<table>
<thead>
<tr>
<th>Title</th>
<th>Bivariate Chebyshev maps of $\mathbf{C}^2$ and their dynamics (Complex Dynamics and its Related Fields)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Uchimura, Keisuke</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 2006, 1494: 87-95</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2006-05</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/58296">http://hdl.handle.net/2433/58296</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher京都大学</td>
</tr>
</tbody>
</table>
Bivariate Chebyshev maps of $C^2$ and their dynamics

東海大学・理学部 内村桂輔 (Keisuke Uchimura)
Department of Mathematics
Tokai University

Abstract
We study the properties of bivariate (two-dimensional) Chebyshev maps $T_n(x, y)$ from $C^2$ to $C^2$ and study the properties and dynamics of the maps.

(A) The properties of $T_n$.
(1) Solutions of $T_n(x, y) = (0, 0)$ are obtained.
(2) A critical set $\det(DT_n) = 0$ is written in a simple formula.
    These properties are similar to those of Chebyshev maps of $C$.

(B) The dynamics of $T_n$.
(1) $T_n$ is strictly critically finite.
(2) Any periodic point of $T_n$ is repelling.
(3) The exact form of the invariant probability measure $\mu$ of maximal entropy associated with $T_n$ is obtained.
(4) External rays for $J_2(T_n)$ and foliations of $J_1(T_n)$ are studied.
    These properties are also similar to those of Chebyshev maps of $C$.

1 Bivariate Chebyshev maps

The Chebyshev map is a typical chaotic map. Generalized Chebyshev maps are studied by several researchers, Koornwinder [1974], Lidle [1975], Veselov [1987] and Hoffman & Withers [1988].

In this paper, we study bivariate Chebyshev maps $T_n$ from $C^2$ to $C^2$, $n \in Z$.

$$T_n(x, y) = (g^{(n)}(x, y), g^{(n)}(y, x)).$$

This definition is due to [V]. Here $g^{(n)}(x, y)$ is a generalized Chebyshev polynomial defined by Lidle [L].
Let
\[ x = t_1 + t_2 + t_3, \quad y = t_1 t_2 + t_1 t_3 + t_2 t_3, \quad 1 = t_1 t_2 t_3. \]

Then
\[ g^{(n)}(x, y) := t_1^n + t_2^n + t_3^n. \]

So
\[ g^{(n)}(y, x) = (1/t_1)^n + (1/t_2)^n + (1/t_3)^n = g^{(-n)}(x, y). \]

For instance,
\[ T_2(x, y) = (x^2 - 2y, y^2 - 2x), \]
\[ T_3(x, y) = (x^3 - 3xy + 3, y^3 - 3xy + 3), \]
\[ T_4(x, y) = (x^4 - 4x^2y + 2y^2 + 4x, y^4 - 4xy^2 + 2x^2 + 4y). \]

\{g^n(x, y)\} satisfy the following recurrence equation:
\[ g^{(n)}(x, y) = x g^{(n-1)}(x, y) - y g^{(n-2)}(x, y) + g^{(n-3)}(x, y). \]

First, we show a branch covering over \( \mathbb{C}^2 \).

The following diagram is commutative.

\[
\begin{array}{ccc}
(C - \{0\})^2 & \xrightarrow{g_n} & (C - \{0\})^2 \\
\downarrow \Psi & & \downarrow \Psi \\
\mathbb{C}^2 & \xrightarrow{T_n} & \mathbb{C}^2 \\
\end{array}
\]

where \( g_n(u, v) = (u^n, v^n) \),

and
\[ (x, y) = \Psi(u, v) = (u + v + \frac{1}{uv}, \frac{1}{u} + \frac{1}{v} + uv). \]

The covering map
\[ \Psi : \mathbb{C}^2 - \Psi^{-1}(D) \to \mathbb{C}^2 - D \]

is a 6-sheated covering map. Branch locus \( D \) of \( \Psi \) is written as
\[ x^2y^2 - 4x^3 - 4y^3 + 18xy - 27 = 0. \]

In the case \( n = 2 \), Ueda[Ue] showed this diagram.

\( T_n(x, y) \) restricted on \( \{x = \bar{y}\} \) is a Chebyshev polynomial defined by Koornwinder [K]
\[ P_{n,0}^{-\frac{1}{2}}(z, \bar{z}) = e^{inz} + e^{-inr} + e^{i(nr-n\sigma)}. \]

Set
\[ z(\sigma, \tau) := e^{i\sigma} + e^{-i\tau} + e^{i(\tau-\sigma)} = u + iv. \]

The mapping
\[ z : (\sigma, \tau) \to (u, v) \]
is a diffeomorphism from $R$ onto $S$. See Koornwinder [K].

**Proposition 1.** There are $n^2$ solutions of $T_n(x, y) = (0, 0)$. All solutions lie in the closed domain $S$ in $\{x = \overline{y}\}$. They are written in the $(\sigma, \tau)$ coordinate.

\[
(1) \quad (\sigma, \tau) = \left(\frac{2(1 + j + h)\pi}{3n}, \frac{2(1 + 2j + h)\pi}{3n}\right)
\]
\[j = 0, 1, \ldots, n - 1, \text{ and } h = 0, 1, \ldots, j.\]

\[
(2) \quad (\sigma, \tau) = \left(\frac{2(2 + j + h)\pi}{3n}, \frac{2(2 + 2j - h)\pi}{3n}\right)
\]
\[j = 0, 1, \ldots, n - 2, \text{ and } h = 0, 1, \ldots, j.\]

**Proof.** By definition,

\[T_n(x, y) = (g^{(n)}(x, y), g^{(n)}(y, x)).\]

$g^{(n)}(x, y)$ and $g^{(n)}(y, x)$ are polynomials of degree $n$ with no common components.

We can find $n^2$ zeros on $S$. See Uchimura [Uc1]. $\Box$

We see that the zeros of $T_n$ and $T_{n+1}$ "mutually separate each other".

Next we consider critical set of $T_n(x, y)$.

\[C_n := \{(x, y) \in \mathbb{C}^2 : \det(DT_n) = 0\}.\]

**Proposition 2.** Let $n \in \mathbb{Z}$. Assume that

\[x = t_1 + t_2 + t_3, \quad y = t_1t_2 + t_1t_3 + t_2t_3, \quad t_1t_2t_3 = 1.\]

Then

\[
\det(DT_n) = n^2 \frac{t_1^n - t_2^n}{t_1 - t_2} \cdot \frac{t_1^n - t_3^n}{t_1 - t_3} \cdot \frac{t_2^n - t_3^n}{t_2 - t_3}.
\]

**Proof.**

\[
\det(DT_n) = \det(D(T_n \circ \Psi))/\det(D\Psi). \quad \square
\]

The similar result is holds for generalized Chebyshev maps from $\mathbb{C}^n$ to $\mathbb{C}^n$.

**Corollary 1.** Any irreducible component of $C_n$ is a rational curve of degree 2 or 4.

**Proof.** From Proposition 2, we have

\[x = t + \epsilon^k t + \frac{1}{\epsilon^{k+2}}, \quad \epsilon = e^{2\pi i/n}.\]
\[ y = \frac{1}{t} + \frac{1}{\epsilon^k t} + \epsilon^k t^2. \]

When \( \epsilon^k = -1 \), the degree of the rational curve is 2.

We see that \( C_n \) and \( C_{n+1} \) "mutually separate each other", and

\[ C_n \cap S \neq \emptyset \quad (S = J_2(T_n)). \]

Note that \( \{T_m : m \in \mathbb{Z}\} \) is a semigroup satisfying

\[ T_m \circ T_n = T_{mn}. \]

### 2 Dynamics of Bivariate Maps

We study the dynamics of \( T_n(x, y) \). Let

\[ K(T_n) := \{(x, y) : \{T_n^m(x, y)\} \text{is bounded for any } m\}. \]

In our setting we have the following proposition.

**Proposition 3.**

\[ K(T_n) = \{|t_1| = |t_2| = 1\} = S \subset \{x = \overline{y}\}. \]

**Proof.**

\[
\begin{array}{ccc}
(t_1, t_2) & \xrightarrow{g^n} & (t_1^n, t_2^n) \\
\downarrow \psi & & \downarrow \psi \\
(x, y) & \xrightarrow{T_n} & (g^n, g^{(-n)})
\end{array}
\]

\[ f \] is called **critically finite** if each irreducible component of the critical set of \( f \) is periodic or preperiodic. Dihn and Sibony [DS] show that generalized Chebyshev maps are critically finite. Here using proposition 2, we give a direct proof.

**Proposition 4.** \( T_n \) is **strictly critically finite.**

**Proof.**

\[
\begin{array}{ccc}
C_n & \xrightarrow{T_n} & T_n(C_n) \\
(t, \epsilon t) & \xrightarrow{T_n} & (t^n, \epsilon^n t^n)
\end{array}
\]

Next we study the second Julia set \( J_2 \) of \( T_n(x, y) \).

**Proposition 5.** All periodic points of \( T_n \) lie on \( S \) and are equidistributed in \( S \).
Proof. From [FS], we know that number of periodic points with period \( k \) equals \( n^{2k} \). For the distribution of periodic points, see [Uc2]. \( \square \)

Proposition 6. Any periodic point of \( T_n \) is repelling.

To prove this proposition we consider the following function.

\[
S_n := T_n \mid \{x = \overline{y}\} : \mathbb{R}^2 \rightarrow \mathbb{R}^2
\]

\[e.g. \quad S_2(z) = z^2 - 2\overline{z} : (u, v) \mapsto (u^2 - 2u - v^2, 2uv + 2v)\]

Lemma 1. Let \( p \) be a periodic point of \( S_n \). Let \( \alpha \) and \( \beta \) be eigen values of \( DS_n(p) \). Then

\[|\alpha|, \quad |\beta| > 1\]

Proposition 7. Let

\[f(x, y) \in \mathbb{R}[x, y].\]

\[T(x, y) := (f(x, y), f(y, x)) : \mathbb{C}^2 \rightarrow \mathbb{C}^2.\]

\[t(z) := T \mid \{x = \overline{y}\} : \mathbb{R}^2 \rightarrow \mathbb{R}^2.\]

Then

\[U^{-1}DT(z, \overline{z})U = Dt(z),\]

where

\[U = \frac{1}{2} \begin{pmatrix} 1 + i & -1 + i \\ 1 + i & 1 - i \end{pmatrix}.\]

From Lemma 1 and Proposition 7, Proposition 6 follows. \( \square \)

Next we study the invariant measure \( \mu \) of maximal entropy for \( T_n \).

Proposition 8. Under the above notation,

\[
\text{supp } \mu = S.
\]

\[
\mu = \left( \frac{2}{\pi} \right)^2 \frac{dx_1dx_2}{\sqrt{-x^2y^2 + 4x^3 + 4y^3 - 18xy + 27}} dx_1dx_2.
\]

\((x = x_1 + ix_2)\)
This is an extension of invariant measure

$$\mu = \frac{1}{\pi} \frac{dx}{\sqrt{(x+1)(3-x)}}$$

for Chebyshev maps in one variable on $[-1, 3]$.

**Proof.** We prove this proposition in the following three steps.

(1) Briand and Duval [BD] shows that

$$\mu_n := \frac{1}{d^{nk}} \sum_{f^n(y) = y, \text{repelling}} \delta_y,$$

then

$$\mu_n \to \mu \quad \text{(weak convergence).}$$

(2) From Proposition 5, we see that the periodic points are repelling and equidistributed in the triangle on the $(s,t)$ plane (see [Uc2]).

(3) Pullback of Lebesgue measure under $\phi$. \hfill $\square$

Next we consider the properties of external rays of $T_n(x, y)$. We use the definitions of external rays by Bedford and Jonsson [BJ]. We extend the map

$$T_n(x, y) : \mathbb{C}^2 \to \mathbb{C}^2$$

to

$$\hat{T}_n(x : y : z) : \mathbb{P}^2 \to \mathbb{P}^2.$$

Let $\Pi := \mathbb{P}^2 - \mathbb{C}^2$ be the line at infinity.

Then

$$\hat{T}_n | \Pi : (x : y : 0) \to (x^n : y^n : 0).$$

Therefore

$$J_{\Pi} = \{(x : y : 0) : |x| = |y| \} \simeq S^1.$$

The stable set of $J_{\Pi}$ for $T_n$ is defined by

$$W^s(J_{\Pi}, T_n) := \{ x \in \mathbb{P}^2 : d(T_n^j x, J_{\Pi}) \to 0, \quad j \to \infty \}.$$ 

Bedford and Jonsson [BJ] state that there exists a Böttcher coordinate $\Psi$ such that

$$\Psi : W^s(J_{\Pi}, f_n) \to W^s(J_{\Pi}, T_n)$$
conjugating $f_n$ to $T_n$, where
\[ f_n(x, y) = (x^n, y^n). \]

They also show that $W^s(J_n, T_n)$ is foliated by stable disks $W_a$. They define a local stable manifold $W^s_{loc}(a)$, $(a \in J_n)$ and then a stable disk $W_a \supset W^s_{loc}(a)$ and an external ray $R(a, \theta)$. They show that $J_0(T_n) = J_1(T_n)$ is laminated by stable disks $W_a$.

Nakane \cite{N} shows the following results on $T_2(x, y)$:

1. The map $\Psi$ defined by Ueda is essentially the inverse of Böttcher coordinate $\phi$.
   \[ \Psi(u, v) = \Psi(t, at), \ |t| > 1. \]

2. The stable disk $W_a$ is the set of points $R(r, \phi, \theta)$
   \[ x = re^{-2\pi i \theta} + \frac{1}{r}e^{2\pi i(\theta - \phi)} + e^{2\pi i \phi}, \]
   \[ y = re^{2\pi i(\phi - \theta)} + \frac{1}{r}e^{2\pi i \theta} + e^{-2\pi i \phi}, \ a = e^{2\pi i \phi}, \ (r > 1). \]

An external ray is written as
   \[ R(\phi, \theta) := \{ R(r, \phi, \theta) : r > 1 \}. \]

From this,
   \[ J_2 = S \subset \{ x = \bar{y} \}. \]

3. Each point $z \in S$ is the landing point of exactly 1, 3, or 6 external rays if $z$ is a cusp point on $\partial S$, $z$ is non-cusp point on $\partial S$ or $z \in int(S)$ respectively.

We can show that Nakane's results are also true for any $T_n(x, y), \ n \neq 0$.

Next we study the structure of foliations $W_a$ of
   \[ J_1(T_n) = W^s(J_n, T_n). \]

**Proposition 9.** For any point $z \in int(S)$, there exist three stable disks $W_a$ such that boundaries of these three disks intersect at $z$. At the point, two external rays on each $W_a$ land from opposite directions.

Metaphorically speaking, three mouths (stable disks) eat a sandwich (the second Julia set $S$).

Two external rays $R(\phi, \theta)$ and $R(\phi, \phi - \theta)$ lie on the stable disk
   \[ W_a \ (a = e^{2\pi i \phi}). \]
Two points $R(r, \phi, \theta)$ and $R(r, \phi, \phi - \theta)$ are "symmetrical" about $\{x = \overline{y}\}$ in the following sense.

1. The midpoint of the segment $R(r, \phi, \theta)R(r, \phi, \phi - \theta)$ lies on the plane $\{x = \overline{y}\}$,
2. The segment connecting two points is perpendicular to $\{x = \overline{y}\}$.

We compare the external rays of $T_n(x, y)$ with those of Chebyshev map $T_n(z)$ in one variable. The external rays $T_n(z)$ is written as

$$R(r, \phi) : u = re^{2\pi i \phi} + \frac{1}{r}e^{2\pi i(-\phi)}, \quad (r > 1).$$

Clearly,

$$R(r, -\phi) : v = re^{2\pi i(-\phi)} + \frac{1}{r}e^{2\pi i \phi},$$

$$v = \overline{u}.$$ 

It is well-known that $R(r, \phi)$ and $R(r, -\phi)$ are "symmetrical" about the real axis.

Note that symmetric group $S_2$ acts on external rays of $T_n(z)$. On the other hand, $S_3$ acts on external rays of $T_n(x, y)$.

Using the notations in Sect. 1, we can write

$$W^s(J_n, T_n) = \{\Psi(t_1, t_2) : |t_1| = \frac{1}{|t_2|} > 1\}.$$ 

Then

$$C_n \cap W^s(J_n, T_n) = \phi.$$ 

Lastly we consider periodic rays $R(\phi, \theta)$ of $T_n(x, y)$.

**Proposition 10.** If one periodic ray lands at the point $z_0 \in S$, all rays which land at $z_0$ are all periodic with the same period.

**References**


[DS] T. Dihn and N. Sibony, *Sur les endomorphismes holomorphes permutable*


