Dynamical properties of holomorphic maps with symmetries on projective spaces (Complex Dynamics and its Related Fields)

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Dynamical properties of holomorphic maps with symmetries on projective spaces

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We consider complex dynamics of a holomorphic map from $\mathbb{P}^k$ to $\mathbb{P}^k$, which has symmetries associated with the symmetric group $S_{k+2}$ acting on $\mathbb{P}^k$, for each $k \geq 1$. Here $\mathbb{P}^k$ denotes the $k$-dimensional complex projective space. Informations about critical orbits lead us to global dynamics results: The Fatou set of each map of this family consists of attractive basins of superattracting points determined by an action of the symmetric group $S_{k+2}$. Furthermore each map of this family satisfies Axiom A.

1 $S_{k+2}$-equivariant maps

For a rational map $f$ and a finite group $G$ acting on $\mathbb{P}^k$ as projective transformations, we say that $f$ is $G$-equivariant if $f$ commutes with each element of $G$, that is, $f \circ r = r \circ f$ for any $r \in G$. Doyle and McMullen [1] introduced a notion of $G$-equivariant functions on $\mathbb{P}^1$ to solve quintic equations. See also Ushiki [2] for $G$-equivariant functions on $\mathbb{P}^1$. Crass [3, 4] extended Doyle and McMullen's algorithm to higher dimensions to solve polynomial equations. Crass [5] found good pairs of $G$ and $f$ for which one may say something about global dynamics.

Crass [5] selected the symmetric group $S_{k+2}$ as a finite group acting on $\mathbb{P}^k$ and found an $S_{k+2}$-equivariant map $g_{k+3}$ which is holomorphic and critically finite, for each $k \geq 1$. Holomorphy means that $f$ is well-defined at any point in $\mathbb{P}^k$. We denote by $C = C(f)$ the critical set of $f$ and say that $f$ is critically finite if each irreducible component of $C(f)$ is periodic or...
eventually periodic. In addition, the complement of $C(g_{k+3})$ is Kobayashi hyperbolic so that we can use Kobayashi metrics to prove our theorems.

### 1.1 Existence of $S_{k+2}$-equivariant maps

An action of $S_{k+2}$ on $P^k$ is induced by the permutation action of $S_{k+2}$ on $C^{k+2}$ for each $k \geq 1$. The transposition $(ij)$ in $S_{k+2}$ corresponds with the involution $u_i \leftrightarrow u_j$ on $C^{k+2} = \{u = (u_1, \ldots, u_{k+2}) | u_i \in C\}$. This action pointwise fixes the hyperplane $\{u_i = u_j\}$. Since $S_{k+2}$ preserves a hyperplane $H$,

$$H = \{\sum_{i=1}^{k+2} u_i = 0\} \cong C^{k+1} = \{x = (x_1, \ldots, x_{k+1}) | x_i \in C\},$$

the permutation action of the symmetric group $S_{k+2}$ on $C^{k+2}$ induces an action of $S_{k+2}$ on $C^{k+1}$ and $T$ is a matrix which corresponds with $(1, k+2)$ in $S_{k+2}$.

$$T = \begin{pmatrix}
-1 & 0 & \cdots & 0 \\
-1 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-1 & 0 & 0 & 1
\end{pmatrix}, \quad A = \begin{pmatrix}
1 & 0 & \cdots & 0 & -1 \\
0 & 1 & \cdots & 0 & -1 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 1 & -1
\end{pmatrix}, \quad Au = x.$$

Induced hyperplanes in $C^{k+1}$ are $\{x_i = 0\}$, $1 \leq i \leq k + 1$, and $\{x_i = x_j\}$, $1 \leq i < j \leq k + 1$. This action of $S_{k+2}$ on $C^{k+1}$ projects naturally to the action of $S_{k+2}$ on $P^k$ and we denote it by $S_{k+2}$ for simplicity.

To get $S_{k+2}$-equivariant maps on $P^k$ which are critically finite, we have the critical set coincide with the union of these hyperplanes.

**Theorem 1 (Crass [5]).** For each $k \geq 1$, $g_{k+3}$ defined below is the unique $S_{k+2}$-equivariant holomorphic map of degree $k + 3$ which is doubly critical on each hyperplane.

$$g = g_{k+3} := [g_{k+3,1} : g_{k+3,2} : \cdots : g_{k+3,k+1}],$$

$$g_{k+3,l} = x_l^3 \sum_{s=0}^{k} (-1)^s \frac{s+1}{s+3} x_l^s A_{k-s},$$

where $A_{k-s}$ is the elementary symmetric function of degree $k-s$ in $x_1, x_2, \ldots, x_{k+1}$ and $A_0 = 1$.

Then $C(g)$ coincide with the union of hyperplanes. Since $g$ is $S_{k+2}$-equivariant and each hyperplane is pointwise fixed by some action of $S_{k+2}$, $g$ preserves each hyperplane. In particular $g$ is critically finite.
1.2 Properties of $S_{k+2}$-equivariant maps

Let us look at properties of an $S_{k+2}$-equivariant map $g_{k+3}$, which is proved in Crass [5] and will be used to prove our results. Let $L^{k-1}$ denote one of hyperplanes \( \{x_i = x_j\} \) and \( \{x_i = 0\} \). Let $L^m$ denote one of intersections of \((k - m)\) distinct $L^{k-1}$'s for $m = 1, 2, \ldots, k - 1$. Clearly $L^m \simeq \mathbb{P}^m$ for $m = 1, 2, \ldots, k$.

First let us look at properties of $g$ itself. The critical set of $g$ consists of the union of hyperplanes and $g$ preserves each hyperplane. In particular $g$ is critically finite. Furthermore $\mathbb{P}^1 \setminus C(g)$ is Kobayashi hyperbolic.

Next let us look at properties of $g$ restricted to $L^m$ for $m = 1, 2, \ldots, k - 1$. Since $g$ preserves each $L^m$, we can also consider dynamics of $g$ restricted to $L^m$. Each restricted map $g|_{L^m}$ has the same properties as above. Let us fix some $L^m$. The critical set of $g|_{L^m}$ consists of union of hyperplanes in $L^m$. Here $L^{m-1}$, a hyperplane in $L^m$, is a intersection of $L^m$ and another $L^{k-1}$. And $g|_{L^m}$ preserves each hyperplane $L^{m-1}$ of $L^m$. In particular $g|_{L^m}$ is critically finite. Furthermore $L^m \setminus C(g|_{L^m})$ is Kobayashi hyperbolic.

Finally let us look at properties of superattracting fixed points of $g$. The set of superattracting points, where the derivative of $g$ vanishes for all directions, coincide with the set of $L^0$'s.

Remark 1. For any $k \geq 1$ and $m \geq 1$, any restricted map $g_{k+3}|_{L^m}$ of $g_{k+3}$ to some $L^m$ is not conjugate to $g_{m+3}$.

1.3 Examples for $k = 1$ and $2$

Let us see hyperplanes of an $S_3$-equivariant function $g_4$ and an $S_4$-equivariant map $g_5$ for make clear what $L^m$ means. We do not write explicit forms of $g_5$ and $g_5|_{L^1}$. See Crass [5] for details.

1.3.1 An $S_3$-equivariant function $g_4$ in $\mathbb{P}^1$

\[
g_3([x_1 : x_2]) = [x_1^2(-x_1 + 2x_2) : x_2^2(2x_1 - x_2)],
\]

\[
C(g_3) = \{x_1 = 0\} \cup \{x_2 = 0\} \cup \{x_1 = x_2\} = "\{0, 1, \infty\}\n\]

in $\mathbb{P}^1 = \{[x_1 : x_2] \ | (x_1, x_2) \in \mathbb{C}^2 \setminus \{0\}\} = \{z = \frac{x_1}{x_2} \ | x_2 \neq 0\} \cup \{\infty\}$.

In this case "hyperplanes" are points in $\mathbb{P}^1$ and $L^0$ denotes one of these superattracting fixed points of $g_3$. 
1.3.2 An $S_4$-equivariant map $g_5$ in $\mathbb{P}^2$

$C(g_5) = \{x_1 = 0\} \cup \{x_2 = 0\} \cup \{x_3 = 0\} \cup \{x_1 = x_2\} \cup \{x_2 = x_3\} \cup \{x_3 = x_1\}$

in $\mathbb{P}^2 = \{[x_1 : x_2 : x_3] \mid (x_1, x_2, x_3) \in \mathbb{C}^3 \setminus \{0\}\}$.

In this case $L^1$ denotes one of irreducible components of $C(g_5)$, which is a hyperplane in $\mathbb{P}^2$. For example let us fix a hyperplane $\{x_1 = 0\}$. Since $g_5$ preserves each $L^1$, we can also consider dynamics of $g_5$ restricted to $\{x_1 = 0\}$. The critical set of $g_5|_{\{x_1 = 0\}}$ in $\{x_1 = 0\} \simeq \mathbb{P}^1$ is

$$C(g_5|_{\{x_1 = 0\}}) = \{[0 : 1 : 0], [0 : 0 : 1], [0 : 1 : 1]\}.$$

When we use $L^0$ after such sentences above, $L^0$ means one of intersections of $\{x_1 = 0\}$ and another $L^1$, which is a superattracting fixed point of $g_5|_{\{x_1 = 0\}}$ in $\mathbb{P}^1$. The set of superattracting points of $g_5$ in $\mathbb{P}^2$ is

$$\{(1 : 0 : 0), [0 : 0 : 1], [0 : 0 : 1], [1 : 1 : 0], [1 : 0 : 1], [0 : 1 : 1]\}.$$

Sometimes $L^0$ denotes one of intersections of two or more $L^1$'s, which is a superattracting fixed point of $g_5$ in $\mathbb{P}^2$.

2 The Fatou sets of $S_{k+2}$-equivariant maps

Theorem 2 (Ueno). For each $k \geq 1$, the Fatou set of $g_{k+3}$ consists of attractive basins of superattracting points which are intersections of $k$ distinct hyperplanes.

Before starting a proof of Theorem 2, let us recall theorems about critically finite holomorphic maps and a notion of Kobayashi metrics. Let $f$ be a holomorphic map from $\mathbb{P}^k$ to $\mathbb{P}^k$ and $U$ a Fatou component. A holomorphic map $h$ is said to be a limit map on $U$ if there is a subsequence $\{f^{n_j}|_{U}\}_{j \geq 0}$ which locally converges to $h$ on $U$. We say that a point $q$ is a Fatou limit point if there is a limit map $h$ on $U$ such that $q \in h(U)$. The set of all Fatou limit points will be called the Fatou limit set. We define the $\omega$-limit set of the critical points by

$$E = \bigcap_{j=1}^{\infty} f^j(D), \quad D = \bigcup_{j=1}^{\infty} f^j(C).$$

Theorem 3. (Ueda [6, Proposition 5.1]) If $f$ is a critically finite holomorphic map from $\mathbb{P}^k$ to $\mathbb{P}^k$, then the Fatou limit set is contained in $E$. 


Let $K_M(x, v)$ be a Kobayashi quasimetric on a complex manifold $M$.

\[ \inf \left\{ |a| : \varphi : D \to M : \text{holomorphic}, \varphi(0) = x, D\varphi \left( a \left( \frac{\partial}{\partial z} \right)_0 \right) = v, a \in \mathbb{C} \right\} \]

for $x \in M, v \in T_xM, z \in D$, where $D$ is the unit disk in $\mathbb{C}$. We say that $M$ is Kobayashi hyperbolic if $K_M$ becomes a metric. Theorem 2 is a corollary of Theorem 4 and Theorem 5 for $k = 1$ and 2.

**Theorem 4.** If $f$ is a critically finite function from $\mathbb{P}^1$ to $\mathbb{P}^1$, then the only Fatou components of $f$ are attractive components of superattracting points.

**Theorem 5.** (Fornaess and Sibony [7, theorem 7.7]) If $f$ is a critically finite holomorphic map from $\mathbb{P}^2$ to $\mathbb{P}^2$ and the complement of $C(f)$ is Kobayashi hyperbolic, then the only Fatou components of $f$ are superattractive components of superattracting points.

We can apply an argument in FS [7] to an $S_{k+2}$-equivariant map $g_{k+3}$ because each $L^{m-1}$ is smooth and $L^m \setminus C(g|_{L^m})$ is Kobayashi hyperbolic for $m = 1, 2, \ldots, k$.

**Proof of Theorem 2.** Take any Fatou component $U$ and any point $x \in U$. It is enough to show that $\{g^n(x)\}_{n \geq 0}$ accumulates to some $L^0$, one of superattracting fixed points. By theorem 3 $\{g^n(x)\}_{n \geq 0}$ accumulates to $C(g)$. Since $C(g)$ is the union of $L^{k-1}$s, there exists a smallest integer $m$ such that $\{g^n(x)\}_{n \geq 0}$ accumulates to some $L^m$. Let $m$ be $k - 1$ for simplicity. By using Kobayashi metrics and an argument in FS [7], we shall show the following result later,

\[ \exists n_k \in \mathbb{N} \text{ s.t. } g^{n_k}(U) \cap L^{k-1} \neq \emptyset. \tag{1} \]

Next let $U_{k-1}$ be $g^{n_k}(U) \cap L^{k-1}$ and do the same thing as above. Then

\[ \exists n_{k-1} \in \mathbb{N}, \exists L^{k-2} \text{ s.t. } g^{n_{k-1}}(U_{k-1}) \cap L^{k-2} \neq \emptyset. \]

Let $U_{k-2}$ be $g^{n_{k-1}}(U) \cap L^{k-2}$ and do the same thing as above. These reductions finally come to some $L^1$. Let $U_2$ be $g^{n_k + n_{k-1} + \cdots + n_2}(U) \subset L^2$, then

\[ \exists n_2 \in \mathbb{N}, \exists L^1 \text{ s.t. } g^{n_2}(U_2) \cap L^1 \neq \emptyset. \]

Let $U_1$ be $g^{n_2}(U_2) \cap L^1$. By Theorem 4 there exists $n_1$ such that $g^n$ sends $U_1$ to an attracting component of some superattracting fixed point $L^0$ in $L^1 \simeq \mathbb{P}^1$. Hence $g^{n_k + n_{k-1} + \cdots + n_1}$ sends $U$ to an attracting component of a superattracting fixed point $L^0$ in $\mathbb{P}^k$. 


To prove (1), let us assume that (1) is not true and derive a contradiction. By Theorem 3 $h(x)$ belongs to $C(g)$ for a limit map $h$ of convergent subsequence $\{g_{n_{s}}|_{U}\}_{s\geq 0}$. So there exists a smallest integer $m$ such that $h(x)$ belongs to some $L^{m}$. If $h$ is open map from $U$ to $L^{m}$, then $h(U)$ is an open set in $L^{m}$ and is contained in $F(g_{1}^{m})$. The same argument of reductions as above implies that $\{g_{n}(x)\}$ accumulates to one of $L^{0}$. That is, there exists $n$ such that $g^{n}$ sends $U$ to an attracting component of $L^{0}$, which is a contradiction.

To show that $h$ is open map from $U$ to $L^{m}$, we shall use Kobayashi metrics. Let $A$ be $P^{k} \setminus g^{-1}(C(g))$ and let $B$ be $P^{k} \setminus C(g)$. Since $B$ is Kobayashi hyperbolic and $A \subset B$, $A$ is also Kobayashi hyperbolic. So we can use Kobayashi metrics $K_{A}$ and $K_{B}$. By $A \subset B$ $K_{B}(y, v) \leq K_{A}(y, v), \forall y \in A, v \in T_{y}P^{k}$.

Since $g$ is an unbranched covering from $A$ to $B$, $K_{A}(y, v) = K_{B}(g(y), Dg(v)), \forall y \in A, v \in T_{y}P^{k}$. 

Thus, $K_{B}(y, v) \leq K_{B}(g(y), Dg(v)), \forall y \in A, v \in T_{y}P^{k}$. 

Since the same argument holds for any $g^{n}$ from $P^{k} \setminus g^{-n}(C(g))$ to $P^{k} \setminus C(g)$, $K_{B}(y, v) \leq K_{B}(g^{n}(y), Dg^{n}(v)), \forall y \in P^{k} \setminus g^{-n}(C(g)), v \in T_{y}P^{k}$. (2)

Since $g^{n}$ is an unbranched covering from $U$ to $g^{n}(U)$ and $g^{n}(U) \subset B$ for any $n$, $K_{B}(g^{n}(x), Dg^{n}(v))$ is bounded, $K_{B}(g^{n}(y), Dg^{n}(v)) \leq K_{g^{n}(U)}(g^{n}(y), Dg^{n}(v)) = K_{U}(y, v) < \infty$.

We claim that for unit vectors $v_{n} \in T_{x}U$ such that $Dg^{n}(x)v_{n}$ keeps parallel to $L^{m}$, $Dh(x)v \neq 0 = (0, 0, \cdots, 0)$ for an accumulation vector $v$ of $v_{n}$. Let $h = \lim_{n \rightarrow \infty}g^{n}$ for simplicity. One can choose a local chart around $h(x)$ so that $h(x) = 0$ and $L^{m} = \{y = (y_{1}, y_{2}, \cdots, y_{k})|y_{1} = \cdots = y_{k-m} = 0\}$. In this chart there exists $r > 0$ such that polydisk $P(0, r)$ is disjoint from $L^{k-1}$ which does not include $L^{m}$. Since $g^{n}(x) \rightarrow 0$ as $n \rightarrow \infty$, we may assume $g^{n}(x) \in P(0, r)$. By assumption that (1) is not true, $g^{n}(x) \notin C(g)$ for any $n \geq 1$. Thus one can define maps $\varphi_{n}$ from $D$ to $P(0, r)$ for $z \in D$, $\varphi_{n}(z) := g^{n}(x) + rz_{k} = g^{n}(x) + (0, \cdots, 0, rz)$. Here $e_{k} = (0, \cdots, 0, 1)$. Then $\varphi_{n}(0) = g^{n}(x)$ and $\varphi_{n}(D) \subset P^{k} \setminus g^{-n}(C(g))$. Let us choose unit vectors $v_{n}$ so that $Dg^{n}(x)v_{n} = \lim_{n \rightarrow \infty}Dg^{n}(x)v_{n}$.

By the definition of Kobayashi metric, $K_{B}(g^{n}(x), Dg^{n}(x)v_{n}) \leq \frac{|Dg^{n}(x)v_{n}|}{r}$. 
Suppose $Dh(x)v = 0$, then $Dg^n(x)v \to 0$ and $Dg^n(x)v_n \to 0$ as $n \to \infty$.

\[ \therefore K_B(g^n(x), Dg^n(x)v_n) \leq \frac{|Dg^n(x)v_n|}{r} \to 0. \]

On the other hand, by (2)

\[ 0 < \inf_{|v|=1} K_B(x, v) \leq K_B(x, v_n) \leq K_B(g^n(x), Dg^n(x)v_n). \]

Hence $K_B(g^n(x), Dg^n(x)v_n)$ is bounded away from 0 uniformly and this contradiction completes the proof.

\[ \square \]

3 \textit{S}_{k+2}-equivariant maps and Axiom A

Theorem 6 (Ueno). For each $k \geq 1$, $g_{k+3}$ satisfies Axiom A.

First let us define hyperbolicity of maps and a notion of Axiom A. See Jonsson [9] for details. Let $f$ be a holomorphic map from $\mathbb{P}^k$ to $\mathbb{P}^k$.

\[ \Omega := \{ x \in \mathbb{P}^k \mid \forall U: \text{neighborhood of } x, \exists n \in \mathbb{N} \text{ s.t. } f^n(U) \cap U \neq \emptyset \}. \]

This set is called the non-wandering set, which is compact and forward invariant. We say that $f$ is hyperbolic on $\Omega$ if there exists a continuous decomposition $T_\Omega = E^u + E^s$ such that $D\widehat{f}(E^u/\overline{x}) \subset E^u/\overline{f(x)}$ and if there exists $c > 0, \lambda > 1$ such that for any $n \geq 1$,

\[ |D\widehat{f}^n(v)| \geq c\lambda^n |v|, \forall v \in E^u, \]

\[ |D\widehat{f}^n(v)| \leq c^{-1}\lambda^{-n} |v|, \forall v \in E^s. \]

Here $\widehat{\Omega}$ is the set of histories in $\Omega$ and $\widehat{f}$ is a diffeomorphism on $\widehat{\Omega}$. If a decomposition and inequalities above hold for $\Omega$ and $f$, then it also holds for $\widehat{\Omega}$ and $\widehat{f}$. We say that $f$ satisfies Axiom A if $f$ is hyperbolic on $\Omega$ and periodic points are dense in $\Omega$.

\textit{Proof of Theorem 6}. We shall show this by induction. For each $S_{k+2}$-equivariant map $g$, it is clear that $g|_{L^1}$ satisfies Axiom A for each $L^1$ from a theorem of critically finite functions. We only show that $g|_{L^2}$ satisfies Axiom A for some $L^2$. An argument for $g|_{L^m}, 3 \leq m \leq k$, is similar as for $g|_{L^2}$. So let us fix some $L^2$. First we shall show that $g|_{L^2}$ is hyperbolic on $\Omega(g|_{L^2})$. 

Next we shall show that periodic points of $g|_{L^2}$ are dense in $\Omega(g|_{L^2})$. Let denote $g|_{L^2}$ and $\Omega(g|_{L^2})$ by $g$ and $\Omega$ for simplicity.

If $g$ is hyperbolic on $\Omega$, $\Omega$ has a decomposition to $S_i$, where $i=1,2,3$ indicate the unstable dimensions. Since $C(g)$ attracts all nearby points, it follows that $\cup L^0 \subset S_0$ and $\cup J(g|_{L^1}) \subset S_1$, where $g|_{L^0}$ is contracting for all direction and $g|_{J(g|_{L^1})}$ is contracting for a certain explicit direction and expanding for an $L^1$-direction. Let us consider a compact, completely invariant subset in the complement of $C$ in $L^2$,

$$S := \{ x \in P^2 | \text{dist}(f^n(x), C) \rightarrow 0 \text{ as } n \rightarrow \infty \}.$$ 

It is clear that $S \cap C = \emptyset$ and $S \supset J_2 \neq \phi$. Here $J_2$ is the second Julia set, in which repelling periodic points are dense. By the definition of $S$,

$$\Omega = (\cup L^0) \cup (\cup J(g|_{L^1})) \cup S.$$ 

If we show that $g$ is expanding on $S$, it follows that $\cup L^0 = S_0$, $\cup J(g|_{L^1}) = S_1$, $S = S_2$. Thus $g$ is hyperbolic on $\Omega$.

Let us show that $g$ is expanding on $S$. Since $f$ is attracting on $C$ and $f(C) = C$, there exists a neighborhood $N$ of $C$ such that $N \subset g^{-1}(N)$ and $B := P^2 \setminus N$ is connected. Let $U$ be one of connected components of $P^2 \setminus g^{-1}(N)$. Let one of $L^1$'s be the line at infinity of $P^2$, then

$$U \subset P^2 \setminus g^{-1}(N) \Subset B \subset C^2 = P^2 \setminus L^1.$$ 

Since the map $g$ from $U$ to $B$ is an unbranched covering,

$$K_U(x, v) = K_B(g(x), Dg(v)), \forall x \in U, v \in T_xC^2.$$ 

Since $B$ and all connected components of $P^2 \setminus g^{-1}(N)$ are in one local chart, there exists a constant number $\rho < 1$ such that for any $U$

$$K_B(x, v) \leq \rho K_U(x, v), \forall x \in U, v \in T_xC^2.$$ 

$$\therefore K_B(x, v) \leq \rho K_B(g(x), Dg(v)), \forall x \in P^2 \setminus g^{-1}(N), v \in T_xC^2.$$ 

Since $g^n(x)$ belongs to $S$, which is contained in $P^2 \setminus g^{-1}(N)$, for any $x$ which belongs to $S$ and for any $n \geq 1$, we have that

$$K_B(x, v) \leq \rho^n K_B(g^n(x), Dg^n(v)), \forall x \in S, v \in T_xC^2.$$ 

$$\therefore K_B(g^n(x), Dg^n(v)) \geq \lambda^n K_B(x, v), \forall x \in S, v \in T_xC^2,$$ 

where $\lambda = \frac{1}{\rho} > 1$.

Since $K_B(x, v)$ is upper semicontinuous and $|v|$ is continuous, $K_B(x, v)$ and $|v|$ may be different only by a constant factor. There exists $c > 0$ such that

$$|Dg^n(x)v| \geq c\lambda^n |v|, \forall x \in S, v \in T_xC^2.$$
Thus $g$ is expanding on $S$ and hyperbolic on $\Omega$.

Next we shall show that periodic points are dense in $\Omega$. It is enough to show that $J_2 = S_2$ since periodic points are dense in $J(g|_{L^1})$ and $J_2$. This follows from the same argument in FS [8, Theorem 3.8]. Let us recall that proof. Let $\sigma$ be $S_2 \setminus J_2$ and suppose that $\sigma$ is not empty. Since $\sigma$ is attracting for inverse branches of $f^m$, $\sigma$ is disjoint from $J_2$ and is closed. Since $f(C) = C$, one can define holomorphic local branches of inverses of $f^n$ in $\mathbb{P}^2 \setminus C$. Then this family $\{f^{-n}\}_{n \geq 0}$ becomes a normal family. For any continuous function $\phi$ on $\mathbb{P}^2$, we define

$$A_\phi^0(x) := \frac{1}{d^n} \sum_{i=1}^{d^n} \phi(f_i^{-n}(x)).$$

In this case $\{A_\phi^n\}_{n \geq 0}$ is locally equicontinuous in $\mathbb{P}^2 \setminus C$ and

$$A_\phi^n(x) \to \mu(\phi) \text{ as } n \to \infty, \forall x \in \mathbb{P}^2 \setminus C,$$

(3)

where $\mu$ is the invariant probability measure whose support is $J_2$. Let $\phi = 1$ in a neighborhood of $J_2$ and $\phi = 0$ in a neighborhood of $\sigma$. Since $f^{-1}(\sigma) = \sigma$, $A_\phi^0 \equiv 0$ in $\sigma$ for any $n$. On the other hand, by (3)

$$A_\phi^n(x) \to \mu(\phi) = 1 \text{ as } n \to \infty, \forall x \in \sigma \subset \mathbb{P}^2 \setminus C.$$ 

This contradiction implies that $\sigma$ is empty. Thus $J_2 = S_2$ and periodic points are dense in $\Omega$.

References


