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Kyoto University
ON 2-SPHERICAL CELL-LIKE 2-DIMENSIONAL PEANO CONTINUUM

by

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We report about joint with Katsuya Eda and Dušan Repovš result:

There exists 2-spherical simply connected cell-like 2-dimensional Peano continuum X.

First of all we fix the terminology. By n-spherical space we mean a space n's homotopy group of which is nontrivial. The space is called cell-like if it has trivial shape. By Peano continuum we mean compact connected locally connected metric space. By dimension we mean Lebesgue dimension.

The space X is constructed as follows. Consider the closed topologist's sine curve on the square $I^2 = [0,1] \times [-\frac{1}{2}, \frac{1}{2}] \subset \mathbb{R}^2$:

$$T = \left\{ (x,y) \in \mathbb{R}^2 \mid 0 < x \leq 1, y = \frac{1}{2} \sin \left( \frac{2\pi}{x} \right) \right\} \cup \{\{0\} \times [-1,1]\}.$$

Let $S^1$ be the circle and $s_0$ be any of its points which we consider as base point. Consider the topological sum of $I^2$ and $T \times S^1$. The space X is the quotient space of this sum obtained by identification of the points $(t, s_0)$ with $t \in T \subset I^2$ and by identification of each set $\{t\} \times S^1$ with $t$ when $t \in \{0\} \times [-\frac{1}{2}, \frac{1}{2}] \subset I^2$.

Let $G$ be any multiplicative group. By commutator $[x,y]$ of two elements $x$ and $y$ of group $G$ we mean the element $xyx^{-1}y^{-1}$.

Commutator length $cl(g)$ of $g \in G$ is the minimal number $n$ such that $g = \prod_{i=1}^{n} [x_i, y_i]$ [1, 4]. If such number does not exists then $cl(g) = \infty$. The commutator length $cl(g)$ is finite if and only if $g \in G'$ ($G'$ is commutator subgroup of $G$). The terms genus for this concept is used in the literature [2].

Obviously, X is a cell-like Peano continuum. It was shown in [5] that this space is simply connected. Therefore it is necessary to show only that X is 2-spherical, i.e. there exists a nontrivial 2-dimensional singular cycle in X.
Let $p$ be the natural projection of $X$ onto $I^2$ which we consider as a subspace of the plane $\mathbb{R}^2$ with axis $OX$ and $OY$. Let $I^2_+ = \{(x, y) \in I^2 | y \geq 0\}$, $I^2_- = \{(x, y) \in I^2 | y \leq 0\}$, $A^+ = p^{-1}(I^2_+)$, $A^- = p^{-1}(I^2_-)$.

Since the pair $\{A^+, A^-\}$ is an excisive couple of subsets we have the Mayer-Vietoris exact sequence ([10], p.188):

$$H_2(X) \xrightarrow{\delta} H_1(A^+ \cap A^-) \xrightarrow{(i_1, i_2)} H_1(A^+) \oplus H_1(A^-).$$

Obviously, the spaces $A^+ \cap A^-, A^+$ and $A^-$ are homotopy equivalent to the Hawaiian earrings. To show that $H_2(X) \neq 0$ it suffices to prove that $i = (i_1, i_2)$ is not a monomorphism. Consider the natural circles $\{S^1_n\}_{n \in \mathbb{N}}$ of the space $A^+ \cap A^-$ with the clockwise orientation (We consider $A^+ \cap A^-$ as a subspace of the plane $XOZ$). Let $a_n$ be the element of $\pi_1(A^+ \cap A^-)$ corresponding to the loop winding once around the circle $S^1_n$ in the positive direction.

Let $a^+$ be element of fundamental group $\pi_1(A^+ \cap A^-)$ generated by loop winding consecutively once around each circle $\{S^1_n\}_{n=1}^{\infty}$ in positive direction odd circles and in negative direction even circles. Element $a^-$ is defined similar way but corresponding loop winds in negative direction all odd circles and in positive direction even circles. Schematically elements $a^+$ and $a^-$ could be expressed as

$$a^+ = a_1 a_2^{-1} a_3 a_4^{-1} \cdots a_{2n-1} a_{2n}^{-1} \cdots$$

and

$$a^- = a_1^{-1} a_2 a_3^{-1} a_4 \cdots a_{2n-1}^{-1} a_{2n} \cdots.$$

Let $a = a^+ a^-$. Since the 1-dimensional homology group is the abelianization of the fundamental group of the corresponding space, we have element $[a] \in H_1(A^+ \cap A^-)$.

Obviously, $a_1 = a_2, a_3 = a_4, \ldots, a_{2n-1} = a_{2n}, \ldots$ in $\pi_1(A^+)$ and $i_1([a]) = 0$.

Since $a_2 = a_3, a_4 = a_5, \ldots, a_{2n} = a_{2n+1}, \ldots$ in $\pi_1(A^-)$ we have $i_2[a] = [a_1^{-1} a_1] = 0$.

Therefore $i(a) = (i_1(a), i_2(a)) = 0$. So it is enough to show that $[a] \neq 0$ in $H_1(A^+ \cap A^-)$ or that $a$ is not a element of commutator subgroup of $\pi_1(A^+ \cap A^-)$. Suppose that $a$ lies in commutator subgroup, then $cl(a) = m$ for some number $m$. To prove that this is not possible we shall need some algebraic lemmas.
Lemma 0.1. For any elements \( \{b_i\}_{i=1}^{n} \) of any group \( G \) there exist elements \( \{x_i\}_{i=1}^{n} \) of the group \( G \) such that:
\[
b_1 b_2 \cdots b_{2n} b_1^{-1} b_2^{-1} \cdots b_{2n}^{-1} = [x_1, x_2] [x_3, x_4] \cdots [x_{2n-1}, x_{2n}].
\]
If group \( G \) is free group and the set of elements \( \{b_i\}_{i=1}^{n} \) is a basis of the group \( G \) then \( \{x_i\}_{i=1}^{n} \) is also a basis of \( G \).

Proof. It is easy to check by induction that the set of elements:
\[
x_1 = b_1,
\]
\[
x_2 = b_2,
\]
\[
x_3 = b_2 b_1 b_3,
\]
\[
x_4 = b_4 b_1^{-1} b_2^{-1},
\]
\[ \cdots \]
\[
x_{2n-1} = b_{2n-2} b_{2n-3} \cdots b_2 b_1 b_{2n-1},
\]
\[
x_{2n} = b_{2n} b_1^{-1} b_2^{-1} \cdots b_{2n-2}^{-1}
\]
satisfy the condition of the lemma.
\( \square \)

Choose a natural number \( n \) such that \( n > m \). Consider the projection \( f \) of the group \( \pi_1(A^+ \cap A^-) \) on the free group \( F_{2n} \) with \( 2n \) generators \( b_1, b_2, \ldots, b_{2n} \), which is defined as follows \( f(a_1) = b_1, f(a_2) = b_2^{-1}, \ldots, f(a_{2n-1}) = b_{2n-1}, f(a_{2n}) = b_{2n}^{-1} \), for \( i > 2n, f(a_i) = e \), where \( e \) is the trivial element of \( F \) (Such projection is generated by continuous mapping of the space \( A^+ \cap A^- \) to the first \( 2n \) circles). Then \( f(a) = b_1 b_2 \cdots b_{2n} b_1^{-1} b_2^{-1} \cdots b_{2n}^{-1} \). Since \( f \) is a homomorphism and by our hypothesis \( cl(a) = m \) it follows that \( cl(f(a)) \leq m \). However, by Lemma 0.1
\[
b_1 b_2 \cdots b_{2n} b_1^{-1} b_2^{-1} \cdots b_{2n}^{-1} = [x_1, x_2] [x_3, x_4] \cdots [x_{2n-1}, x_{2n}]
\]
and by the following proposition:

Proposition 0.2. ( [9], p.55, [2], p.137). If \( F \) is a free group with a basis of distinct elements \( x_1, x_2, \ldots x_{2n} \) and there are elements \( u_1, u_2, \ldots, u_{2m} \) of \( F \) such that
\[
[x_1, x_2] [x_3, x_4] \cdots [x_{2n-1}, x_{2n}] = [u_1, u_2] [u_3, u_4] \cdots [u_{2m-1}, u_{2m}]
\]
then \( m \geq n \).

it follows that \( cl(f(a)) = n \). This contradicts our choice of number \( n \). Therefore the element \( [a] \) is a nontrivial element of \( Ker(i) \) and \( H_2(X) \neq 0 \).

Since \( \pi_1(X) = 0 \), it follows by the by Hurewicz Theorem that \( \pi_2 = H_2 \) and \( \pi_2(X) \neq 0 \).

Problem 0.3. Does there exists a noncontractible finite-dimensional Peano continuum all homotopy groups of which are trivial?
REFERENCES


