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Complex Dynamics and its Related Fields

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Dynamical properties of holomorphic maps with symmetries on projective spaces

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We consider complex dynamics of a holomorphic map from $\mathbb{P}^k$ to $\mathbb{P}^k$, which has symmetries associated with the symmetric group $S_{k+2}$ acting on $\mathbb{P}^k$, for each $k \geq 1$. Here $\mathbb{P}^k$ denotes the $k$-dimensional complex projective space. Informations about critical orbits lead us to global dynamics results: The Fatou set of each map of this family consists of attractive basins of superattracting points determined by an action of the symmetric group $S_{k+2}$. Furthermore each map of this family satisfies Axiom A.

1 $S_{k+2}$-equivariant maps

For a rational map $f$ and a finite group $G$ acting on $\mathbb{P}^k$ as projective transformations, we say that $f$ is $G$-equivariant if $f$ commutes with each element of $G$, that is, $f \circ r = r \circ f$ for any $r \in G$. Doyle and McMullen [1] introduced a notion of $G$-equivariant functions on $\mathbb{P}^1$ to solve quintic equations. See also Ushiki [2] for $G$-equivariant functions on $\mathbb{P}^1$. Crass [3, 4] extended Doyle and McMullen’s algorithm to higher dimensions to solve polynomial equations. Crass [5] found good pairs of $G$ and $f$ for which one may say something about global dynamics.

Crass [5] selected the symmetric group $S_{k+2}$ as a finite group acting on $\mathbb{P}^k$ and found an $S_{k+2}$-equivariant map $g_{k+3}$ which is holomorphic and critically finite, for each $k \geq 1$. Holomorphy means that $f$ is well-defined at any point in $\mathbb{P}^k$. We denote by $C = C(f)$ the critical set of $f$ and say that $f$ is critically finite if each irreducible component of $C(f)$ is periodic or
eventually periodic. In addition, the complement of \( C(g_{k+3}) \) is Kobayashi hyperbolic so that we can use Kobayashi metrics to prove our theorems.

### 1.1 Existence of \( S_{k+2} \)-equivariant maps

An action of \( S_{k+2} \) on \( \mathbb{P}^k \) is induced by the permutation action of \( S_{k+2} \) on \( C^{k+2} \) for each \( k \geq 1 \). The transposition \((ij)\) in \( S_{k+2} \) corresponds with the involution \( u_i \leftrightarrow u_j \) on \( C^{k+2}_u = \{ u = (u_1, u_2, \ldots, u_{k+2}) | u_i \in \mathbb{C} \} \). This action pointwise fixes the hyperplane \( \{ u_i = u_j \} \). Since \( S_{k+2} \) preserves a hyperplane \( H = \{ \sum_{i=1}^{k+2} u_i = 0 \} \), \( H \cong A \) \( \mathbb{C}_{X}^{k+1} = \{ x = (x_1, x_2, \ldots, x_{k+1}) | x_i \in \mathbb{C} \} \), the permutation action of the symmetric group \( S_{k+2} \) on \( C_x^{k+1} \) induces an action of \( \langle S_{k+2}, P \rangle \) on \( \mathbb{C}_X^{k+1} \), where \( S_{k+1} \) is the permutation action on \( \mathbb{C}_X^{k+1} \) and \( P \) is a matrix which corresponds with \( (1, k+2) \) in \( S_{k+2} \).

\[
T = \begin{pmatrix}
-1 & 0 & \ldots & 0 \\
-1 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-1 & 0 & \ldots & 1
\end{pmatrix},
A = \begin{pmatrix}
1 & 0 & \ldots & 0 & -1 \\
0 & 1 & \ldots & 0 & -1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 1 & -1
\end{pmatrix},
Au = x.
\]

Induced hyperplanes in \( C_x^{k+1} \) are \( \{ x_i = 0 \}, 1 \leq i \leq k+1 \), and \( \{ x_i = x_j \}, 1 \leq i < j \leq k+1 \). This action of \( \langle S_{k+2}, T \rangle \) on \( C_x^{k+1} \) projects naturally to the action of \( \langle S_{k+2} \rangle \) on \( \mathbb{P}_X^k \) and we denote it by \( S_{k+2} \) for simplicity.

To get \( S_{k+2} \)-equivariant maps on \( \mathbb{P}_X^k \) which are critically finite, we have the critical set coincide with the union of these hyperplanes.

**Theorem 1 (Crass [5]).** For each \( k \geq 1 \), \( g_{k+3} \) defined below is the unique \( S_{k+2} \)-equivariant holomorphic map of degree \( k+3 \) which is douubly critical on each hyperplane.

\[
g = g_{k+3} := [g_{k+3,1} : g_{k+3,2} : \ldots : g_{k+3,k+1}],
\]

\[
g_{k+3,l} = x_1^3 \sum_{s=0}^{k} (-1)^s \frac{s+1}{s+3} x_l^s A_{k-s},
\]

where \( A_{k-s} \) is the elementary symmetric function of degree \( k-s \) in \( x_1, x_2, \ldots, x_{k+1} \) and \( A_0 = 1 \).

Then \( C(g) \) coincide with the union of hyperplanes. Since \( g \) is \( S_{k+2} \)-equivariant and each hyperplane is pointwise fixed by some action of \( S_{k+2} \), \( g \) preserves each hyperplane. In particular \( g \) is critically finite.
1.2 Properties of $S_{k+2}$-equivariant maps

Let us look at properties of an $S_{k+2}$-equivariant map $g_{k+3}$, which is proved in Crass [5] and will be used to prove our results. Let $L^{k-1}$ denote one of hyperplanes $\{x_i = x_j\}$ and $\{x_i = 0\}$. Let $L^m$ denote one of intersections of $(k - m)$ distinct $L^{k-1}$'s for $m = 1, 2, \cdots, k - 1$. Clearly $L^m \simeq P^m$ for $m = 1, 2, \cdots, k$.

First let us look at properties of $g$ itself. The critical set of $g$ consists of the union of hyperplanes and $g$ preserves each hyperplane. In particular $g$ is critically finite. Furthermore $P^k \setminus C(g)$ is Kobayashi hyperbolic.

Next let us look at properties of $g$ restricted to $L^m$ for $m = 1, 2, \cdots, k - 1$. Since $g$ preserves each $L^m$, we can also consider dynamics of $g$ restricted to $L^m$. Each restricted map $g|_{L^m}$ has the same properties as above. Let us fix some $L^m$. The critical set of $g|_{L^m}$ consists of union of hyperplanes in $L^m$. Here $L^{m-1}$, a hyperplane in $L^m$, is a intersection of $L^m$ and another $L^{k-1}$. And $g|_{L^m}$ preserves each hyperplane $L^{m-1}$ of $L^m$. In particular $g|_{L^m}$ is critically finite. Furthermore $L^m \setminus C(g|_{L^m})$ is Kobayashi hyperbolic.

Finally let us look at properties of superattracting fixed points of $g$. The set of superattracting points, where the derivative of $g$ vanishes for all directions, coincides with the set of $L^0$'s.

Remark 1. For any $k \geq 1$ and $m \geq 1$, any restricted map $g_{k+3}|_{L^m}$ of $g_{k+3}$ to some $L^m$ is not conjugate to $g_{m+3}$.

1.3 Examples for $k = 1$ and $2$

Let us see hyperplanes of an $S_3$-equivariant function $g_4$ and an $S_4$-equivariant map $g_5$ for make clear what $L^m$ means. We do not write explicit forms of $g_3$ and $g_4|_{L^1}$. See Crass [5] for details.

1.3.1 An $S_3$-equivariant function $g_4$ in $P^1$

$$g_3([x_1 : x_2]) = [x_1^2(-x_1 + 2x_2) : x_2^2(2x_1 - x_2)],$$

$$C(g_3) = \{x_1 = 0\} \cup \{x_2 = 0\} \cup \{x_1 = x_2\}^* = \{0, 1, \infty\}$$

in $P^1 = \{(x_1 : x_2) | (x_1, x_2) \in \mathbb{C}^2 \setminus \{0\}\}^* = \{z = \frac{x_1}{x_2} | x_2 \neq 0\} \cup \{\infty\}$.

In this case "hyperplanes" are points in $P^1$ and $L^0$ denotes one of these superattracting fixed points of $g_3$. 
1.3.2 An $S_4$-equivariant map $g_5$ in $P^2$

$C(g_5) = \{x_1 = 0\} \cup \{x_2 = 0\} \cup \{x_3 = 0\} \cup \{x_1 = x_2\} \cup \{x_2 = x_3\} \cup \{x_3 = x_1\}$

in $P^2 = \{(x_1 : x_2 : x_3) \mid (x_1, x_2, x_3) \in C^3 \setminus \{0\}\}$.

In this case $L^1$ denotes one of irreducible components of $C(g_5)$, which is a hyperplane in $P^2$. For example let us fix a hyperplane $\{x_1 = 0\}$. Since $g_5$ preserves each $L^1$, we can also consider dynamics of $g_5$ restricted to $\{x_1 = 0\}$. The critical set of $g_5|_{\{x_1=0\}}$ in $\{x_1 = 0\} \simeq P^1$ is

$C(g_5|_{\{x_1=0\}}) = \{[0:1:0],[0:0:1],[0:1:1]\}$.

When we use $L^0$ after such sentences above, $L^0$ means one of intersections of $\{x_j = 0\}$ and another $L^1$, which is a superattracting fixed point of $g_5|_{\{x_1=0\}}$ in $P^1$. The set of superattracting points of $g_5$ in $P^2$ is

$\{[1:0:0],[0:1:0],[0:0:1],[1:1:1],[1:1:0],[1:0:1],[0:1:1]\}$.

Sometimes $L^0$ denotes one of intersections of two or more $L^1$'s, which is a superattracting fixed point of $g_5$ in $P^2$.

2  The Fatou sets of $S_{k+2}$-equivariant maps

Theorem 2 (Ueno). For each $k \geq 1$, the Fatou set of $g_{k+3}$ consists of attractive basins of superattracting points which are intersections of $k$ distinct hyperplanes.

Before starting a proof of Theorem 2, let us recall theorems about critically finite holomorphic maps and a notion of Kobayashi metrics. Let $f$ be a holomorphic map from $P^k$ to $P^k$ and $U$ a Fatou component. A holomorphic map $h$ is said to be a limit map on $U$ if there is a subsequence $\{f^{(n)}|_{U}\}_{n \geq 0}$ which locally converges to $h$ on $U$. We say that a point $q$ is a Fatou limit point if there is a limit map $h$ on $U$ such that $q \in h(U)$. The set of all Fatou limit points will be called the Fatou limit set. We define the $\omega$-limit set of the critical points by

$E = \cap_{j=1}^{\infty} f^j(D), \quad D = \cup_{j=1}^{\infty} f^j(C)$.

Theorem 3. (Ueda [6, Proposition 5.1]) If $f$ is a critically finite holomorphic map from $P^k$ to $P^k$, then the Fatou limit set is contained in $E$. 
Let $K_M(x,v)$ be a Kobayashi quasimetric on a complex manifold $M$.

$$\inf \left\{ |a| : \varphi : D \to M : \text{holomorphic}, \varphi(0) = x, D\varphi \left( a \left( \frac{\partial}{\partial z} \right)_0 \right) = v, a \in \mathbb{C} \right\}$$

for $x \in M$, $v \in T_x M$, $z \in D$, where $D$ is the unit disk in $\mathbb{C}$. We say that $M$ is Kobayashi hyperbolic if $K_M$ becomes a metric. Theorem 2 is a corollary of Theorem 4 and Theorem 5 for $k = 1$ and 2.

**Theorem 4.** If $f$ is a critically finite function from $\mathbb{P}^1$ to $\mathbb{P}^1$, then the only Fatou components of $f$ are attractive components of superattracting points.

**Theorem 5.** (Fornaess and Sibony [7, theorem 7.7]) If $f$ is a critically finite holomorphic map from $\mathbb{P}^2$ to $\mathbb{P}^2$ and the complement of $C(f)$ is Kobayashi hyperbolic, then the only Fatou components of $f$ are superattractive components of superattracting points.

We can apply an argument in FS [7] to an $S_{k+2}$-equivariant map $g_{k+3}$ because each $L^{m-1}$ is smooth and $L^m \setminus C(g|_{L^m})$ is Kobayashi hyperbolic for $m = 1, 2, \ldots, k$.

**Proof of Theorem 2.** Take any Fatou component $U$ and any point $x \in U$. It is enough to show that $\{g^n(x)\}_{n \geq 0}$ accumulates to some $L^0$, one of superattracting fixed points. By theorem 3 $\{g^n(x)\}_{n \geq 0}$ accumulates to $C(g)$.

Since $C(g)$ is the union of $L^{k-1}$s, there exists a smallest integer $m$ such that $\{g^n(x)\}_{n \geq 0}$ accumulates to some $L^m$. Let $m$ be $k - 1$ for simplicity. By using Kobayashi metrics and an argument in FS [7], we shall show the following result later,

$$\exists n_k \in \mathbb{N} \text{ s.t. } g^n(U) \cap L^{k-1} \neq \emptyset. \quad (1)$$

Next let $U_{k-1} = g^n(U) \cap L^{k-1}$ and do the same thing as above. Then

$$\exists n_{k-1} \in \mathbb{N}, \exists L^{k-2} \text{ s.t. } g^{n_{k-1}}(U_{k-1}) \cap L^{k-2} \neq \emptyset.$$  

Let $U_{k-2} = g^{n_{k-1}}(U) \cap L^{k-2}$ and do the same thing as above. These reductions finally come to some $L^1$. Let $U_2$ be $g^{n_k + n_{k-1} + \cdots + n_3}(U) \subset L^2$, then

$$\exists n_2 \in \mathbb{N}, \exists L^1 \text{ s.t. } g^n(U_2) \cap L^1 \neq \emptyset.$$  

Let $U_1 = g^{n_k}(U_2) \cap L^1$. By Theorem 4 there exists $n_1$ such that $g^n$ sends $U_1$ to an attracting component of some superattracting fixed point $L^0$ in $L^1 \simeq \mathbb{P}^1$. Hence $g^{n_k + n_{k-1} + \cdots + n_1}$ sends $U$ to an attracting component of a superattracting fixed point $L^0$ in $\mathbb{P}^k$.  

To prove (1), let us assume that (1) is not true and derive a contradiction. By Theorem 3 $h(x)$ belongs to $C(g)$ for a limit map $h$ of convergent subsequence $\{g^n|_U\}_{s \geq 0}$. So there exists a smallest integer $m$ such that $h(x)$ belongs to some $L^m$. If $h$ is open map from $U$ to $L^m$, then $h(U) \cap L^m$ is an open set in $L^m$ and is contained in $F(g|_{L^m})$. The same argument of reductions as above implies that $\{g^n(x)\}$ accumulates to one of $L^0$. That is, there exists $n$ such that $g^n$ sends $U$ to an attracting component of $L^0$, which is a contradiction.

To show that $h$ is open map from $U$ to $L^m$, we shall use Kobayashi metrics. Let $A$ be $P^k \setminus g^{-1}(C(g))$ and let $B$ be $P^k \setminus C(g)$. Since $B$ is Kobayashi hyperbolic and $A \subset B$, $A$ is also Kobayashi hyperbolic. So we can use Kobayashi metrics $K_A$ and $K_B$. By $A \subset B$

$$K_B(y, v) \leq K_A(y, v), \forall y \in A, v \in T_y P^k.$$ 
Since $g$ is an unbranched covering from $A$ to $B$,

$$K_A(y, v) = K_B(g(y), Dg(v)), \forall y \in A, v \in T_y P^k.$$ 

Thus $K_B(y, v) \leq K_B(g(y), Dg(v)), \forall y \in A, v \in T_y P^k.$

Since the same argument holds for any $g^n$ from $P^k \setminus g^{-n}(C(g))$ to $P^k \setminus C(g)$,

$$K_B(y, v) \leq K_B(g^n(y), Dg^n(v)), \forall y \in P^k \setminus g^{-n}(C(g)), v \in T_y P^k.$$ 

Since $g^n$ is an unbranched covering from $U$ to $g^n(U)$ and $g^n(U) \subset B$ for any $n$, $K_B(g^n(x), Dg^n(v))$ is bounded,

$$K_B(g^n(y), Dg^n(v)) \leq K_B(U)(g^n(y), Dg^n(v)) = K_U(y, v) < \infty.$$ 

We claim that for unit vectors $v_n \in T_x U$ such that $Dg^n(x)v_n$ keeps parallel to $L^m$, $Dh(x)v \neq 0 = (0, 0, \cdots, 0)$ for an accumulation vector $v$ of $v_n$. Let $h = \lim_{n \to \infty} g^n$ for simplicity. One can choose a local chart around $h(x)$ so that $h(x) = 0$ and $L^m = \{y = (y_1, y_2, \cdots, y_k) | y_1 = \cdots = y_{k-m} = 0\}$. In this chart there exists $r > 0$ such that polydisk $P(0, r)$ is disjoint from $L^{k-1}$ which does not include $L^m$. Since $g^n(x) \to 0$ as $n \to \infty$, we may assume $g^n(x) \in P(0, r)$. By assumption that (1) is not true, $g^n(x) \notin C(g)$ for any $n \geq 1$. Thus one can define maps $\varphi_n$ from $D$ to $P(0, r)$ for $z \in D$,

$$\varphi_n(z) := g^n(z) + rze_k = g^n(z) + (0, \cdots, 0, rz).$$ 
Here $e_k = (0, \cdots, 0, 1)$. Then $\varphi_n(0) = g^n(x)$ and $\varphi_n(D) \subset P^k \setminus g^{-n}(C(g))$. Let us choose unit vectors $v_n$ so that $Dg^n(x)v_n = |Dg^n(x)v_n|e_k$ by the definition of Kobayashi metric,

$$K_B(g^n(x), Dg^n(x)v_n) \leq \frac{|Dg^n(x)v_n|}{r}.$$
Suppose $Dh(x)v = 0$, then $Dg^n(x)v \to 0$ and $Dg^n(x)v_n \to 0$ as $n \to \infty$.

\[
\therefore K_B(g^n(x), Dg^n(x)v_n) \leq \frac{|Dg^n(x)v_n|}{r} \to 0.
\]

On the other hand, by (2)

\[
0 < \inf_{|v|=1} K_B(x, v) \leq K_B(x, v_n) \leq K_B(g^n(x), Dg^n(x)v_n).
\]

Hence $K_B(g^n(x), Dg^n(x)v_n)$ is bounded away from 0 uniformly and this contradiction completes the proof.

\[\square\]

3  $S_{k+2}$-equivariant maps and Axiom A

Theorem 6 (Ueno). For each $k \geq 1$, $g_{k+3}$ satisfies Axiom A.

First let us define hyperbolicity of maps and a notion of Axiom A. See Jonsson [9] for details. Let $f$ be a holomorphic map from $\mathbb{P}^k$ to $\mathbb{P}^k$.

\[\Omega := \{ x \in \mathbb{P}^k | \forall U : \text{neighborhood of } x, \exists n \in \mathbb{N} \text{ s.t. } f^n(U) \cap U \neq \emptyset \}.\]

This set is called the non-wandering set, which is compact and forward invariant. We say that $f$ is hyperbolic on $\Omega$ if there exists a continuous decomposition $T_{\hat{\Omega}} = E^u + E^s$ such that $D\hat{f}(E^u_{\hat{\Omega}}) \subset E^u_{\hat{\Omega}}$ and if there exists $c > 0, \lambda > 1$ such that for any $n \geq 1$,

\[
|D\hat{f}^n(v)| \geq c\lambda^n|v|, \forall v \in E^u.
\]

\[
|D\hat{f}^n(v)| \leq c^{-1}\lambda^{-n}|v|, \forall v \in E^s.
\]

Here $\hat{\Omega}$ is the set of histories in $\Omega$ and $\hat{f}$ is a diffeomorphism on $\hat{\Omega}$. If a decomposition and inequalities above hold for $\Omega$ and $f$, then it also holds for $\hat{\Omega}$ and $\hat{f}$. We say that $f$ satisfies Axiom A if $f$ is hyperbolic on $\Omega$ and periodic points are dense in $\Omega$.

Proof of Theorem 6. We shall show this by induction. For each $S_{k+2}$-equivariant map $g$, it is clear that $g_{L^1}$ satisfies Axiom A for each $L^1$ from a theorem of critically finite functions. We only show that $g_{L^2}$ satisfies Axiom A for some $L^2$. An argument for $g_{L^m}, 3 \leq m \leq k$, is similar as for $g_{L^2}$. So let us fix some $L^2$. First we shall show that $g_{L^2}$ is hyperbolic on $\Omega(g_{L^2})$. 


Next we shall show that periodic points of $g|_{L^2}$ are dense in $\Omega(g|_{L^2})$. Let denote $g|_{L^2}$ and $\Omega(g|_{L^2})$ by $g$ and $\Omega$ for simplicity.

If $g$ is hyperbolic on $\Omega$, $\Omega$ has a decomposition to $S_i$, where $i=1,2,3$ indicates the unstable dimensions. Since $C(g)$ attracts all nearby points, it follows that $\cup L^0 \subset S_0$ and $\cup J(g|_{L^1}) \subset S_1$, where $g|_{L^0}$ is contracting for all direction and $g|_{J(g|_{L^1})}$ is contracting for a certain explicit direction and expanding for an $L^1$-direction. Let us consider a compact, completely invariant subset in the complement of $C$ in $L^2$,

$$S := \{ x \in \mathbb{P}^2 \mid \text{dist}(f^n(x), C) \to 0 \text{ as } n \to \infty \}. $$

It is clear that $S \cap C = \emptyset$ and $S \supset J_2 \neq \emptyset$. Here $J_2$ is the second Julia set, in which repelling periodic points are dense. By the definition of $S$,

$$\Omega = (\cup L^0) \cup (\cup J(g|_{L^1})) \cup S.$$ 

If we show that $g$ is expanding on $S$, it follows that $\cup L^0 = S_0$, $\cup J(g|_{L^1}) = S_1$, $S = S_2$. Thus $g$ is hyperbolic on $\Omega$.

Let us show that $g$ is expanding on $S$. Since $f$ is attracting on $C$ and $f(C) = C$, there exists a neighborhood $N$ of $C$ such that $N \Subset g^{-1}(N)$ and $B := \mathbb{P}^2 \setminus N$ is connected. Let $U$ be one of connected components of $\mathbb{P}^2 \setminus g^{-1}(N)$. Let one of $L^1$'s be the line at infinity of $\mathbb{P}^2$, then

$$U \subset \mathbb{P}^2 \setminus g^{-1}(N) \Subset B \subset \mathbb{C}^2 = \mathbb{P}^2 \setminus L^1.$$ 

Since the map $g$ from $U$ to $B$ is an unbranched covering,

$$K_U(x, v) = K_B(g(x), Dg(v)), \forall x \in U, v \in T_x \mathbb{C}^2.$$ 

Since $B$ and all connected components of $\mathbb{P}^2 \setminus g^{-1}(N)$ are in one local chart, there exists a constant number $\rho < 1$ such that for any $U$

$$K_B(x, v) \leq \rho K_U(x, v), \forall x \in U, v \in T_x \mathbb{C}^2.$$ 

Thus,

$$K_B(x, v) \leq \rho K_B(g(x), Dg(v)), \forall x \in \mathbb{P}^2 \setminus g^{-1}(N), v \in T_x \mathbb{C}^2.$$ 

Since $g^n(x)$ belongs to $S$, which is contained in $\mathbb{P}^2 \setminus g^{-1}(N)$, for any $x$ which belongs to $S$ and for any $n \geq 1$, we have that

$$K_B(x, v) \leq \rho^n K_B(g^n(x), Dg^n(v)), \forall x \in S, v \in T_x \mathbb{C}^2.$$ 

Thus,

$$K_B(g^n(x), Dg^n(v)) \geq \lambda^n K_B(x, v), \forall x \in S, v \in T_x \mathbb{C}^2, \lambda = \frac{1}{\rho} > 1.$$ 

Since $K_B(x, v)$ is upper semicontinuous and $|v|$ is continuous, $K_B(x, v)$ and $|v|$ may be different only by a constant factor. There exists $c > 0$ such that

$$|Dg^n(x)v| \geq c\lambda^n |v|, \forall x \in S, v \in T_x \mathbb{C}^2.$$
Thus $g$ is expanding on $S$ and hyperbolic on $\Omega$.

Next we shall show that periodic points are dense in $\Omega$. It is enough to show that $J_2 = S_2$ since periodic points are dense in $J(g|_{L^1})$ and $J_2$. This follows from the same argument in FS [8, Theorem3.8]. Let us recall that proof. Let $\sigma$ be $S_2 \setminus J_2$ and suppose that $\sigma$ is not empty. Since $\sigma$ is attracting for inverse branches of $f^n$, $\sigma$ is disjoint from $J_2$ and is closed. Since $f(C) = C$, one can define holomorphic local branches of inverses of $f^n$ in $P^2 \setminus C$. Then this family $\{f_i^{-n}\}_{i,n \geq 0}$ becomes a normal family. For any continuous function $\phi$ on $P^2$, we define

$$A^n_{\phi}(x) := \frac{1}{d^2n} \sum_{i=1}^{d^n} \phi(f_i^{-n}(x)).$$

In this case $\{A^n_{\phi}\}_{n \geq 0}$ is locally equicontinuous in $P^2 \setminus C$ and

$$A^n_{\phi}(x) \to \mu(\phi) \text{ as } n \to \infty, \forall x \in P^2 \setminus C, \quad (3)$$

where $\mu$ is the invariant probability measure whose support is $J_2$. Let $\phi = 1$ in a neighborhood of $J_2$ and $\phi = 0$ in a neighborhood of $\sigma$. Since $f^{-1}(\sigma) = \sigma$, $A^n_{\phi} \equiv 0$ in $\sigma$ for any $n$. On the other hand, by (3)

$$A^n_{\phi}(x) \to \mu(\phi) = 1 \text{ as } n \to \infty, \forall x \in \sigma \subset P^2 \setminus C.$$

This contradiction implies that $\sigma$ is empty. Thus $J_2 = S_2$ and periodic points are dense in $\Omega$.

□

References


