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Bivariate Chebyshev maps of $\mathbf{C}^2$ and their dynamics

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Abstract

We study the properties of bivariate (two-dimensional) Chebyshev maps $T_n(x,y)$ from $\mathbf{C}^2$ to $\mathbf{C}^2$ and study the properties and dynamics of the maps.

(A) The properties of T_n.

(1) Solutions of $T_n(x,y)=(0,0)$ are obtained.

(2) A critical set $\det(DT_n)=0$ is written in a simple formula.

These properties are similar to those of Chebyshev maps of $\mathbf{C}$.

(B) The dynamics of T_n.

(1) T_n is strictly critically finite.

(2) Any periodic point of T_n is repelling.

(3) The exact form of the invariant probability measure μ of maximal entropy associated with T_n is obtained.

(4) External rays for $J_2(T_n)$ and foliations of $J_1(T_n)$ are studied.

These properties are also similar to those of Chebyshev maps of $\mathbf{C}$.

1 Bivariate Chebyshev maps

The Chebyshev map is a typical chaotic map. Generalized Chebyshev maps are studied by several researchers, Koornwinder [1974], Lidle [1975], Veselov [1987] and Hoffman & Withers [1988].

In this paper, we study bivariate Chebyshev maps T_n from $\mathbf{C}^2$ to $\mathbf{C}^2$, $n \in \mathbf{Z}$.

$$T_n(x,y)=(g^{(n)}(x,y), g^{(n)}(y,x)).$$

This definition is due to [V]. Here $g^{(n)}(x,y)$ is a generalized Chebyshev polynomial defined by Lidle [L].

Let

$$x = t_1 + t_2 + t_3, \quad y = t_1t_2 + t_1t_3 + t_2t_3, \quad 1 = t_1t_2t_3.$$

Then

$$g^{(n)}(x,y) := t_1^n + t_2^n + t_3^n.$$

So

$$g^{(n)}(y,x) = (1/t_1)^n + (1/t_2)^n + (1/t_3)^n = g^{(-n)}(x,y).$$

For instance,

$$T_2(x,y) = (x^2 - 2y, y^2 - 2x),$$
$$T_3(x,y) = (x^3 - 3xy + 3, y^3 - 3xy + 3),$$
$$T_4(x,y) = (x^4 - 4x^2y + 2y^2 + 4x, y^4 - 4xy^2 + 2x^2 + 4y).$$

$\{g^n(x,y)\}$ satisfy the following recurrence equation:

$$g^{(n)}(x,y) = xg^{(n-1)}(x,y) - yg^{(n-2)}(x,y) + g^{(n-3)}(x,y).$$

First, we show a branch covering over $\mathbf{C}^2$.
The following diagram is commutative.

$$\begin{array}{ccc} (\mathbf{C}-\{0\})^2 & \xrightarrow{g_n} & (\mathbf{C}-\{0\})^2 \\ \downarrow \Psi & & \downarrow \Psi \\ \mathbf{C}^2 & \xrightarrow{T_n} & \mathbf{C}^2 \end{array}$$

where $\qquad g_n(u,v) = (u^n, v^n),$

and

$$(x,y) = \Psi(u,v) = (u + v + \frac{1}{uv}, \frac{1}{u} + \frac{1}{v} + uv).$$

The covering map

$$\Psi : \mathbf{C}^2 - \Psi^{-1}(D) \to \mathbf{C}^2 - D$$

is a 6-sheated covering map. Branch locus D of Ψ is written as

$$x^2y^2 - 4x^3 - 4y^3 + 18xy - 27 = 0.$$

In the case $n = 2$, Ueda[Ue] showed this diagram.

$T_n(x,y)$ restricted on $\{x = \overline{y}\}$ is a Chebyshev polynomial defined by Koornwinder [K]

$$P_{n,0}^{-\frac{1}{2}}(z,\bar{z}) = e^{in\sigma} + e^{-in\tau} + e^{i(n\tau - n\sigma)}.$$

Set

$$z(\sigma,\tau) := e^{i\sigma} + e^{-i\tau} + e^{i(\tau-\sigma)} = u + iv.$$

The mapping

$$z : (\sigma,\tau) \to (u,v)$$

is a diffeomorphisam from R onto S. See Koornwinder [K].

Proposition 1. *There are n^2 solutions of $T_n(x,y)=(0,0)$. All solutions lie in the closed domain S in $\{x=\overline{y}\}$. They are written in the (σ,τ) coordinate.*

$$(1)\quad (\sigma,\tau)=(\frac{2(1+j+h)\pi}{3n},\frac{2(1+2j+h)\pi}{3n})$$

$$j=0,1,...,n-1, and \quad h=0,1,...,j.$$

$$(2)\quad (\sigma,\tau)=(\frac{2(2+j+h)\pi}{3n},\frac{2(2+2j-h)\pi}{3n})$$

$$j=0,1,...,n-2, and \quad h=0,1,...,j.$$

Proof. By definition,

$$T_n(x,y)=(g^{(n)}(x,y),g^{(n)}(y,x)).$$

$g^{(n)}(x,y)$ and $g^{(n)}(y,x)$ are polynomials of degree n with no common components. We can find n^2 zeros on S. See Uchimura [Uc1]. □

We see that the zeros of T_n and T_{n+1} "mutually separate each other".

Next we consider critical set of $T_n(x,y)$.

$$C_n:=\{(x,y)\in \mathbf{C}^2 : det(DT_n)=0\}.$$

Proposition 2. *Let $n\in\mathbf{Z}$. Assume that*

$$x=t_1+t_2+t_3,\quad y=t_1t_2+t_1t_3+t_2t_3,\quad t_1t_2t_3=1.$$

Then

$$Det(DT_n)=n^2\frac{t_1^n-t_2^n}{t_1-t_2}\cdot\frac{t_1^n-t_3^n}{t_1-t_3}\cdot\frac{t_2^n-t_3^n}{t_2-t_3}.$$

Proof.

$$Det(DT_n)=Det(D(T_n\circ\Psi))/Det(D\Psi).$$ □

The similar result is holds for generalized Chebyshev maps from $\mathbf{C}^n$ to $\mathbf{C}^n$.

Corollary 1. *Any irreducible component of C_n is a rational curve of degree 2 or 4.*

Proof. From Proposition 2, we have

$$x=t+\epsilon^k t+\frac{1}{\epsilon^k t^2},\qquad \epsilon=e^{\frac{2\pi i}{n}}$$

$$y = \frac{1}{t} + \frac{1}{\epsilon^k t} + \epsilon^k t^2.$$

When $\epsilon^k = -1$, the degree of the rational curve is 2. □

We see that C_n and C_{n+1} "mutually separate each other", and

$$C_n \cap S \neq \phi \qquad (S = J_2(T_n)).$$

Note that $\{T_m : m \in \mathbf{Z}\}$ is a semigroup satisfying

$$T_m \circ T_n = T_{mn}.$$

2 Dynamics of Bivariate Maps

We study the dynamics of $T_n(x, y)$. Let

$$K(T_n) := \{(x, y) : \{T_n^m(x, y)\} \text{is bounded for any } m\}.$$

In our setting we have the following proposition.

Proposition 3.

$$K(T_n) = \{|\, t_1 \,|=|\, t_2 \,|= 1\} = S \subset \{x = \overline{y}\}.$$

Proof

$$\begin{array}{ccc} (t_1, t_2) & \xrightarrow{g_n} & (t_1^n, t_2^n) \\ \downarrow \Psi & & \downarrow \Psi \\ (x, y) & \xrightarrow{T_n} & (g^{(n)}, g^{(-n)}) \end{array}$$

□

f is called *critically finite* if each irreducible component of the critical set of f is periodic or preperiodic. Dihn and Sibony [DS] show that generalized Chebyshev maps are critically finite. Here using proposition 2, we give a direct proof.

Proposition 4. *T_n is strictlly critically finite.*

Proof.

$$\begin{array}{ccccc} C_n & \xrightarrow{T_n} & T_n(C_n) & \xrightarrow{T_n} & T_n(C_n) \\ (t, \epsilon t) & & (t^n, t^n) & & (t^{n^2}, t^{n^2}) \end{array}$$

□

Next we study the second Julia set J_2 of $T_n(x, y)$.

Proposition 5. *All periodic points of T_n lie on S and are equidistributed in S.*

Proof. From [FS], we know that number of periodic points with period k equals n^{2k}. For the distribution of periodic points, see [Uc2]. □

Proposition 6. *Any periodic point of T_n is repelling.*

To prove this proposition we consider the following function.

$$S_n := T_n \mid \{x = \overline{y}\} : \mathbf{R}^2 \to \mathbf{R}^2$$

$$e.g. \qquad S_2(z) = z^2 - 2\overline{z} : (u, v) \mapsto (u^2 - 2u - v^2, 2uv + 2v).$$

Lemma 1. *Let p be a periodic point of S_n. Let α and β be eigen values of $DS_n(p)$. Then*

$$\mid \alpha \mid, \quad \mid \beta \mid > 1.$$

Proposition 7. *Let*

$$f(x, y) \in \mathbf{R}[x, y].$$

$$T(x, y) := (f(x, y), f(y, x)) : \mathbf{C}^2 \to \mathbf{C}^2.$$

$$t(z) := T \mid \{x = \overline{y}\} : \mathbf{R}^2 \to \mathbf{R}^2.$$

Then

$$U^{-1} DT(z, \overline{z}) U = Dt(z),$$

where

$$U = \frac{1}{2} \begin{pmatrix} 1+i & -1+i \\ 1+i & 1-i \end{pmatrix}.$$

From Lemma 1 and Proposition 7, Proposition 6 follows. □

Next we study the invariant measure μ of maximal entropy for T_n.

Proposition 8. *Under the above notation,*

$$supp \ \mu = S.$$

$$\mu = (\frac{2}{\pi})^2 \frac{dx_1 dx_2}{\sqrt{-x^2\overline{x}^2 + 4x^3 + 4\overline{x}^3 - 18x\overline{x} + 27}}.$$

$$(x = x_1 + ix_2)$$

This is an extension of invariant measure

$$\mu = \frac{1}{\pi}\frac{dx}{\sqrt{(x+1)(3-x)}}$$

for Chebyshev maps in one variable on $[-1,3]$.

Proof. We prove this proposition in the following three steps.

(1) Briend and Duval [BD] shows that

$$\text{let} \qquad \mu_n := \frac{1}{d^{nk}} \sum_{f^n(y)=y, y \text{repelling}} \delta_y,$$

then

$$\mu_n \to \mu \quad \text{(weak convergence)}.$$

(2) From Proposition 5, we see that the periodic points are repelling and equidistributed in the triangle on the (s,t) plane (see [Uc2]).

(3) Pullback of Lebesgue measure under ϕ. □

Next we consider the properties of external rays of $T_n(x,y)$. We use the definitions of external rays by Bedford and Jonsson [BJ] . We extend the map

$$T_n(x,y) : \mathbf{C}^2 \to \mathbf{C}^2$$

$$\text{to} \qquad \hat{T}_n(x:y:z) : \mathbf{P}^2 \to \mathbf{P}^2.$$

$$\text{Let} \qquad \Pi := \mathbf{P}^2 - \mathbf{C}^2 \qquad \text{be the line at infinity.}$$

Then

$$\hat{T}_n \mid \Pi : (x:y:0) \to (x^n : y^n : 0).$$

Therefore

$$J_\Pi = \{(x:y:0) : \mid x \mid = \mid y \mid\} \simeq S^1.$$

The stable set of J_Π for T_n is defined by

$$W^s(J_\Pi, T_n) := \{x \in \mathbf{P}^2 : d(T_n^j x, J_\Pi) \to 0, \quad j \to \infty\}.$$

Bedford and Jonsson [BJ] state that there exists a Böttcher coordinate Ψ such that

$$\Psi : W^s(J_\Pi, f_n) \to W^s(J_\Pi, T_n)$$

conjuating f_n to T_n, where

$$f_n(x,y) = (x^n, y^n).$$

They also show that $W^s(J_{\Pi}, T_n)$ is foliated by stable disks W_a. They define a local stable manifold $W^s_{loc}(a)$, $(a \in J_{\Pi})$ and then a stable disk $W_a \supset W^s_{loc}(a)$ and an external ray $R(a,\theta)$. They show that $J_0(T_n) = J_1(T_n)$ is laminated by stable disks W_a.

Nakane [N] shows the following results on $T_2(x,y)$:

(1) The map Ψ defined by Ueda is essentially the inverse of Böttcher coordinate ϕ.

$$\Psi(u,v) = \Psi(t, at), \mid t \mid > 1.$$

(2) The stable disk W_a is the set of points $R(r,\phi,\theta)$

$$x = re^{-2\pi i\theta} + \frac{1}{r}e^{2\pi i(\theta-\phi)} + e^{2\pi i\phi},$$

$$y = re^{2\pi i(\phi-\theta)} + \frac{1}{r}e^{2\pi i\theta} + e^{-2\pi i\phi}, \quad a = e^{2\pi i\phi}, \quad (r > 1).$$

An external ray is written as

$$R(\phi,\theta) := \{R(r,\phi,\theta) : r > 1\}.$$

From this,

$$J_2 = S \subset \{x = \bar{y}\}.$$

(3) Each point $z \in S$ is the landing point of exactly 1, 3, or 6 external rays if z is a cusp point on ∂S, z is non-cusp point on ∂S or $z \in int(S)$ respectively.

We can show that Nakane's results are also true for any $T_n(x,y)$, $\quad n \neq 0$.

Next we study the structure of foliations W_a of

$$J_1(T_n) = W^s(J_{\Pi}, T_n).$$

Proposition 9. *For any point $z \in int(S)$, there exist three stable disks W_a such that boundaries of these three disks intersect at z. At the point, two external rays on each W_a land from opposite directions.*

Metaphorically speaking, three mouths (stable disks) eat a sandwich (the second Julia set S).

Two external rays $R(\phi,\theta)$ and $R(\phi,\phi-\theta)$ lie on the stable disk

$$W_a \qquad (a = e^{2\pi i\phi}).$$

Two points $R(r,\phi,\theta)$ and $R(r,\phi,\phi-\theta)$ are "symmetrical" about $\{x=\bar{y}\}$ in the following sense.
(1) The midpoint of the segment $\overline{R(r,\phi,\theta)R(r,\phi,\phi-\theta)}$ lies on the plane $\{x=\bar{y}\}$,
(2) The segment connecting two points is perpendicular to $\{x=\bar{y}\}$.

We compare the external rays of $T_n(x,y)$ with those of Chebyshev map $T_n(z)$ in one variable. The external rays $T_n(z)$ is written as

$$R(r,\phi): u = re^{2\pi i\phi} + \frac{1}{r}e^{2\pi i(-\phi)}, \qquad (r>1).$$

Clearly,

$$R(r,-\phi): v = re^{2\pi i(-\phi)} + \frac{1}{r}e^{2\pi i\phi},$$
$$v = \bar{u}.$$

It is well-known that $R(r,\phi)$ and $R(r,-\phi)$ are "symmetrical" about the real axis. Note that symmetric group S_2 acts on external rays of $T_n(z)$. On the other hand, S_3 acts on external rays of $T_n(x,y)$.

Using the notations in Sect. 1, we can write

$$W^s(J_\Pi, T_n) = \{\Psi(t_1,t_2) : \mid t_1 \mid = \frac{1}{\mid t_2 \mid} > 1\}.$$

Then

$$C_n \cap W^s(J_\Pi, T_n) = \phi.$$

Lastly we consider periodic rays $R(\phi,\theta)$ of $T_n(x,y)$.

Proposition 10. *If one periodic ray lands at the point $z_0 \in S$, all rays which land at z_0 are all periodic with the same period.*

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