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Kyoto University
On cubic polynomials with a parabolic fixed point of a capture type

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Abstract
We consider the location of each critical point of a cubic polynomial map with a parabolic fixed point. We show that, for any given number of iterations, there exists a cubic polynomial map with a parabolic fixed point such that the immediate parabolic basin contains just one of the critical points and the image of another critical point under the specified number of iterations.

1 Introduction

Let \( f \) be any cubic polynomial. If \( f \) has a parabolic fixed point \( \alpha \), then a cycle of Fatou components of \( f \) is called the immediate parabolic basin for \( \alpha \) if the cycle contains a parabolic petal for \( \alpha \).

Roughly speaking, in this note we consider the dynamically location of each critical points of \( f \) with a parabolic fixed point whose basin contains both the critical points. We denote by \( c_0 \) and \( c_1 \) the critical points of \( f \). Using the Haissinsky pinching deformation, we prove the following result:

**Theorem 1.1.** For any positive integer \( n \), there exists a cubic polynomial map \( f \) with a parabolic fixed point such that the immediate parabolic basin contains \( c_0 \) and \( f_\text{at}(c_1) \), and does not contain \( f_\text{at}^k(c_1) \) for any integer \( k \) with \( 0 \leq k < n \).

Now, suppose that \( f \) has a parabolic fixed point, and the parabolic basin contains \( c_0 \) and \( c_1 \). By analogy with Milnor [3], we shall define the types of this parabolic fixed point. For \( j = 0,1 \), we denote by \( U_j \) the Fatou component which contains \( c_j \). Without loss of generality, we may assume that \( U_0 \) is contained in the immediate basin of the parabolic fixed point. Following from [3], there exist four possibilities as follows.
Case 1: The Fatou component is adjacent, i.e., $U_0 = U_1$.

Case 2: The Fatou component is bitransitive. Namely, $U_0 \neq U_1$, and moreover there exist the smallest positive integers $p, q > 0$ such that $f^{op}(U_0) = U_1$ and $f^{oq}(U_1) = U_0$.

Case 3: The immediate parabolic basin captures $U_1$. Namely, the immediate parabolic basin does not contain $U_1$, but $f^{ok}(U_1)$ for some integer $k \geq 1$.

Case 4: Each of $U_0$ and $U_1$ is contained in the disjoint cycle of the immediate parabolic basin. Namely, $U_0$ and $U_1$ is contained in the immediate parabolic basin, and it follows that $f^{on}(U_0) \cap f^{om}(U_1) = \emptyset$ for any integers $n, m \geq 0$.

We define the types of the parabolic fixed point $\alpha$ as follows:

Definition 1.2. In Case 1, 2, 3 or 4, we say that $\alpha$ is a parabolic fixed point of an adjacent, bitransitive, capture, or disjoint type, respectively.

We will consider the type of the parabolic fixed point $\alpha$ the cubic polynomial map obtained by the Haissinsky pinching deformation, which is illustrated in the next section.

2 The Haissinsky Pinching deformation

Suppose that $f$ is any cubic polynomial map with an attracting fixed point $\alpha$. Let $B_f(\alpha)$ be the attracting basin for $\alpha$. We consider the Haissinsky pinching deformation of $f$ defined by pinching curves in $B_f(\alpha)$.

Following from [1], for any integer $q \geq 1$, there exist a smooth open arc $\gamma$ and a neighborhood $U \subset B_f(\alpha)$ of $\gamma$ satisfying the following conditions.

- $\overline{\gamma} \setminus \gamma$ consists of the attracting fixed point $\alpha$ and a repelling periodic point $\beta$ of period $q$.
- $f^{oq}(\gamma) = \gamma$, $f^{oq}(U) = U$, and $f^{oq}|_U$ is univalent.
- $f^{on}(U) \cap f^{om}(U) = \emptyset$ for any $0 \leq n < m < q$.
- There exist a number $\sigma > 0$ and a conformal map $\Phi_\sigma : U \to \{|z| < \pi\}$ such that $\Phi_\sigma \circ f^{oq}(z) = \Phi_\sigma(z) + \sigma$ for all $z \in U$.

We call the union $S := \bigcup_{k \geq 0} f^{o-k}(\gamma)$ the support of pinching, and define $S_0 := \bigcup_{k \geq 0} f^{ok}(\gamma)$. It follows from [1] that we have a sequence of quasiconformal maps $(h_t)_{t \geq 0}$ satisfying the following conditions.
• $h_t$ converges uniformly on $\hat{\mathbb{C}}$ to a local quasiconformal map $h_\infty$ on $\hat{\mathbb{C}}\backslash S$.

• $f_t := h_t \circ f \circ h_t^{-1}$ converges uniformly on $\hat{\mathbb{C}}$ to a cubic polynomial $f_\infty$.

• $h_\infty(\alpha)$ is a parabolic fixed point of $f_\infty$.

• $h_\infty(S_0) = h_\infty(\alpha)$.

For further details, see [1] or [2].

3 Proof of Theorem 1.1

We first prove the following lemma needed later.

Lemma 3.1. Let $n$ be any positive integer, and let $\lambda$ be any complex number in $\mathbb{D}\setminus\{0\}$. Then there exists a cubic polynomial $f$, with $f^n(c_1) = c_0$, such that $f$ has an attracting fixed point of multiplier $\lambda$ whose attracting basin is simply connected.

Proof. Consider a monic and centered cubic polynomial

$$P_{A,B}(z) = z^3 - 3Az + \sqrt{B}, \quad (A, B) \in \mathbb{C}^2.$$  

Suppose that $P_{A,B}$ has a fixed point of multiplier $\lambda$. Then the fixed point is $\alpha_{A,\lambda} := \sqrt{A + \lambda}/3$, and hence, $P_{A,B}$ is affine conjugate to the cubic polynomial map

$$Q_{A,\lambda}(z) = z^3 + 3\alpha_{A,\lambda}z^2 + \lambda z$$

with critical points $c_{A,\lambda}^\pm := -\alpha_{A,\lambda} \pm \sqrt{A}$.

Suppose that $\lambda \in (-1,0)$, and the parameter $A$ is any real number $> -\lambda/3$ such that the attracting basin for zero is simply connected.

For each integer $k \geq 0$, we denote by $z_{A,\lambda}(k)$ the unique point on $\mathbb{R}_+$ such that $Q_{A,\lambda}^k(z_{A,\lambda}(k)) = c_{A,\lambda}^+$. For any integer $k > 0$ and for any real number $A'$ with $A' > A$, we have $z_{A,\lambda}(k) < z_{A,\lambda}(k + 1)$ and $z_{A,\lambda}(k) > z_{A',\lambda}(k)$. Thus since $Q_{A,\lambda}(c_{A,\lambda}^-) \to +\infty$ as $A \to +\infty$, for any integer $n > 0$ there exists a real number $A$ such that $Q_{A,\lambda}^n(c_{A,\lambda}^-) = c_{A,\lambda}^+$. 

Let $\lambda'$ be any complex number in $\mathbb{D}\setminus\{0\}$. Then it follows from [5] that there exists a quasiconformal map $h$ such that the cubic polynomial map $g := h \circ Q_{A,\lambda} \circ h^{-1}$ has an attracting fixed point with multiplier $\lambda'$. \qed

We use the Haissinsky pinching deformation of $f$ obtained from this lemma.
Proof of Theorem 1.1. Without loss of generality, we may assume that \( f(z) = z^3 + 3\alpha z^2 + \lambda z, \) \( c_0 = c_{A,\lambda}^+ \) and \( c_1 = c_{A,\lambda}^- \).

Suppose that \( \lambda \) is any real number with \(-1 < \lambda < 0\), and \( A \) is a real number \(-\lambda/3\) such that the attracting basin for zero is simply connected. Recall that \( B_f(0) \) is the attracting basin for zero. Let \( \varphi_f \) be the Koenigs map such that \( \varphi_f(0) = 0 \), and \( \varphi_f(z) = \lambda z \) for all \( z \in B_f(0) \). We may assume that \( \varphi_f(c_0) = 1 \).

Define the half-line \( \hat{\gamma} := i\mathbb{R}^+ \), so that \( \hat{\gamma} \) is periodic of period two under the iterates of the map \( L(z) := \lambda z \). We denoted by \( \gamma \) the connected component of the preimage of \( \hat{\gamma} \) under \( \varphi_f \) whose closure contains zero. Thus, we have the support of pinching \( S := \bigcup_{k \geq 0} f^{o-k}(\hat{\gamma}) \), and denote by \( f_\infty \) the limit of the Haissinsky pinching deformation of \( f \) defined by \( S \).

Let \( n \) be any positive integer. From Lemma 3.1, we have a parameter \( A \) such that \( f^{o-n}(c_1) = c_0 \). For each integer \( k \geq 1 \), we denote by \( \alpha(k) \) the point on \( \mathbb{R}_+ \) such that \( f^{o-k}(\alpha(k)) = 0 \), and by \( S_{\alpha(k)} \) the connected component of \( S \) which contains \( \alpha(k) \).

At first consider the case \( n \geq 2 \). Since for each integer \( k \geq 1 \) the component \( S_{\alpha(k)} \) separates the origin and \( f^{o-k}(c_1) \), it follows that \( f_\infty \) has a parabolic fixed point of a capture type.

Next, consider the case \( n = 1 \). Since no connected component of \( S \) separates the origin and \( c_1 \), it follows that \( f_\infty \) has a parabolic fixed point of a bitransitive type.

In order to obtain a polynomial with a parabolic fixed point of a capture type, we will use the Branner-Hubbard deformation of \( f \) obtained by wringing the almost complex structure on the attracting basin for zero (cf. [5]). In particular, we consider the Branner-Hubbard deformation which does not change the multiplier of the origin.

Let \( s = 1 + 2\pi i / \log \lambda \), and let \( l \) be the quasi-conformal map defined as \( l(z) := z |z|^{s-1} \).

Recall that \( \varphi_f \) is the Koenigs map defined on \( B_f(0) \). We define the holomorphic map \( \psi_f : \mathbb{D} \rightarrow \mathbb{C} \) as the inverse map of \( \varphi_f \) such that \( \psi_f(0) = 0 \).

Let \( \sigma_0 \) be the standard almost complex structure of \( \hat{\mathbb{C}} \), and let \( \sigma \) be the almost complex structure defined as follows:

\[
\sigma = \begin{cases} 
\sigma_0 & \text{on } \hat{\mathbb{C}} \setminus B_f(0) \\
(1 \circ \varphi_f)^*(\sigma_0) & \text{on } \psi_f(\mathbb{D}) \\
(1 \circ \varphi_f \circ f^o)^*(\sigma_0) & \text{on } f^{-k}(\psi_f(\mathbb{D})) \setminus f^{-k+1}(\psi_f(\mathbb{D}))
\end{cases}
\]

where \( k \) is an integer \( \geq 1 \).

From the Measurable Riemann Mapping Theorem, we obtain the quasi-conformal map \( h \) such that \( h^* \sigma_0 = \sigma \). Suppose that \( h(0) = 0 \), \( h(1) = 1 \)
and $h(\infty) = \infty$. Then, we obtain a cubic polynomial map $g = h \circ f \circ h^{-1}$ with the attracting fixed point zero. It follows from [5] that the multiplier is $g'(0) = h(\lambda) = \lambda |\lambda|^{s-1} = \lambda$, and that the Koenigs map $\varphi_g = l \circ \varphi_f \circ h^{-1}$.

Following from the argument similar to the above discussion, we define $S' \subset B_g(0)$ as the support of the pinching deformation, and denote by $g_\infty$ the limit of the pinching deformation of $g$ defined by the support $S'$.

There exists a cycle of connected components of $B_f(0) \setminus h^{-1}(S')$ under the iterates of $f$.

If $c_1$ is not contained in this cycle, then one of the critical points of $g_\infty$ is not contained in the immediate parabolic basin of $g_\infty$.

We consider the inverse image of $i\mathbb{R}$ under $\varphi \circ h^{-1}$. We introduce a preliminary definition as follows. For any point $z$ of the backward orbit of the origin, we denote by $D_f(z;r)$ the connected component of the set $\{w : |\varphi_f(w)| < r\}$ which contains the point $z$.

Since $f$ has no critical point in the open set $D_f(0;|\lambda|^{-1})$ except $c_0$, it follows that $f$ maps $D_f(0;|\lambda|^{-1}) \setminus \{c_0\}$ to $D_f(0;1) \setminus \{c_0\}$ in two-to-one correspondence. Thus $f$ has the unique preimage $\alpha'$ of the origin such that $\alpha' \neq 0$ and $\alpha' \in D_f(0;|\lambda|^{-1}) \setminus \{c_0\}$.

We extend $\psi_f$ to the conformal map $\psi_{f,0}$ defined on $D_f(0;|\lambda|^{-1}) \setminus \{c_0\}$ to a subset of $D_f(0;|\lambda|^{-1})$. Moreover, we define $\psi_{f,1}$ as the conformal map defined on $D_f(0;|\lambda|^{-1}) \setminus \{c_0\}$ such that $\varphi_f \circ \psi_{f,1} \equiv$ identity map and $\psi_{f,1}(0) = \alpha'$.

The end points of the image of the set $\{yi \mid -|\lambda|^{-1} < y < |\lambda|^{-1}\}$ under $\psi_{f,0} \circ h^{-1}$ is contained in the boundary of $\psi_{f,1}(D_f(0;|\lambda|^{-1}) \setminus \{c_0\})$. Hence, the connected component of the preimage of $i\mathbb{R}$ under $\varphi_f \circ h^{-1}$ which contains zero passes through the boundary of $\psi_{f,1}(D_f(0;|\lambda|^{-1}) \setminus \{c_0\})$, and does not separate $c_0$ and $c_1$. On the other hand, the connected component of the preimage of $i\mathbb{R}$ under $\varphi_f \circ h^{-1}$ which contains $\alpha'$ separates $c_0$ and $c_1$. Therefore, the cycle of the Fatou components of $g$ does not contain one of the critical points of $g$, and hence $g_\infty$ has a parabolic fixed point of a capture type. \qed 

### 4 Notes

Consider the family of cubic polynomials $P_{A,B}(z) := z^3 - 3Az + \sqrt{B}$ with $P_{A,B}(\sqrt{A}) = \sqrt{A}$. We have $B = A(1 - 2A)^2$. The connectedness locus of the family of $P_{A,A(1 - 2A)^2}(z) = z^3 - 3Az + \sqrt{A} - 2A\sqrt{A}, A \in \mathbb{C}$, is showed in Figure 2.
Figure 1: Sketch for the pinching curves.

Figure 2: The connectedness locus of the family of cubic polynomials $P_{A,A(1-2A)^3}$, $A \in \mathbb{C}$. 
$P_{A,A(1-2A)^2}$ is affine conjugate to the cubic polynomial map

$$F_A(z) := (P_{A,A(1-2A)^2}(\sqrt{A}z + \sqrt{A}) - \sqrt{A})/\sqrt{A} = Az^3 + 3Az^2 - 4A.$$ 

Suppose that $0 < |A| < 1/4$. Then the map $F_A$ satisfies the inequality $|F_A(z) + 4A| < |4A|$, that is, $F_A$ maps the disk of radius $|F_A(0)|$ centered at $F_A(0)$ into itself. Hence $F_A$ has an attracting fixed point in the disk.

Let $\alpha_A$ be the attracting fixed point.

**Proposition 4.1.** If $A$ turns around the origin once, then the multiplier of the attracting fixed point of $F_A$ turns around the origin twice.

**Proof.** Let $D$ be the disk of radius $|F_A(0)|$ centered at $F_A(0)$. If $A$ turns around the origin once, then the center of $D$ turns around the origin once.

Set $0 < r < 1/4$, $\theta \in [0,1]$, and $A = re^{2\pi i \theta}$. Since the radius of $D$ is the constant $|F_A(0)|$, the attracting fixed point $\alpha_A$ also turns around the origin once. Thus the multiplier $F'_A(\alpha_A) = 3A\alpha_A(\alpha_A + 2)$ turns around the origin twice. \[ \square \]

**References**


