Bivariate Chebyshev maps of $\mathbf{C}^2$ and their dynamics

Complex Dynamics and its Related Fields

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Abstract
We study the properties of bivariate (two-dimensional) Chebyshev maps $T_n(x, y)$ from $\mathbb{C}^2$ to $\mathbb{C}^2$ and study the properties and dynamics of the maps.

(A) The properties of $T_n$.
(1) Solutions of $T_n(x, y) = (0, 0)$ are obtained.
(2) A critical set $\det(DT_n) = 0$ is written in a simple formula.
These properties are similar to those of Chebyshev maps of $\mathbb{C}$.

(B) The dynamics of $T_n$.
(1) $T_n$ is strictly critically finite.
(2) Any periodic point of $T_n$ is repelling.
(3) The exact form of the invariant probability measure $\mu$ of maximal entropy associated with $T_n$ is obtained.
(4) External rays for $J_2(T_n)$ and foliations of $J_1(T_n)$ are studied.
These properties are also similar to those of Chebyshev maps of $\mathbb{C}$.

1 Bivariate Chebyshev maps

The Chebyshev map is a typical chaotic map. Generalized Chebyshev maps are studied by several researchers, Koornwinder [1974], Lidle [1975], Veselov [1987] and Hoffman & Withers [1988].

In this paper, we study bivariate Chebyshev maps $T_n$ from $\mathbb{C}^2$ to $\mathbb{C}^2$, $n \in \mathbb{Z}$.

$$T_n(x, y) = (g^{(n)}(x, y), g^{(n)}(y, x)).$$

This definition is due to [V]. Here $g^{(n)}(x, y)$ is a generalized Chebyshev polynomial defined by Lidle [L].
Let
\[ x = t_1 + t_2 + t_3, \quad y = t_1 t_2 + t_1 t_3 + t_2 t_3, \quad 1 = t_1 t_2 t_3. \]
Then
\[ g^{(n)}(x, y) := t_1^n + t_2^n + t_3^n. \]
So
\[ g^{(n)}(y, x) = (1/t_1)^n + (1/t_2)^n + (1/t_3)^n = g^{(-n)}(x, y). \]
For instance,
\[
T_2(x, y) = (x^2 - 2y, y^2 - 2x), \\
T_3(x, y) = (x^3 - 3xy + 3, y^3 - 3xy + 3), \\
T_4(x, y) = (x^4 - 4x^2 y + 2y^2 + 4x, y^4 - 4xy^2 + 2x^2 + 4y).
\]
\{g^n(x, y)\} satisfy the following recurrence equation:
\[ g^{(n)}(x, y) = xg^{(n-1)}(x, y) - yg^{(n-2)}(x, y) + g^{(n-3)}(x, y). \]
First, we show a branch covering over \( \mathbb{C}^2 \).
The following diagram is commutative.
\[
\begin{array}{ccc}
(C - \{0\})^2 & \xrightarrow{g_n} & (C - \{0\})^2 \\
\downarrow \Psi & & \downarrow \Psi \\
\mathbb{C}^2 & \xrightarrow{T_n} & \mathbb{C}^2
\end{array}
\]
where
\[ g_n(u, v) = (u^n, v^n), \]
and
\[ (x, y) = \Psi(u, v) = (u + v + \frac{1}{uv}, \frac{1}{u} + \frac{1}{v} + uv). \]
The covering map
\[ \Psi: \mathbb{C}^2 - \Psi^{-1}(D) \to \mathbb{C}^2 - D \]
is a 6-sheated covering map. Branch locus \( D \) of \( \Psi \) is written as
\[ x^2 y^2 - 4x^3 - 4y^3 + 18xy - 27 = 0. \]
In the case \( n = 2 \), Ueda[Ue] showed this diagram.
\( T_n(x, y) \) restricted on \( \{x = \overline{y}\} \) is a Chebyshev polynomial defined by Koornwinder [K]
\[ P_{n, 0}^{-\frac{1}{2}}(z, \overline{z}) = e^{in\sigma} + e^{-in\tau} + e^{i(n\tau - n\sigma)}. \]
Set
\[ z(\sigma, \tau) := e^{i\sigma} + e^{-i\tau} + e^{i(\tau - \sigma)} = u + iv. \]
The mapping
\[ z: (\sigma, \tau) \to (u, v) \]
is a diffeomorphism from $R$ onto $S$. See Koornwinder [K].

**Proposition 1.** There are $n^2$ solutions of $T_n(x, y) = (0, 0)$. All solutions lie in the closed domain $S$ in $\{x = \bar{y}\}$. They are written in the $(\sigma, \tau)$ coordinate.

1. \((\sigma, \tau) = \left( \frac{2(1 + j + h)\pi}{3n}, \frac{2(1 + 2j + h)\pi}{3n} \right) \)
   
   \(j = 0, 1, ..., n - 1, \text{and } h = 0, 1, ..., j\).

2. \((\sigma, \tau) = \left( \frac{2(2 + j + h)\pi}{3n}, \frac{2(2 + 2j - h)\pi}{3n} \right) \)
   
   \(j = 0, 1, ..., n - 2, \text{and } h = 0, 1, ..., j\).

**Proof.** By definition,

\[ T_n(x, y) = (g^{(n)}(x, y), g^{(n)}(y, x)). \]

$g^{(n)}(x, y)$ and $g^{(n)}(y, x)$ are polynomials of degree $n$ with no common components. We can find $n^2$ zeros on $S$. See Uchimura [Uc1]. □

We see that the zeros of $T_n$ and $T_{n+1}$ "mutually separate each other".

Next we consider critical set of $T_n(x, y)$.

\[ C_n := \{(x, y) \in \mathbb{C}^2 : \text{det}(DT_n) = 0\}. \]

**Proposition 2.** Let $n \in \mathbb{Z}$. Assume that

\[ x = t_1 + t_2 + t_3, \quad y = t_1t_2 + t_1t_3 + t_2t_3, \quad t_1t_2t_3 = 1. \]

Then

\[ \text{Det}(DT_n) = n^2 \frac{t_1^n - t_2^n}{t_1 - t_2} \cdot \frac{t_1^n - t_3^n}{t_1 - t_3} \cdot \frac{t_2^n - t_3^n}{t_2 - t_3}. \]

**Proof.**

\[ \text{Det}(DT_n) = \frac{\text{Det}(D(T_n \circ \Psi))}{\text{Det}(D\Psi)}. \]

□

The similar result is holds for generalized Chebyshev maps from $\mathbb{C}^n$ to $\mathbb{C}^n$.

**Corollary 1.** Any irreducible component of $C_n$ is a rational curve of degree 2 or 4.

**Proof.** From Proposition 2, we have

\[ x = t + \epsilon^k t + \frac{1}{\epsilon^k t^2}, \quad \epsilon = e^{\frac{2\pi i}{n}} \]
$y = \frac{1}{t} + \frac{1}{\epsilon^k t} + \epsilon^k t^2.$

When $\epsilon^k = -1$, the degree of the rational curve is 2.

We see that $C_n$ and $C_{n+1}$ "mutually separate each other", and

$$C_n \cap S \neq \phi \quad (S = J_2(T_n)).$$

Note that $\{T_m : m \in \mathbb{Z}\}$ is a semigroup satisfying

$$T_m \circ T_n = T_{mn}.$$

2 Dynamics of Bivariate Maps

We study the dynamics of $T_n(x, y)$. Let

$$K(T_n) := \{(x, y) : \{T_n^m(x, y)\} \text{is bounded for any } m\}.$$ 

In our setting we have the following proposition.

Proposition 3.

$$K(T_n) = \{ |t_1| = |t_2| = 1 \} = S \subset \{x = \overline{y}\}.$$ 

Proof

\[
\begin{array}{c}
(t_1, t_2) \xrightarrow{g_n} (t_1^n, t_2^n) \\
\downarrow \psi \quad \downarrow \psi \\
(x, y) \xrightarrow{T_n} (g^n, g^{(-n)})
\end{array}
\]

$f$ is called critically finite if each irreducible component of the critical set of $f$ is periodic or preperiodic. Dihn and Sibony [DS] show that generalized Chebyshev maps are critically finite. Here using proposition 2, we give a direct proof.

Proposition 4. \quad $T_n$ is strictly critically finite.

Proof.

\[
\begin{array}{c}
C_n \xrightarrow{T_n} T_n(C_n) \xrightarrow{T_n} T_n(C_n) \\
(t, \epsilon t) \quad (t^n, \epsilon^n t^n) \quad (t^{n^2}, \epsilon^{n^2})
\end{array}
\]

Next we study the second Julia set $J_2$ of $T_n(x, y)$.

Proposition 5. \quad All periodic points of $T_n$ lie on $S$ and are equidistributed in $S$. 

Proof. From [FS], we know that number of periodic points with period \( k \) equals \( n^{2k} \). For the distribution of periodic points, see [Uc2]. \( \square \)

**Proposition 6.** Any periodic point of \( T_n \) is repelling.

To prove this proposition we consider the following function.

\[
S_n := T_n \mid \{ x = \overline{y} \} : \mathbb{R}^2 \to \mathbb{R}^2
\]

\[\text{e.g. } S_2(z) = z^2 - 2\overline{z} : (u, v) \mapsto (u^2 - 2u - v^2, 2uv + 2v).\]

**Lemma 1.** Let \( p \) be a periodic point of \( S_n \). Let \( \alpha \) and \( \beta \) be eigen values of \( DS_n(p) \). Then

\[
| \alpha |, \quad | \beta | > 1.
\]

**Proposition 7.** Let

\[
f(x, y) \in \mathbb{R}[x, y].
\]

\[
T(x, y) := (f(x, y), f(y, x)) : \mathbb{C}^2 \to \mathbb{C}^2.
\]

\[
t(z) := T \mid \{ x = \overline{y} \} : \mathbb{R}^2 \to \mathbb{R}^2.
\]

Then

\[
U^{-1}DT(z, \overline{z})U = Dt(z),
\]

where

\[
U = \frac{1}{2} \begin{pmatrix} 1 + i & -1 + i \\ 1 + i & 1 - i \end{pmatrix}.
\]

From Lemma 1 and Proposition 7, Proposition 6 follows. \( \square \)

Next we study the invariant measure \( \mu \) of maximal entropy for \( T_n \).

**Proposition 8.** Under the above notation,

\[
\text{supp } \mu = S.
\]

\[
\mu = \left( \frac{2}{\pi} \right)^2 \frac{dx_1dx_2}{\sqrt{-x^2x^2 + 4x^3 + 4x^3 - 18x + 27}}.
\]

\( (x = x_1 + ix_2) \)
This is an extension of invariant measure

\[
\mu = \frac{1}{\pi} \frac{dx}{\sqrt{(x+1)(3-x)}}
\]

for Chebyshev maps in one variable on \([-1, 3]\).

**Proof.** We prove this proposition in the following three steps.

(1) Briand and Duval [BD] shows that

let \( \mu_n := \frac{1}{d^{nk}} \sum \delta_{y} \), where \( f^{n}(y) = y \), repeatedly

then \( \mu_n \rightarrow \mu \) (weak convergence).

(2) From Proposition 5, we see that the periodic points are repelling and equidistributed in the triangle on the (s,t) plane (see [Uc2]).

(3) Pullback of Lebesgue measure under \( \phi \).

Next we consider the properties of external rays of \( T_n(x, y) \). We use the definitions of external rays by Bedford and Jonsson [BJ]. We extend the map

\( T_n(x, y) : C^2 \rightarrow C^2 \)

to \( \hat{T}_n(x : y : z) : P^2 \rightarrow P^2 \).

Let \( \Pi := P^2 - C^2 \) be the line at infinity.

Then

\( \hat{T}_n | \Pi : (x : y : 0) \rightarrow (x^n : y^n : 0) \).

Therefore

\( J_\Pi = \{(x : y : 0) : |x| = |y| \} \simeq S^1 \).

The stable set of \( J_\Pi \) for \( T_n \) is defined by

\( W^s(J_\Pi, T_n) = \{x \in P^2 : d(T_n^j x, J_\Pi) \rightarrow 0, \ j \rightarrow \infty \} \).

Bedford and Jonsson [BJ] state that there exists a Böttcher coordinate \( \Psi \) such that

\( \Psi : W^s(J_\Pi, f_n) \rightarrow W^s(J_\Pi, T_n) \)
conjuring \( f_n \) to \( T_n \), where
\[
f_n(x, y) = (x^n, y^n).
\]
They also show that \( W^s(J, T_n) \) is foliated by stable disks \( W_a \). They define a local stable manifold \( W^s_{loc}(a), \ a \in J \) and then a stable disk \( W_a \supset W^s_{loc}(a) \) and an external ray \( R(a, \theta) \). They show that \( J_0(T_n) = J_1(T_n) \) is laminated by stable disks \( W_a \).

Nakane [N] shows the following results on \( T_2(x, y) \):

1. The map \( \Psi \) defined by Ueda is essentially the inverse of Böttcher coordinate \( \phi \).
\[
\Psi(u, v) = \Psi(t, at), \ |t| > 1.
\]
2. The stable disk \( W_a \) is the set of points \( R(r, \phi, \theta) \)
\[
x = re^{-2\pi i \theta} + \frac{1}{r}e^{2\pi i (\theta - \phi)} + e^{2\pi i \phi},
\]
\[
y = re^{2\pi i (\phi - \theta)} + \frac{1}{r}e^{2\pi i \theta} + e^{-2\pi i \phi}, \ a = e^{2\pi i \phi}, \ (r > 1).
\]
An external ray is written as
\[
R(\phi, \theta) := \{R(r, \phi, \theta) : r > 1\}.
\]

From this,
\[
J_2 = S \subset \{x = \bar{y}\}.
\]

3. Each point \( z \in S \) is the landing point of exactly 1, 3, or 6 external rays if \( z \) is a cusp point on \( \partial S \), \( z \) is non-cusp point on \( \partial S \) or \( z \in int(S) \) respectively.

We can show that Nakane's results are also true for any \( T_n(x, y) \), \( n \neq 0 \).

Next we study the structure of foliations \( W_a \) of
\[
J_1(T_n) = W^s(J, T_n).
\]

**Proposition 9.** For any point \( z \in int(S) \), there exist three stable disks \( W_a \) such that boundaries of these three disks intersect at \( z \). At the point, two external rays on each \( W_a \) land from opposite directions.

Metaphorically speaking, three mouths (stable disks) eat a sandwich (the second Julia set \( S \)).

Two external rays \( R(\phi, \theta) \) and \( R(\phi, \phi - \theta) \) lie on the stable disk
\[
W_a \quad (a = e^{2\pi i \phi}).
\]
Two points $R(r, \phi, \theta)$ and $R(r, \phi, \phi - \theta)$ are "symmetrical" about $\{x = y\}$ in the following sense.

1. The midpoint of the segment $\overline{R(r, \phi, \theta)R(r, \phi, \phi - \theta)}$ lies on the plane $\{x = y\}$,
2. The segment connecting two points is perpendicular to $\{x = y\}$.

We compare the external rays of $T_n(x, y)$ with those of Chebyshev map $T_n(z)$ in one variable. The external rays $T_n(z)$ is written as

$$R(r, \phi) : u = re^{2\pi i \phi} + \frac{1}{r}e^{2\pi i (-\phi)}, \quad (r > 1).$$

Clearly,

$$R(r, -\phi) : v = re^{2\pi i (-\phi)} + \frac{1}{r}e^{2\pi i \phi}, \quad v = \overline{u}.$$ 

It is well-known that $R(r, \phi)$ and $R(r, -\phi)$ are "symmetrical" about the real axis.

Note that symmetric group $S_2$ acts on external rays of $T_n(z)$. On the other hand, $S_3$ acts on external rays of $T_n(x, y)$.

Using the notations in Sect. 1, we can write

$$W^s(J_n, T_n) = \{\Psi(t_1, t_2) : |t_1| = \frac{1}{|t_2|} > 1\}.$$ 

Then

$$C_n \cap W^s(J_n, T_n) = \phi.$$ 

Lastly we consider periodic rays $R(\phi, \theta)$ of $T_n(x, y)$.

**Proposition 10.** If one periodic ray lands at the point $z_0 \in S$, all rays which land at $z_0$ are all periodic with the same period.

**References**


[DS] T. Dihn and N. Sibony, *Sur les endomorphismes holomorphes permutables*


