$.

**Remark 3.1.** (1) For a space $X = F_1 + F_2$, $\mathcal{F} = \{F_1, F_2\}$ is a determining cover of $X$, but $\mathcal{F}^\omega (= \mathcal{F} \times \mathcal{F} \times \cdots )$ is not a determining cover of $X^\omega$.

(2) Let $X$ be the sequential fan $S_\omega$ (or the Arens' space $S_2$). Then, for any (countable) determining closed cover $\mathcal{F}$ by (compact) metric subsets in $X$, $[\mathcal{F}]^\omega$ is not a determining cover of $X^\omega$ by means of Theorem 3.2(2) below.

As a generalization of sequential spaces, we recall that a space $X$ has countable tightness, $t(X) \leq \omega$, if whenever $a \in clA$, $a \in clC$ for some countable $C \subset A$ (equivalently, $X$ has a determining cover by countable subsets); see [8]. While, as a generalization of countably bi-quasi-$k$-spaces, let us consider the following property (P), referring to [6, (3.1)].

(P): For each decreasing sequence $(A_n)$ in $X$ with $\bigcap\{clA_n : n \in N\} \neq \emptyset$, there exists a countably compact set $K$ of $X$ with $K \cap A_n \neq \emptyset$ for all $n \in N$.

**Theorem 3.2.** (1) Let $X^\omega$ be a sequential space. Let $\mathcal{P}$ be a determining cover of $X$. Then $\mathcal{P}^\omega$ (hence, $[\mathcal{P}]^\omega$) is a determining cover of $X^\omega$ ([13]).

(2) Let $X^\omega$ be a quasi-$k$-space. Let $\mathcal{P}$ be a dominating or point-countable determining cover of $X$. Then the following hold.

(a) $[\mathcal{P}]^\omega$ is a determining cover of $X^\omega$.

(b) If $t(X) \leq \omega$, then $X$ has property (P), hence $[\mathcal{P}]^\omega$ is a determining cover of $X^\omega$. 


A space $X$ is a bi-$k$-space [8] if, whenever $A$ is a filterbase with $x \in cl A$ for every $A \in A$, there exists a $k$-sequence $(B_n)$ in $X$ such that $x \in cl(A \cap B_n)$ for all $A \in A$ and $n \in N$. Locally compact spaces, first countable spaces, or paracompact $M$-spaces are bi-$k$-spaces. Bi-$k$-spaces are $k$-spaces which are countably bi-quasi-$k$. Every countable product of bi-$k$-spaces is a bi-$k$-space ([8]), hence a $k$-space.

**Corollary 3.3.** Let $X$ be a bi-$k$-space, and let $\mathcal{P}$ be a determining cover of $X$. Then the following hold.

(a) $[\mathcal{P}]^\omega$ is a determining cover of $X^\omega$ if $X$ is sequential, or $\mathcal{P}$ is a point-countable cover.

(b) $[\mathcal{P}]^{0\omega}$ is a determining cover of $X^\omega$ if $\mathcal{P}$ is a dominating cover, a point-countable closed cover, or a point-countable cover with $t(X) \leq \omega$.

**Corollary 3.4.** Let $X$ have a dominating or point-countable determining closed cover $\mathcal{F}$ by first countable spaces. Then the following properties are equivalent ([20]).

(a) $X^\omega$ is a quasi-$k$-space.

(b) $X^\omega$ is a sequential space.

(c) $\mathcal{F}^\omega$ is a determining cover of $X^\omega$.

(d) $\mathcal{F}^\omega$ (actually, $[\mathcal{F}]^{\omega}$) is a determining cover of $X^\omega$.

(e) $\mathcal{F}^\circ$ is an open cover of $X$.

(f) $X$ is first countable.

**Corollary 3.5.** Let $X$ satisfy (a), (b), or (c) below. If $X^\omega$ is a quasi-$k$-space, then $X$ is metric.

(a) $X$ has a dominating cover by metric spaces.

(b) $X$ is a paracompact space having a point-countable determining closed cover by metric spaces.

(c) $X$ has a point-countable determining cover by locally separable, metric spaces.

**Corollary 3.6.** Let $X$ have a dominating or point-countable determining closed cover $\mathcal{F}$ by locally compact spaces (resp. bi-$k$-spaces). Then the implications (a) $\iff$ (b) $\iff$ (c), and (d) $\iff$ (e) $\Rightarrow$ (b) hold. When $t(X) \leq \omega$, (a) $\sim$ (e) are equivalent.

(a) $X^\omega$ is a quasi-$k$-space.

(b) $X^\omega$ is a $k$-space.

(c) $\mathcal{F}^\omega$ is a determining cover of $X^\omega$.

(d) $\mathcal{F}^\circ$ is an open cover of $X$.

(e) $X$ is a locally compact space (resp. bi-$k$-space).
Remark 3.7. (CH). "$t(X) \leq \omega$" is essential in Corollary 3.6 (the implication (b) $\Rightarrow$ (d) or (e)), and so is in Theorem 3.2(2). Actually, under (CH), there exists a space $X$ having a countable dominating cover $\mathcal{F}$ by compact subsets, and $X^\omega$ is a $k$-space, but $X$ is not locally compact ([2]) (hence, $[\mathcal{F}]^\omega$ is a determining cover of $X^\omega$, but $[\mathcal{F}]^\circ$ is not an open cover of $X$, and $X$ doesn’t have property (P)).

Finally, let us give questions on products of weak topologies. First, let us review some related matoes.

Remark 3.8. (1) Let $X$ be a sequential space (resp. paracompact space). Then $X^\omega$ is a sequential space (resp. $k$-space) iff $X$ is a quasi-$k$-space (see [12] for the finite products).

(2) Let $\mathcal{P}$ be a determining cover of $X$. Then (a) and (b) below hold.

(a) Let $X^2$ be a $k$-space. Then $\mathcal{P}^2$ is a determining cover of $X^2$ if $X$ is a sequential space, or each element of $\mathcal{P}$ is a $k$-space (see [19] or [20]), in particular, $\mathcal{P}$ is a closed cover.

(b) Let $X^\omega$ be a $k$-space. Then $[\mathcal{P}]^\omega$ is a determining cover of $X^\omega$ if $\mathcal{P}$ is a dominating or point-countable cover, or $X$ is sequential (for example, the elements of $\mathcal{P}$ are sequential).

In view of Remark 3.8, the author has Question 3.9 below, in particular. For (1), $X^2 \in [\mathcal{P}]^2$. Also, the compactness of $X$ is essential even if $\mathcal{P}$ is a countable HCP closed cover by separable metric subsets. If (1) is affirmative, then so is the question for $X$ being a bi-$k$-space (generally, space with $X^\omega$ a $k$-space). For (2), any $\mathcal{F}^n$ ($n \in N$) is a determining cover of $X^n$. (3) is affirmative if $X$ is sequential, or $\mathcal{P}$ is dominating or point-countable. If $X$ is paracompact, then any $\mathcal{F}^n$ ($n \in N$) is a determining cover of $X^n$.

Question 3.9. (1) Let $X$ be a compact space, and let $\mathcal{P}$ be a countable determining cover of $X$. Is $\mathcal{P}^2$ a determining cover of $X^2$?

(2) Let $X$ be a compact space (or space with $X^\omega$ a $k$-space), and let $\mathcal{F}$ be a determining closed cover of $X$. Is $[\mathcal{F}]^\omega$ a determining cover of $X^\omega$?

(3) Let $\mathcal{F}$ be a determining cover of $X$ by compact subsets. Let $X^\omega$ be a quasi-$k$-space (in particular, let $X$ be a countably compact space). Is $[\mathcal{F}]^\omega$ a determining cover of $X^\omega$?

References


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