数理解析研究所講究録
1494巻 2006年 117-126

Dynamical properties of holomorphic maps with symmetries on projective spaces

Kohei Ueno (上野康平)

Graduate School of Human and Environmental Studies
Kyoto University (京都大学大学人間環境学研究科)

We consider complex dynamics of a holomorphic map from $\mathbf{P}^k$ to $\mathbf{P}^k$, which has symmetries associated with the symmetric group S_{k+2} acting on $\mathbf{P}^k$, for each $k \geq 1$. Here $\mathbf{P}^k$ denotes the k-dimensional complex projective space. Informations about critical orbits lead us to global dynamics results: The Fatou set of each map of this family consists of attractive basins of superattracting points determined by an action of the symmetric group S_{k+2}. Furthermore each map of this family satisfies Axiom A.

1 S_{k+2}-equivariant maps

For a rational map f and a finite group G acting on $\mathbf{P}^k$ as projective transformations, we say that f is G-equivariant if f commutes with each element of G, that is, $f \circ r = r \circ f$ for any $r \in G$. Doyle and McMullen [1] introduced a notion of G-equivariant functions on $\mathbf{P}^1$ to solve quintic equations. See also Ushiki [2] for G-equivariant functions on $\mathbf{P}^1$. Crass [3, 4] extended Doyle and McMullen's algorithm to higher dimensions to solve polynomial equations. Crass [5] found good pairs of G and f for which one may say something about global dynamics.

Crass [5] selected the symmetric group S_{k+2} as a finite group acting on $\mathbf{P}^k$ and found an S_{k+2}-equivariant map g_{k+3} which is holomorphic and critically finite, for each $k \geq 1$. Holomorphy means that f is well-defined at any point in $\mathbf{P}^k$. We denote by $C = C(f)$ the critical set of f and say that f is critically finite if each irreducible component of $C(f)$ is periodic or eventually periodic. In addition, the complement of $C(g_{k+3})$ is Kobayashi hyperbolic so that we can use Kobayashi metrics to prove our theorems.

1.1 Existence of S_{k+2}-equivariant maps

An action of S_{k+2} on $\mathbf{P}^k$ is induced by the permutation action of S_{k+2} on $\mathbf{C}^{k+2}$ for each $k \geq 1$. The transposition (ij) in S_{k+2} corresponds with the involution "$u_i \leftrightarrow u_j$" on $\mathbf{C}_u^{k+2} = \{u = (u_1, u_2, \cdots, u_{k+2}) | u_i \in \mathbf{C}\}$. This action pointwise fixes the hyperplane $\{u_i = u_j\}$. Since S_{k+2} preserves a hyperplane H,

$$H = \{\sum_{i=1}^{k+2} u_i = 0\} \overset{A}{\simeq} \mathbf{C}_x^{k+1} = \{x = (x_1, x_2, \cdots, x_{k+1}) \mid x_i \in \mathbf{C}\},$$

the permutation action of the symmetric group S_{k+2} on $\mathbf{C}_u^{k+2}$ induces an action of "S_{k+2}" $=< S_{k+1}, T >$ on $\mathbf{C}_x^{k+1}$, where S_{k+1} is the permutation action on $\mathbf{C}_x^{k+1}$ and T is a matrix which corresponds with $(1,\ k+2)$ in S_{k+2}.

$$T = \begin{pmatrix} -1 & 0 & \dots & 0 \\ -1 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}, \ A = \begin{pmatrix} 1 & 0 & \dots & 0 & -1 \\ 0 & 1 & \dots & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}, \ Au = x.$$

Induced hyperplanes in $\mathbf{C}_x^{k+1}$ are $\{x_i = 0\}$, $1 \leq i \leq k+1$, and $\{x_i = x_j\}$, $1 \leq i < j \leq k+1$. This action of "S_{k+2}" on $\mathbf{C}_x^{k+1}$ projects naturally to the action of "S_{k+2}" on $\mathbf{P}_x^k$ and we denote it by S_{k+2} for simplicity.

To get S_{k+2}-equivariant maps on $\mathbf{P}^k$ which are critically finite, we have the critical set coincide with the union of these hyperplanes.

Theorem 1 (Crass [5]). *For each $k \geq 1$, g_{k+3} defined below is the unique S_{k+2}-equivariant holomorphic map of degree $k+3$ which is douubly critical on each hyperplane.*

$$g = g_{k+3} := [g_{k+3,1} : g_{k+3,2} : \cdots : g_{k+3,k+1}],$$

$$g_{k+3,l} = x_l^3 \sum_{s=0}^{k} (-1)^s \frac{s+1}{s+3} x_l^s A_{k-s},$$

where A_{k-s} is the elementary symmetric function of degree k-s in $x_1, x_2, \cdots, x_{k+1}$ and $A_0 = 1$.

Then $C(g)$ coincide with the union of hyperplanes. Since g is S_{k+2}-equivariant and each hyperplane is pointwise fixed by some action of S_{k+2}, g preserves each hyperplane. In particular g is critically finite.

1.2 Properties of S_{k+2}-equivariant maps

Let us look at properties of an S_{k+2}-equivariant map g_{k+3}, which is proved in Crass [5] and will be used to prove our results. Let L^{k-1} denote one of hyperplanes $\{x_i = x_j\}$ and $\{x_i = 0\}$. Let L^m denote one of intersections of $(k-m)$ distinct L^{k-1}'s for $m = 1, 2, \cdot\cdot, k-1$. Clealy $L^m \simeq \mathbf{P}^m$ for $m = 1, 2, \cdot\cdot, k$.

First let us look at properties of g itself. The critical set of g consists of the union of hyperplanes and g preserves each hyperplane. In particular g is critically finite. Furthermore $\mathbf{P}^k \setminus C(g)$ is Kobayashi hyperbolic.

Next let us look at properties of g restrcted to L^m for $m = 1, 2, \cdot\cdot, k-1$. Since g preserves each L^m, we can also consider dynamics of g restrcted to L^m. Each restrcted map $g|_{L^m}$ has the same properties as above. Let us fix some L^m. The critical set of $g|_{L^m}$ consists of union of hyperplanes in L^m. Here L^{m-1}, a hyperplane in L^m, is a intersection of L^m and another L^{k-1}. And $g|_{L^m}$ preserves each hyperplane L^{m-1} of L^m. In particular $g|_{L^m}$ is critically finite. Furthermore $L^m \setminus C(g|_{L^m})$ is Kobayashi hyperbolic.

Finally let us look at properties of superattracting fixed points of g. The set of superattracting points, where the derivative of g vanishes for all directions, coinsides with the set of L^0's.

Remark 1. *For any $k \geq 1$ and $m \geq 1$, any restrcted map $g_{k+3}|_{L^m}$ of g_{k+3} to some L^m is not conjugate to g_{m+3}.*

1.3 Examples for $k = 1$ and 2

Let us see hyperplanes of an S_3-equivariant function g_4 and an S_4-equivariant map g_5 for make clear what L^m means. We do not write explicit forms of g_5 and $g_5|_{L^1}$. See Crass [5] for details.

1.3.1 An S_3-equivariant function g_4 in $\mathbf{P}^1$

$$g_3([x_1 : x_2]) = [x_1^3(-x_1 + 2x_2) : x_2^3(2x_1 - x_2)],$$
$$C(g_3) = \{x_1 = 0\} \cup \{x_2 = 0\} \cup \{x_1 = x_2\}\text{" = "}\{0, 1, \infty\}$$
$$\text{in } \mathbf{P}^1 = \{[x_1 : x_2] \mid (x_1, x_2) \in \mathbf{C}^2 \setminus \{0\}\}\text{" = "}\{z = \frac{x_1}{x_2} \mid x_2 \neq 0\} \cup \{\infty\}.$$

In this case "hyperplanes" are points in $\mathbf{P}^1$ and L^0 denotes one of these superattracting fixed points of g_3.

1.3.2 An S_4-equivariant map g_5 in $\mathbf{P}^2$

$$C(g_5) = \{x_1 = 0\} \cup \{x_2 = 0\} \cup \{x_3 = 0\} \cup \{x_1 = x_2\} \cup \{x_2 = x_3\} \cup \{x_3 = x_1\}$$
$$\text{in } \mathbf{P}^2 = \{[x_1 : x_2 : x_3] \mid (x_1, x_2, x_3) \in \mathbf{C}^3 \setminus \{\mathbf{0}\}\}.$$

In this case L^1 denotes one of irreducible components of $C(g_5)$, which is a hyperplane in $\mathbf{P}^2$. For example let us fix a hyperplane $\{x_1 = 0\}$. Since g_5 preserves each L^1, we can also consider dynamics of g_5 restrcted to $\{x_1 = 0\}$. The critical set of $g_5|_{\{x_1=0\}}$ in $\{x_1 = 0\} \simeq \mathbf{P}^1$ is

$$C(g_5|_{\{x_1=0\}}) = \{[0:1:0], [0:0:1], [0:1:1]\}.$$

When we use L^0 after such sentences above, L^0 means one of intersections of $\{x_1 = 0\}$ and another L^1, which is a superattracting fixed point of $g_5|_{\{x_1=0\}}$ in $\mathbf{P}^1$. The set of superattracting points of g_5 in $\mathbf{P}^2$ is

$$\{[1:0:0], [0:1:0], [0:0:1], [1:1:1], [1:1:0], [1:0:1], [0:1:1]\}.$$

Sometimes L^0 denotes one of intersections of two or more L^1's, which is a superattracting fixed point of g_5 in $\mathbf{P}^2$.

2 The Fatou sets of S_{k+2}-equivariant maps

Theorem 2 (Ueno). *For each $k \geq 1$, the Fatou set of g_{k+3} consists of attractive basins of superattracting points which are intersections of k distinct hyperplanes.*

Before starting a proof of Theorem 2, let us recall theorems about critically finite holomorphic maps and a notion of Kobayashi metrics. Let f be a holomorphic map from $\mathbf{P}^k$ to $\mathbf{P}^k$ and U a Fatou component. A holomorphic map h is said to be a limit map on U if there is a subsequence $\{f^{n_s}|_U\}_{s\geq 0}$ which locally converges to h on U. We say that a point q is a Fatou limit point if there is a limit map h on U such that $q \in h(U)$. The set of all Fatou limit points will be called the Fatou limit set. We define the ω-limit set of the critical points by

$$E = \cap_{j=1}^{\infty} f^j(D), \ D = \cup_{j=1}^{\infty} f^j(C).$$

Theorem 3. *(Ueda [6, Proposition 5.1]) If f is a critically finite holomorphic map from $\mathbf{P}^k$ to $\mathbf{P}^k$, then the Fatou limit set is contained in E.*

Let $K_M(x, v)$ be a Kobayashi quasimetric on a complex manifold M,

$$\inf\left\{|a| \,\middle|\, \varphi : \mathbf{D} \to M : \text{holomorphic}, \varphi(0) = x, D\varphi\left(a\left(\frac{\partial}{\partial z}\right)_0\right) = v, a \in \mathbf{C}\right\}$$

for $x \in M$, $v \in T_xM$, $z \in \mathbf{D}$, where $\mathbf{D}$ is the unit disk in $\mathbf{C}$. We say that M is Kobayashi hyperbolic if K_M becomes a metric. Theorem 2 is a corollary of Theorem 4 and Theorem 5 for $k = 1$ and 2.

Theorem 4. *If f is a critically finite function from $\mathbf{P}^1$ to $\mathbf{P}^1$, then the only Fatou components of f are attractive components of superattracting points.*

Theorem 5. *(Fornaess and Sibony [7, theorem 7.7]) If f is a critically finite holomorphic map from $\mathbf{P}^2$ to $\mathbf{P}^2$ and the complement of $C(f)$ is Kobayashi hyperbolic, then the only Fatou components of f are superattractive components of superattracting points.*

We can apply an argument in FS [7] to an S_{k+2}-equivariant map g_{k+3} because each L^{m-1} is smooth and $L^m \backslash C(g|_{L^m})$ is Kobayashi hyperbolic for $m = 1, 2, \cdots, k$.

Proof of Theorem 2. Take any Fatou component U and any point $x \in U$. It is enough to show that $\{g^n(x)\}_{n \geq 0}$ accumlates to some L^0, one of superattracting fixed points. By theorem 3 $\{g^n(x)\}_{n \geq 0}$ accumulates to $C(g)$. Since $C(g)$ is the union of L^{k-1}'s, there exists a smallest integer m such that $\{g^n(x)\}_{n \geq 0}$ accumulates to some L^m. Let m be $k - 1$ for simplicity. By using Kobayashi metrics and an argument in FS [7], we shall show the following result later,

$$\exists n_k \in \mathbf{N} \text{ s.t. } g^{n_k}(U) \cap L^{k-1} \neq \emptyset. \tag{1}$$

Next let U_{k-1} be $g^{n_k}(U) \cap L^{k-1}$ and do the same thing as above. Then

$$\exists n_{k-1} \in \mathbf{N},\ \exists\, L^{k-2} \text{ s.t. } g^{n_{k-1}}(U_{k-1}) \cap L^{k-2} \neq \emptyset.$$

Let U_{k-2} be $g^{n_{k-1}}(U) \cap L^{k-2}$ and do the same thing as above. These reductions finally come to some L^1. Let U_2 be $g^{n_k + n_{k-1} + \cdots + n_3}(U) \subset L^2$, then

$$\exists n_2 \in \mathbf{N},\ \exists L^1 \text{ s.t. } g^{n_2}(U_2) \cap L^1 \neq \emptyset.$$

Let U_1 be $g^{n_2}(U_2) \cap L^1$. By Theorem 4 there exists n_1 such that g^{n_1} sends U_1 to an attractive component of some superattracting fixed point L^0 in $L^1 \simeq \mathbf{P}^1$. Hence $g^{n_k + n_{k-1} + \cdots + n_1}$ sends U to an attracting component of a superattracting fixed point L^0 in $\mathbf{P}^k$.

To prove (1), let us assume that (1) is not true and derive a contradiction. By Theorem 3 $h(x)$ belongs to $C(g)$ for a limit map h of convergent subsequence $\{g^{n_s}|_U\}_{s\geq 0}$. So there exists a smallest integer m such that $h(x)$ belongs to some L^m. If h is open map from U to L^m, then $h(U) \cap L^m$ is an open set in L^m and is contained in $F(g|_{L^m})$. The same argument of reductions as above implies that $\{g^{n_k}(x)\}$ accumlates to one of L^0. That is, there exists n such that g^n sends U to an attracting component of L^0, which is a contradiction.

To show that h is open map from U to L^m, we shall use Kobayashi metrics. Let A be $\mathbf{P}^k \setminus g^{-1}(C(g))$ and let B be $\mathbf{P}^k \setminus C(g)$. Since B is Kobayashi hyperbolic and $A \subset B$, A is also Kobayashi hyperbolic. So we can use Kobayashi metrics K_A and K_B. By $A \subset B$

$$K_B(y, v) \leq K_A(y, v), \ \forall y \in A, \ v \in T_y\mathbf{P}^k.$$

Since g is an unbranched covering from A to B,

$$K_A(y, v) = K_B(g(y), Dg(v)), \ \forall y \in A, \ v \in T_y\mathbf{P}^k.$$

$$\therefore K_B(y, v) \leq K_B(g(y), Dg(v)), \ \forall y \in A, \ v \in T_y\mathbf{P}^k.$$

Since the same argument holds for any g^n from $\mathbf{P}^k \setminus g^{-n}(C(g))$ to $\mathbf{P}^k \setminus C(g)$,

$$K_B(y, v) \leq K_B(g^n(y), Dg^n(v)), \ \forall y \in \mathbf{P}^k \setminus g^{-n}(C(g)), \ v \in T_y\mathbf{P}^k. \quad (2)$$

Since g^n is an unbranched covering from U to $g^n(U)$ and $g^n(U) \subset B$ for any n, $K_B(g^n(x), Dg^n(v))$ is bounded,

$$K_B(g^n(y), Dg^n(v)) \leq K_{g^n(U)}(g^n(y), Dg^n(v)) = K_U(y, v) < \infty.$$

We claim that for unit vectors $v_n \in T_xU$ such that $Dg^n(x)v_n$ keeps parallel to L^m, $Dh(x)v \neq \mathbf{0} = (0, 0, \cdots, 0)$ for an accumlation vector v of v_n. Let $h = \lim_{n\to\infty} g^n$ for simplicity. One can choose a local chart around $h(x)$ so that $h(x) = \mathbf{0}$ and $L^m = \{y = (y_1, y_2, \cdots, y_k) | y_1 = \cdots = y_{k-m} = 0\}$. In this chart there exists $r > 0$ such that polydisk $P(\mathbf{0}, r)$ is disjoint from L^{k-1} which does not include L^m. Since $g^n(x) \to \mathbf{0}$ as $n \to \infty$, we may assume $g^n(x) \in P(\mathbf{0}, r)$. By assumption that (1) is not true, $g^n(x) \notin C(g)$ for any $n \geq 1$. Thus one can define maps φ_n from $\mathbf{D}$ to $P(\mathbf{0}, r)$ for $z \in \mathbf{D}$,

$$\varphi_n(z) := g^n(x) + rze_k = g^n(x) + (0, \cdots, 0, rz).$$

Here $e_k = (0, \cdots, 0, 1)$. Then $\varphi_n(0) = g^n(x)$ and $\varphi_n(\mathbf{D}) \subset \mathbf{P}^k \setminus g^{-n}(C(g))$. Let us choose unit vectors v_n so that $Dg^n(x)v_n = |Dg^n(x)v_n| e_k$. By the definition of Kobayashi metric,

$$K_B(g^n(x), Dg^n(x)v_n) \leq \frac{|Dg^n(x)v_n|}{r}.$$

Suppose $Dh(x)v = \mathbf{0}$, then $Dg^n(x)v \to \mathbf{0}$ and $Dg^n(x)v_n \to \mathbf{0}$ as $n \to \infty$.

$$\therefore\ K_B(g^n(x), Dg^n(x)v_n) \leq \frac{|Dg^n(x)v_n|}{r} \to 0.$$

On the other hand, by (2)

$$0 < \inf_{|v|=1} K_B(x, v) \leq K_B(x, v_n) \leq K_B(g^n(x), Dg^n(x)v_n).$$

Hence $K_B(g^n(x), Dg^n(x)v_n)$ is bounded away from 0 uniformly and this contradiction completes the proof.

□

3 S_{k+2}-equivariant maps and Axiom A

Theorem 6 (Ueno). *For each $k \geq 1$, g_{k+3} satisfies Axiom A.*

First let us define hyperbolicity of maps and a notion of Axiom A. See Jonsson [9] for details. Let f be a holomorphic map from $\mathbf{P}^k$ to $\mathbf{P}^k$.

$$\Omega := \{x \in \mathbf{P}^k \mid \forall U: \text{neighborhood of } x,\ \exists n \in \mathbf{N} \text{ s.t. } f^n(U) \cap U \neq \emptyset\}.$$

This set is called the non-wandering set, which is compact and forward invariant. We say that f is hyperbolic on Ω if there exists a continuous decomposition $T_{\widehat{\Omega}} = E^u + E^s$ such that $D\widehat{f}(E^{u/s}_{\widehat{x}}) \subset E^{u/s}_{\widehat{f}(\widehat{x})}$ and if there exists $c > 0$, $\lambda > 1$ such that for any $n \geq 1$,

$$|D\widehat{f}^n(v)| \geq c\lambda^n|v|,\ \forall v \in E^u,$$

$$|D\widehat{f}^n(v)| \leq c^{-1}\lambda^{-n}|v|,\ \forall v \in E^s.$$

Here $\widehat{\Omega}$ is the set of histories in Ω and $\widehat{f}$ is a diffeomorphism on $\widehat{\Omega}$. If a decomposition and inequalities above hold for Ω and f, then it also holds for $\widehat{\Omega}$ and $\widehat{f}$. We say that f satisfies Axiom A if f is hyperbolic on Ω and periodic points are dense in Ω.

Proof of Theorem 6. We shall show this by induction. For each S_{k+2}-equivariant map g, it is clear that $g|_{L^1}$ satisfies Axiom A for each L^1 from a theorem of critically finite functions. We only show that $g|_{L^2}$ satisfies Axiom A for some L^2. An argument for $g|_{L^m}$, $3 \leq m \leq k$, is similer as for $g|_{L^2}$. So let us fix some L^2. First we shall show that $g|_{L^2}$ is hyperbolic on $\Omega(g|_{L^2})$.

Next we shall show that periodic points of $g|_{L^2}$ are dense in $\Omega(g|_{L^2})$. Let denote $g|_{L^2}$ and $\Omega(g|_{L^2})$ by g and Ω for simplicity.

If g is hyperbolic on Ω, Ω has a decomposition to S_i, where i=1,2,3 indicate the unstable dimensions. Since $C(g)$ attracts all nearby points, it follows that $\cup L^0 \subset S_0$ and $\cup J(g|_{L^1}) \subset S_1$, where $g|_{L^0}$ is contracting for all direction and $g|_{J(g|_{L^1})}$ is contracting for a certain explicit direction and expanding for an L^1-direction. Let us consider a compact, completely invariantsubset in the complement of C in L^2,

$$S := \{x \in \mathbf{P}^2 \mid dist(f^n(x), C) \not\to 0 \text{ as } n \to \infty\}.$$

It is clear that $S \cap C = \emptyset$ and $S \supset J_2 \neq \phi$. Here J_2 is the second Julia set, in which repelling periodic points are dense. By the definition of S, $\Omega = (\cup L^0) \cup (\cup J(g|_{L^1})) \cup S$. If we show that g is expanding on S, it follows that $\cup L^0 = S_0$, $\cup J(g|_{L^1}) = S_1$, $S = S_2$. Thus g is hyperbolic on Ω.

Let us show that g is expanding on S. Since f is attracting on C and $f(C) = C$, there exists a neighborhood N of C such that $N \Subset g^{-1}(N)$ and $B := \mathbf{P}^2 \setminus N$ is connected. Let U be one of connected components of $\mathbf{P}^2 \setminus g^{-1}(N)$. Let one of L^1's be the line at infinitry of $\mathbf{P}^2$, then

$$U \subset \mathbf{P}^2 \setminus g^{-1}(N) \Subset B \subset \mathbf{C}^2 = \mathbf{P}^2 \setminus L^1.$$

Since the map g from U to B is an unbranched covering,

$$K_U(x, v) = K_B(g(x), Dg(v)), \ \forall x \in U, \ v \in T_x\mathbf{C}^2.$$

Since B and all connected components of $\mathbf{P}^2 \setminus g^{-1}(N)$ are in one local chart, there exists a constant number $\rho < 1$ such that for any U

$$K_B(x, v) \le \rho K_U(x, v), \ \forall x \in U, \ v \in T_x\mathbf{C}^2.$$

$$\therefore K_B(x, v) \le \rho K_B(g(x), Dg(v)), \ \forall x \in \mathbf{P}^2 \setminus g^{-1}(N), \ v \in T_x\mathbf{C}^2.$$

Since $g^n(x)$ belongs to S, which is contained in $\mathbf{P}^2 \setminus g^{-1}(N)$, for any x which belongs to S and for any $n \ge 1$, we have that

$$K_B(x, v) \le \rho^n K_B(g^n(x), Dg^n(v)), \ \forall x \in S, \ v \in T_x\mathbf{C}^2.$$

$$\therefore K_B(g^n(x), Dg^n(v)) \ge \lambda^n K_B(x, v), \ \forall x \in S, \ v \in T_x\mathbf{C}^2, \ \lambda = \frac{1}{\rho} > 1.$$

Since $K_B(x, v)$ is upper semicontinuous and $|v|$ is continuous, $K_B(x, v)$ and $|v|$ may be different only by a constant factor. There exists $c > 0$ such that

$$|Dg^n(x)v| \ge c\lambda^n |v|, \ \forall x \in S, \ v \in T_x\mathbf{C}^2.$$

Thus g is expanding on S and hyperbolic on Ω.

Next we shall show that periodic points are dense in Ω. It is enough to show that $J_2 = S_2$ since periodic points are dense in $J(g|_{L^1})$ and J_2. This follows from the same argument in FS [8, Theorem3.8]. Let us recall that proof. Let σ be $S_2 \setminus J_2$ and suppose that σ is not empty. Since σ is attracting for inverse branches of f^n, σ is disjoint from J_2 and is closed. Since $f(C) = C$, one can define holomorphic local branches of inverses of f^n in $\mathbf{P}^2 \setminus C$. Then this family $\{f_i^{-n}\}_{i,n\geq 0}$ becomes a normal family. For any continuous function ϕ on $\mathbf{P}^2$, we define

$$A_\phi^n(x) := \frac{1}{d^{2n}} \sum_{i=1}^{d^{2n}} \phi(f_i^{-n}(x)).$$

In this case $\{A_\phi^n\}_{n\geq 0}$ is locally equicontinuous in $\mathbf{P}^2 \setminus C$ and

$$A_\phi^n(x) \to \mu(\phi) \ \text{ as } \ n \to \infty, \ \forall x \in \mathbf{P}^2 \setminus C, \tag{3}$$

where μ is the invariant probability measure whose support is J_2. Let $\phi = 1$ in a neghiborhood of J_2 and $\phi = 0$ in a neghiborhood of σ. Since $f^{-1}(\sigma) = \sigma$, $A_\phi^n \equiv 0$ in σ for any n. On the other hand, by (3)

$$A_\phi^n(x) \to \mu(\phi) = 1 \ \text{ as } \ n \to \infty, \ \forall x \in \sigma \subset \mathbf{P}^2 \setminus C.$$

This contradiction implies that σ is empty. Thus $J_2 = S_2$ and periodic points are dense in Ω.

□

References

[1] P.Doyle and C.McMullen, Solving the quintic by iteration, Acta Math, 163, 1989, 151-180

[2] S.Ushiki, Julia set with polyhedral symmetry, Dynamical systems and related topics, Nagoya, 1990, 515-538

[3] S.Crass, Solving the sextic by iteration, Experiment Mathematics 8, No.3, 1999, 209-240

[4] S.Crass, Solving the quintic by iteration in three dimensions, Experiment Mathematics 10, No.1, 2001, 1-24

[5] S.Crass, A family of critically finite maps with symmetry, Publicacions Mattematiques, Vol.49, No.1, 2005, 127-157

[6] T.Ueda, Critical orbits of holomorphic maps on projective spaces, J.Geom.Analy., Vol.8, No.2, 1998, 319-334

[7] J.E.Fornaess and N.Sibony, Complex dynamics in higher dimension I, Asterisque, No.222, 1994, 201-231

[8] J.E.Fornaess and N.Sibony, Hyperbolic maps on $\mathbf{P}^2$, Math Annalen, 11, 1998, 305-333

[9] M.Jonsson, Hyperbolic dynamics of endomorphisms, preprint