数理解析研究所講究録
1492巻 2006年 78-83

GROUPS OF MEASURE-PRESERVING HOMEOMORPHISMS OF NONCOMPACT 2-MANIFOLDS

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1. Introduction

A. Fathi [5] made a comprehensive study on topological and algebraic properties of groups of measure-preserving homeomorphisms of compact n-manifolds. R. Belranga [1, 2, 3] has extended this work to the noncompact manifolds. In this article we combine R. Belranga's results with our works on groups of homeomorphisms of noncompact 2-manifolds and obtain some conclusions on topological properties of groups of measure-preserving homeomorphisms of noncompact 2-manifolds [12].

2. Preliminaries on spaces of Radon measures

First we recall some basic facts on spaces of Radon measures and actions of homeomorphism groups. Suppose X is a connected locally connected locally compact separable metrizable space.

2.1. Spaces of Radon measures.

Let $\mathcal{B}(X)$ denote the σ-algebra of Borel subsets of X and $\mathcal{K}(X)$ denote the set of all compact subsets of X

Definition 2.1.

(1) A Radon measure on X is a measure μ on $(X, \mathcal{B}(X))$ such that $\mu(K) < \infty$ for any $K \in \mathcal{K}(X)$.

(2) $\mathcal{M}(X)$ = the space of Radon measures on X.

(3) The weak topology w on $\mathcal{M}(X)$ is the weakest topology such that

$$\Phi_f : \mathcal{M}(X) \to \mathbb{R} : \ \Phi_f(\mu) = \int_M f\, d\mu \quad \text{is continuous}$$

for any continuous function $f : M \to \mathbb{R}$ with compact support.

(4) $\mathcal{M}^A(X) = \{\mu \in \mathcal{M}(X) \mid \mu(A) = 0\}$ $(A \in \mathcal{B}(X))$.

Remark 2.1. Let $K \in \mathcal{K}(X)$.

(1) The map $\mathcal{M}(X)_w \ni \mu \longmapsto \mu(K) \in \mathbb{R}$ is upper semicontinuous.

(2) The map $\mathcal{M}^{\mathrm{Fr}\,K}(X)_w \ni \mu \longmapsto \mu(K) \in \mathbb{R}$ is continuous.

Lemma 2.1. *Let $\mu \in \mathcal{M}(M)$.*

The space $\mathcal{M}(M)_w$ admits a canonical contraction $\varphi_t(\nu) = (1-t)\nu + t\mu$ $(0 \le t \le 1)$.

Definition 2.2.

(1) A Radon measure $\mu \in \mathcal{M}(X)$ is said to be good if
(i) $\mu(p) = 0$ for any $p \in M$ and (ii) $\mu(U) > 0$ for any nonempty open subset U of X.

(2) $\mathcal{M}_g^A(X) = \{\mu \in \mathcal{M}^A(X) \mid \mu : \text{good}\} \quad (A \in \mathcal{B}(X))$.

Definition 2.3. Let $\mu \in \mathcal{M}_g^A(X)$.

(1) A Radon measure $\nu \in \mathcal{M}_g^A(X)$ is μ-biregular
if μ and ν have same null sets (i.e., $\mu(B) = 0$ iff $\nu(B) = 0$ for $B \in \mathcal{B}(X)$).

(2) $\mathcal{M}_g^A(X; \mu\text{-reg}) = \{\nu \in \mathcal{M}_g^A(X) \mid \nu(X) = \mu(X),\ \nu : \mu\text{-biregular}\}$.

2.2. Homeomorphism groups.

Let $\mathcal{H}(X)$ denote the group of homeomorphisms of X with the compact-open topology. It is known that $\mathcal{H}(X)$ is a separable completely metrizable topological group.

Definition 2.4. Let $A, C \subset X$.

(1) $\mathcal{H}_C(X, A) = \{h \in \mathcal{H}(X) \mid h|_C = id_C,\ h(A) = A\}$

(2) $\mathcal{H}_C^c(X, A) = \{h \in \mathcal{H}_C(X, A) \mid \operatorname{Supp} h : \text{compact}\}$

(3) For a subgroup G of $\mathcal{H}(X)$ let G_0 denote the connected component of id_M in G.

Let $\mu \in \mathcal{M}(X)$.

Definition 2.5. For $h \in \mathcal{H}(X)$ a measure $h_*\mu \in \mathcal{M}(X)$ is defined by $(h_*\mu)(B) = \mu(h^{-1}(B))$ $(B \in \mathcal{B}(X))$.

Definition 2.6. Let $h \in \mathcal{H}(X)$.

(1) h is μ-preserving if $h_*\mu = \mu$ (i.e., $\mu(h(B)) = \mu(B)$ for any $B \in \mathcal{B}(X)$).

(2) h is μ-biregular if h preserves μ-null sets (i.e., $\mu(h(B)) = 0$ iff $\mu(B) = 0$ for $B \in \mathcal{B}(X)$).

Definition 2.7.

(1) $\mathcal{H}_C(X, A;\ \mu\text{-reg}) = \{h \in \mathcal{H}_C(X, A) \mid h : \mu\text{-biregular}\}$

(2) $\mathcal{H}_C(X, A;\ \mu) = \{h \in \mathcal{H}_C(X, A) \mid h : \mu\text{-preserving}\}$

2.3. Actions of homeomorphism groups on spaces of Radon measures.

The topological group $\mathcal{H}(X, A)$ acts continuously on the space $\mathcal{M}_g^A(X)_w$ by $h \cdot \mu = h_*\mu$. For $\mu \in \mathcal{M}_g^A(X)$ the orbit map at μ is the map $\pi : \mathcal{H}(X, A) \to \mathcal{M}_g^A(X)_w : h \mapsto h_*\mu$. The group $\mathcal{H}(X, A; \mu)$ is the stabilizer of μ and it is the fiber of π at μ.

3. Compact manifold case

In this section we list topological properties of groups of measure-preserving homeomorphisms of compact n-manifolds. Suppose M is a compact connected n-manifold.

Theorem 3.1. **(Transitivity)** (von Neumann-Oxtoby-Ulam [7]) *Let* $\mu, \nu \in \mathcal{M}_g^{\partial}(M)$.
There exists $h \in \mathcal{H}_{\partial}(M)_0$ *with* $h_*\mu = \nu$ *iff* $\mu(M) = \nu(M)$.

Let $\mu \in \mathcal{M}_g^{\partial}(M)$. The topological group $\mathcal{H}(M; \mu\text{-reg})$ acts continuously on the space $\mathcal{M}_g^{\partial}(M; \mu\text{-reg})_w$ by $h \cdot \mu = h_*\mu$. The orbit map at μ is the map $\pi : \mathcal{H}(M; \mu\text{-reg}) \to \mathcal{M}_g^{\partial}(M; \mu\text{-reg})_w : h \mapsto h_*\mu$. The group $\mathcal{H}(M; \mu)$ is the stabilizer of μ and it is the fiber of π at μ.

Theorem 3.2. **(Sections of orbit maps)** (A. Fathi [5])
The orbit map $\pi : \mathcal{H}(M; \mu\text{-reg}) \to \mathcal{M}_g^{\partial}(M; \mu\text{-reg})_w$ *admits a continuous section* $\sigma : \mathcal{M}_g^{\partial}(M; \mu\text{-reg})_w \to \mathcal{H}_{\partial}(M; \mu\text{-reg})_0$.

Corollary 3.1.

(1) $\mathcal{H}(M; \mu\text{-reg}) \cong \mathcal{H}(M; \mu) \times \mathcal{M}_g^{\partial}(M; \mu\text{-reg})_w$.
(2) $\mathcal{H}(M; \mu)$ *is a strong deformation retract (SDR) of* $\mathcal{H}(M; \mu\text{-reg})$.

A. Fathi [5] also studied the properties of the inclusion $\mathcal{H}(M; \mu\text{-reg}) \subset \mathcal{H}(M)$.

Definition 3.1. A subset B of a space Y is homotopy dense (HD)
if there exists a homotopy $\varphi_t : Y \to Y$ such that $\varphi_0 = id_Y$ and $\varphi_t(Y) \subset B$ $(0 < t \leq 1)$.

Theorem 3.3. (A. Fathi [5])
$\mathcal{H}(M; \mu\text{-reg})$ *is "HD for maps from finite-dimensional spaces" in* $\mathcal{H}(M)$.
In particular, the inclusion $\mathcal{H}(M; \mu\text{-reg}) \subset \mathcal{H}(M)$ *is a weak homotopy equivalence.*

When M is a compact 2-manifold, the group $\mathcal{H}(M)$ is an ANR [6] and a ℓ_2-manifold cf.[4]. This implies the next consequences.

Corollary 3.2. *Suppose* M *is a compact connected 2-manifold.*

(1) (i) $\mathcal{H}(M; \mu\text{-reg})$ *is HD in* $\mathcal{H}(M)$. (ii) $\mathcal{H}(M; \mu\text{-reg})$ *is an ANR.*
(2) (i) $\mathcal{H}(M; \mu)$ *is a* ℓ_2*-manifold.* (ii) $\mathcal{H}(M; \mu)$ *is a SDR of* $\mathcal{H}(M)$.

The assertion (3) in Corollary 3.2 is deduced from the following characterization of ℓ_2-manifolds for topological groups.

Theorem 3.4. (T. Dobrowolski - H. Toruńczyk [4]) *Let* G *be a topological group.*
The space G *is a* ℓ_2*-manifold iff* G *is a separable, non-locally compact, completely metrizable ANR.*

4. Non-compact manifold case

R. Belranga [1, 2, 3] has extended some results in the previous setion to noncompact manifolds. To treat noncompact manifolds we have to include informations on ends of manifolds.

4.1. **End compactification.**

First we recall basic facts on end compactifications. Suppose X is a connected locally connected locally compact separable metrizable space. Let $\mathcal{C}(X)$ denote the set of connected components of X.

Definition 4.1.

(1) An end e of X is an assignment $e : \mathcal{K}(X) \ni K \longmapsto e(K) \in \mathcal{C}(X - K)$ such that $e(K_1) \supset e(K_2)$ for $K_1 \subset K_2$.

(2) $E(X)$ = the space of ends of X

(3) The end compactification of X is the space $\overline{X} = X \cup E(X)$ endowed with the topology prescribed by the following conditions :

 (i) X is an open subspace of $\overline{X}$.

 (ii) The fundamental open neighborhoods of $e \in E(X)$ are given by

$$N(e, K) = e(K) \cup \{e' \in E(X) \mid e'(K) = e(K)\} \quad (K \in \mathcal{K}(X)).$$

Let $\mu \in \mathcal{M}(X)$.

Definition 4.2.

(1) An end $e \in E(X)$ is said to be μ-finite if $\mu(e(K)) < \infty$ for some $K \in \mathcal{K}(X)$.

(2) $E_f(X; \mu) = \{e \in E(X) \mid e : \mu\text{-finite}\}$

Definition 4.3.

(1) $h \in \mathcal{H}(X)$ is μ-end-regular if h is μ-biregular and preserves the μ-finite ends of X.

(2) $\mathcal{H}_C(X; \mu\text{-end-reg}) = \{h \in \mathcal{H}_C(X) \mid h : \mu\text{-end-regular}\}$

4.2. **Finite-end weak topology.**

Let $\mu \in \mathcal{M}_g^A(X)$.

Definition 4.4.

(1) $\nu \in \mathcal{M}_g^A(X)$ is μ-end-biregular if

 (i) ν is μ-biregular and (ii) $E_f(X, \nu) = E_f(X, \mu)$ (i.e., ν and μ have same finite ends).

(2) $\mathcal{M}_g^A(X; \mu\text{-end-reg}) = \{\nu \in \mathcal{M}_g^A(X) \mid \nu(X) = \mu(X),\ \nu : \mu\text{-end-biregular}\}$

Consider the subspaces $X \overset{i}{\subset} X \cup E_f(X; \mu) \subset X \cup E(X) = \overline{X}$. The inclusion $X \overset{i}{\subset} X \cup E_f(X; \mu)$ induces a natural injection

$$\begin{array}{rccc} i_* : & \mathcal{M}_g^A(X, \mu\text{-end-reg}) & \longrightarrow & \mathcal{M}_g^A(X \cup E_f(X; \mu))_w \\ & \nu & \longmapsto & i_*\nu \end{array}$$

Definition 4.5. The finite-end weak topology ew on $\mathcal{M}_g^A(X, \mu\text{-end-reg})$ is the topology induced by the injection $i_* : \mathcal{M}_g^A(X, \mu\text{-end-reg}) \to \mathcal{M}_g^A(X \cup E_f(X; \mu))_w$ (i.e., the weakest topology such that i_* is continuous).

Lemma 4.1. *The space $\mathcal{M}_g^A(X;\mu\text{-end-reg})_{ew}$ admits a contraction $\varphi_t(\nu) = (1-t)\nu + t\mu$ $(0 \leq t \leq 1)$.*

The topological group $\mathcal{H}(X, A; \mu\text{-end-reg})$ acts continuously on $\mathcal{M}_g^A(X;\mu\text{-end-reg})_{ew}$ by $h \cdot \nu = h_*\nu$. The orbit map at μ is the map $\pi : \mathcal{H}(X, A; \mu\text{-end-reg}) \to \mathcal{M}_g^A(X;\mu\text{-end-reg})_{ew} : h \mapsto h_*\mu$. The group $\mathcal{H}(X, A; \mu)$ is the stabilizer of μ and it is the fiber of π at μ.

4.3. Results of R. Berlanga.

Suppose M is a noncompact connected separable metrizable n-manifold and $\mu \in \mathcal{M}_g^\partial(M)$. R. Berlanga [1, 2, 3] obtained the following conclusions on the action of $\mathcal{H}(M;\mu\text{-end-reg})$ on $\mathcal{M}_g^\partial(M;\mu\text{-end-reg})_{ew}$.

Theorem 4.1. **(Transitivity** [1]**)** *Let $\mu, \nu \in \mathcal{M}_g^\partial(M)$.*
There exists $h \in \mathcal{H}_\partial(M)_0$ with $h_*\mu = \nu$ iff $\mu(M) = \nu(M)$ and $E_f(M,\mu) = E_f(M,\nu)$.

Theorem 4.2. **(Sections of orbit maps** [3]**)**
The orbit map $\pi : \mathcal{H}(M;\mu\text{-end-reg}) \to \mathcal{M}_g^\partial(M;\mu\text{-end-reg})_{ew}$, $\pi(h) = h_*\mu$ admits a continuous section $\sigma : \mathcal{M}_g^\partial(M;\mu\text{-end-reg})_{ew} \to \mathcal{H}_\partial(M;\mu\text{-end-reg})_0$.

Corollary 4.1.
(1) *$\mathcal{H}_\partial(M;\mu\text{-end-reg}) \cong \mathcal{H}_\partial(M;\mu) \times \mathcal{M}_g^\partial(M;\mu\text{-end-reg})_{ew}$.*
(2) *$\mathcal{H}_\partial(M;\mu)$ is a SDR of $\mathcal{H}_\partial(M, \mu\text{-end-reg})$.*

At this moment we have no general results on the inclusion $\mathcal{H}_\partial(M, \mu\text{-end-reg}) \subset \mathcal{H}_\partial(M)$.

Problem 4.1. Is $\mathcal{H}_\partial(M, \mu\text{-end-reg})$ HD in $\mathcal{H}_\partial(M)$?

4.4. Homeomorphism groups of noncompact 2-manifolds.

We have made a comprehensive study on topological properties of groups of homeomorphisms of noncompact 2-manifolds [8, 9, 10, 11]. The results are summarized as follows.

Theorem 4.3. [9, 10] *Suppose M is a noncompact connected 2-manifold.*
(1) *$\mathcal{H}(M)_0$ is a ℓ_2-manifold.*
(2) $\mathcal{H}(M)_0 \simeq \begin{cases} \mathbb{S}^1 & (M = \mathbb{R}^2,\ \mathbb{S}^1 \times \mathbb{R},\ \mathbb{S}^1 \times [0,1),\ \text{the open Möbius band}) \\ * & (\text{otherwise}) \end{cases}$
(3) *$\mathcal{H}^{\mathrm{PL,c}}(M)_0$ is HD in $\mathcal{H}(M)_0$ (for any PL-structure on M).*

We also studied topological properties of the space of embeddings into 2-manifolds [8, 11]. Suppose M is a connected 2-manifold and X is a compact connected subpolyhedron of M. Let $\mathcal{E}(X, M)$ denote the space of embeddings of X into M with the compact-open topology and $\mathcal{E}(X, M)_0$ denote the connected component of the inclusion $i : X \subset M$ in $\mathcal{E}(X, M)$. We showed that $\mathcal{E}(X, M)$ is a ℓ_2-manifold and classified the homotopy type of $\mathcal{E}(X, M)_0$.

As an application of Theorem 4.3 we can give an affirmative answer to Problem 4.1 in dimension 2 and obtain some consequences on topological properties of groups of measure-preserving homeomorphisms of noncompact 2-manifolds [12].

Lemma 4.2. *Suppose M is a PL n-manifold and $\mu \in \mathcal{M}_g^\partial(M)$.*

There exists an isotopy φ_t of M such that $\varphi_0 = id_M$ and $\mathcal{H}^{\rm PL}(M) \subset \mathcal{H}(M; \mu\text{-end-reg})$ for the new PL - structure on M induced by φ_1 from the old PL - structure.

Corollary 4.2. *Suppose M is a noncompact connected 2-manifold.*

(1) (i) $\mathcal{H}(M, \mu\text{-end-reg})_0$ *is HD in* $\mathcal{H}(M)_0$. (ii) $\mathcal{H}(M, \mu\text{-end-reg})_0$ *is an ANR.*

(2) (i) $\mathcal{H}(M; \mu)_0$ *is a* ℓ_2*-manifold.* (ii) $\mathcal{H}(M; \mu)_0$ *is a SDR of* $\mathcal{H}(M)_0$.

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