<table>
<thead>
<tr>
<th>Title</th>
<th>Julia sets of quartic polynomials and polynomial semigroups (Complex Dynamics and its Related Fields)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Katagata, Koh</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2006), 1494: 49-61</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2006-05</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/58298">http://hdl.handle.net/2433/58298</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
Julia sets of quartic polynomials and polynomial semigroups

Koh Katagata
Interdisciplinary Graduate School of Science and Engineering,
Shimane University, Matsue 690-8504, Japan

Abstract
For a polynomial of degree two or more, the Julia set and the filled-in Julia set are either connected or else have uncountably many components. If the Julia set is totally disconnected, then the polynomial is topologically conjugate to the shift map. In the case of neither connected nor totally disconnected Julia set of a quartic polynomial, there exists a homeomorphism between the set of all components of the filled-in Julia set and some subset of the corresponding symbol space. Furthermore the polynomial is topologically conjugate to the shift map with respect to the homeomorphism. Moreover there exists a homeomorphism between the Julia set of the polynomial and that of a certain polynomial semigroup.

1 Preparations and the main results
Let \( \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \) be the Riemann sphere and let \( f : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) be a polynomial of degree \( d \geq 2 \). The filled-in Julia set \( K_f \) is defined as

\[
K_f = \{ z \in \mathbb{C} : \{f^n(z)\}_{n=0}^{\infty} \text{ is bounded} \}.
\]

The topological boundary of \( K_f \) is called the Julia set \( J_f \), and its complement \( \hat{\mathbb{C}} \setminus J_f \) is called the Fatou set \( F_f \). In this case, \( \infty \) is a superattracting fixed point. We call \( A_f(\infty) = \hat{\mathbb{C}} \setminus K_f \) the basin of attraction.

Definition 1.1. A rational semigroup \( G \) is a semigroup generated by a family of non-constant rational functions \( \{g_1, g_2, \ldots, g_n, \ldots\} \) defined on \( \hat{\mathbb{C}} \). We denote this situation by

\[
G = \langle g_1, g_2, \ldots, g_n, \ldots \rangle.
\]
A rational semigroup $G$ is called a polynomial semigroup if each $g \in G$ is a polynomial.

**Definition 1.2.** Let $G$ be a rational semigroup. The Fatou set $F_G$ of $G$ is defined as

$$F_G = \{ z \in \hat{\mathbb{C}} : G \text{ is normal in a neighborhood of } z \}.$$ 

Its complement $\hat{\mathbb{C}} \setminus F_G$ is called the Julia set $J_G$ of $G$. Note that $F_g = F_{\langle g \rangle}$ and $J_g = J_{\langle g \rangle}$.

**Definition 1.3.** Let $N_0 = \{0\} \cup \mathbb{N}$ be the set of non-negative integers and let $\Sigma_q = \{1, 2, \ldots, q\}^{N_0}$ be the symbol space of $q$-symbols. For $s = (s_n)$ and $t = (t_n)$ in $\Sigma_q$, a metric $\rho$ on $\Sigma_q$ is defined as

$$\rho(s, t) = \sum_{n=0}^{\infty} \frac{\delta(s_n, t_n)}{2^n}, \quad \text{where} \quad \delta(k, l) = \begin{cases} 1 & \text{if } k \neq l, \\ 0 & \text{if } k = l. \end{cases}$$

Then $\Sigma_q$ is a compact metric space. We define the shift map $\sigma : \Sigma_q \to \Sigma_q$ as

$$\sigma((s_0, s_1, s_2, \ldots)) = (s_1, s_2, \ldots).$$

The shift map $\sigma$ is continuous with respect to the metric $\rho$.

In the case of a polynomial of degree two or more, the connectivity of the Julia set is affected by the behavior of finite critical points.

**Theorem 1.4 ([1]).** Let $f$ be a polynomial of degree $d \geq 2$. If all finite critical points of $f$ are in $A_f(\infty)$, then $J_f$ is totally disconnected and $J_f = K_f$. Furthermore $f|_{J_f}$ is topologically conjugate to the shift map $\sigma|_{\Sigma_q}$. On the other hand, if all finite critical points of $f$ are in $K_f$, then $J_f$ and $K_f$ are connected.

**Definition 1.5.** The Green's function associated with $K_f$ is defined as

$$G(z) = \lim_{n \to \infty} \frac{1}{d^n} \log^+ |f^n(z)|,$$

where $\log^+ x = \max\{\log x, 0\}$. $G(z)$ is zero for $z \in K_f$ and $G(z)$ is positive for $z \in \mathbb{C} \setminus K_f$. Note the identity $G(f(z)) = dG(z)$.

**Definition 1.6.** We call the triple $(f, U, V)$ of bounded simply connected domains $U$ and $V$ such that $\overline{U} \subset V$ and a holomorphic proper map $f : U \to V$ of degree $d$ a polynomial-like map of degree $d$. The filled-in Julia set $K_f$ of a polynomial-like map $(f, U, V)$ is defined as

$$K_f = \{ z \in U : \{f^n(z)\}_{n=0}^{\infty} \subset U \}.$$
Definition 1.7. Let $(X, d)$ be a metric space. For a compact subset $A \subset X$ and $\delta > 0$, let $A[\delta]$ be a $\delta$-neighborhood of $A$. For compact subsets $A, B \subset X$, we define the Hausdorff metric $d_H$ as

$$d_H(A, B) = \inf \{ \delta : A \subset B[\delta] \text{ and } B \subset A[\delta] \}.$$ 

Situation: Let $f$ be a quartic polynomial and let $c_1, c_2$ and $c_3$ be finite critical points of $f$. $G$ is the Green's function associated with the filled-in Julia set $K_f$. Suppose that $G(c_1) = G(c_2) = 0$ and $G(c_3) > 0$, that is, $c_1, c_2 \in K_f$ and $c_3 \in A_f(\infty)$.

Let $U$ be a bounded component of $\mathbb{C} \setminus G^{-1}(G(f(c_3)))$. Suppose that $U_A$ and $U_B$ be bounded components of $\mathbb{C} \setminus G^{-1}(G(c_3))$ such that $c_1 \in U_A$ and $c_2 \in U_B$. Then $U_A$ and $U_B$ are proper subsets of $U$. Furthermore $(f|_{U_A}, U_A, U)$ and $(f|_{U_B}, U_B, U)$ are polynomial-like maps of degree 2. We set $f_1 = f|_{U_A}$ and $f_2 = f|_{U_B}$.

Under this situation, we define the A-B kneading sequence $(\alpha_n)_{n \geq 0}$ of $c_i$ as

$$\alpha_n = \begin{cases} A & \text{if } f^n(c_i) \in U_A, \\ B & \text{if } f^n(c_i) \in U_B. \end{cases}$$

We assume that the A-B kneading sequence of $c_1$ is $(AAA \cdots)$ and the A-B kneading sequence of $c_2$ is $(BBB \cdots)$. Note that $K_{f_1}$ and $K_{f_2}$ are connected (see [3]).

Let $\text{Comp}(K_f)$ be the set of all components of $K_f$. Since $G(c_3) > 0$, $\text{Comp}(K_f)$ is an uncountable set. $\text{Comp}(K_f)$ becomes a metric space with the Hausdorff metric $d_H$. We define a map $F : \text{Comp}(K_f) \rightarrow \text{Comp}(K_f)$ as $F(K) = f(K)$ for $K \in \text{Comp}(K_f)$. This map $F$ is continuous with respect to the Hausdorff metric $d_H$.

Let $\Sigma_6 = \{1, 2, 3, 4, A, B\}^{\mathbb{N}_0}$ be the symbol space. We define a subset $\Sigma$ of $\Sigma_6$ as follows: $s = (s_n) \in \Sigma$ if and only if

1. $s_n = A \Rightarrow s_{n+1} = A,$
2. $s_n = B \Rightarrow s_{n+1} = B,$
3. $s_n = A$ and $s_{n-1} \neq A \Rightarrow s_{n-1} = 3$ or 4,
4. $s_n = B$ and $s_{n-1} \neq B \Rightarrow s_{n-1} = 1$ or 2,
5. if $s \in \Sigma_4 = \{1, 2, 3, 4\}^{\mathbb{N}_0}$, then there exist subsequences $(s_n(k))_{k=1}^{\infty}$ and $(s'_n(l))_{l=1}^{\infty}$ such that $s_n(k) = 1$ or 2 for all $k \in \mathbb{N}$ and $s'_n(l) = 3$ or 4 for all $l \in \mathbb{N}$.  

It is our goal to prove the following theorems.

**Theorem 1.8.** Let $f$ be a quartic polynomial. Suppose that its finite critical points $c_1, c_2 \in K_f$ and $c_3 \in A_f(\infty)$ differ mutually and suppose that $J_f$ is disconnected but not totally disconnected. Moreover, suppose that the $A-B$ kneading sequence of $c_1$ is $(AAA\cdots)$ and the $A-B$ kneading sequence of $c_2$ is $(BBB\cdots)$. Then there exists a homeomorphism $\Lambda : \text{Comp}(K_f) \to \Sigma$ such that $\Lambda \circ F = \sigma \circ \Lambda$.

**Theorem 1.9.** Under the assumption of Theorem 1.8, there exist quadratic polynomials $g_1$ and $g_2$ and a homeomorphism $h$ on $K_f$ such that

$$h(J_f) = J_G,$$

where $G = \langle g_1, g_2 \rangle$ is a polynomial semigroup.

## 2 Proof of Theorem 1.8

A conformal map $\Psi$ with the following properties exists (see [6, p.88]): there exist $r > 1$ and $W \subset \mathbb{C} \setminus K_f$ with $c_3 \in \partial W$ and $\mathbb{C} \setminus \overline{W} = U_A \cup U_B$ such that $\Psi : \mathbb{C} \setminus \mathbb{D}_r \to W$ is conformal and $\Psi^{-1} \circ f \circ \Psi(z) = z^4$, where $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$. For $t \in [0, 1)$, $R(t) = \Psi(\{z \in \mathbb{C} : |z| > r \text{ and } \arg(z) = 2\pi t\})$ is called the external ray with angle $t$ for $K_f$.

**Remark 2.1.** $W$ is an unbounded component of $\mathbb{C} \setminus G^{-1}(G(c_3))$ and its boundary $\partial W$ is $G^{-1}(G(c_3))$.

Let $R$ be the intersection of the external ray passes through $f(c_3)$ and $\mathbb{C} \setminus \overline{U}$. Two of four rays $f^{-1}(R)$ have a limit point $c_3$. $\Psi^{-1}(f^{-1}(R))$ is four half-lines extended from $\partial \mathbb{D}_r$ with adjacent angles $\pi/2$. There are three invariant half-lines extended from the unit circle under $z \mapsto z^4$ and their angles are $0, 1/3$ and $2/3$. At least two of three invariant half-lines do not overlap with $\Psi^{-1}(f^{-1}(R))$. Let $\tilde{R}_1$ be the intersection of one of these invariant half-lines and $\mathbb{C} \setminus \overline{\mathbb{D}_r}$. Let $R_1$ be the image of $\tilde{R}_1$ under $\Psi$. We extend $R_1$ to become the invariant ray under $f$. Let $R_0$ be a component of $f^{-1}(R_1)$ which satisfies $R_1 \cap R_0 \neq \emptyset$. Then $R_1 \subset R_0$ and $f$ maps $J_0 = R_0 \setminus R_1$ onto $J_1 = R_1 \cap \overline{U}$. Inductively, let $R_{-n}$ be a component of $f^{-1}(R_{-(n-1)})$ which satisfies $R_{-(n-1)} \cap R_{-n} \neq \emptyset$. Then $R_{-(n-1)} \subset R_{-n}$ and $f$ maps $J_{-n}$ onto $J_{-(n-1)}$, where

$$J_{-n} = \begin{cases} R_{-n} \setminus R_{-(n-1)} & \text{if } n \geq 0, \\ R_1 \cap \overline{U} & \text{if } n = -1. \end{cases}$$
At this time, a ray

\[ R_{\infty} = \bigcup_{n=0}^{\infty} R_{-n} = R_{1} \cup \bigcup_{n=0}^{\infty} J_{-n} \]

is invariant under \( f \).

**Lemma 2.2** ([8]). Let \( F \) be a rational map and let \( X \) denote the closure of the union of the postcritical set and possible rotation domains of \( F \). Suppose that \( \gamma : (-\infty, 0] \to \hat{\mathbb{C}} \setminus X \) is a curve with

\[ F^{nk}(\gamma(-\infty, -k]) = \gamma(-\infty, 0] \]

for all positive integers \( k \). Then \( \lim_{t \to -\infty} \gamma(t) \) exists and is a repelling or parabolic periodic point of \( F \) whose period divides \( n \).

We can apply Lemma 2.2 to \( R_{\infty} \setminus R_{1} = \bigcup_{n=0}^{\infty} J_{-n} \), setting \( \gamma \) such that \( \gamma(-(k+1), -k] = J_{-k} \) for all positive integers \( k \). Therefore \( R_{\infty} \) lands at a repelling or parabolic fixed point of \( f \). If \( R_{\infty} \) lands at a point on \( K_{f_{1}} \), then we describe \( R_{\infty} \) with \( R_{A1} \). Similarly, if \( R_{\infty} \) lands at a point on \( K_{f_{2}} \), then we describe \( R_{\infty} \) with \( R_{B1} \). In fact, we can obtain both \( R_{A1} \) and \( R_{B1} \) by choosing \( R_{1} \) well.

To the next, let \( R_{A2} \) and \( R_{B2} \) be components of \( f^{-1}(R_{A1}) \) and \( f^{-1}(R_{B1}) \) which satisfy \( R_{A2} \cap U_{A} \neq \emptyset \) and \( R_{B2} \cap U_{B} \neq \emptyset \) and differ from \( R_{A1} \) and \( R_{B1} \) respectively. We set \( V_{A} = U \setminus (K_{f_{1}} \cup R_{A1}) \) and \( V_{B} = U \setminus (K_{f_{2}} \cup R_{B1}) \). Let \( I_{1}, I_{2}, I_{3} \) and \( I_{4} \) be branches of \( f^{-1} \) such that

\[ I_{1} : V_{A} \to U_{1}, \quad I_{2} : V_{A} \to U_{2}, \]
\[ I_{3} : V_{B} \to U_{3}, \quad I_{4} : V_{B} \to U_{4}, \]

where \( U_{1} \) and \( U_{2} \) are components of \( U_{A} \setminus K_{f_{1}} \cup R_{A1} \cup R_{A2} \) respectively. Similarly, \( U_{3} \) and \( U_{4} \) are components of \( U_{B} \setminus K_{f_{2}} \cup R_{B1} \cup R_{B2} \) respectively.

We define a map \( \Lambda : \text{Comp}(K_{f}) \to \Sigma \) as follows: for \( K \in \text{Comp}(K_{f}) \),

\[ [\Lambda(K)]_{n} = \begin{cases} 
  i & \text{if } f^{n}(K) \subset U_{i}, \\
  A & \text{if } f^{n}(K) = K_{f_{1}}, \\
  B & \text{if } f^{n}(K) = K_{f_{2}},
\end{cases} \]

where \( n \in \mathbb{N}_{0} \) and \( i = 1, 2, 3, 4 \).

**Lemma 2.3.** \( \Lambda : \text{Comp}(K_{f}) \to \Sigma \) is continuous.
Proof. For any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $1/2^N < \epsilon$. We take $K \in \text{Comp}(K_f)$ arbitrarily and set $s = \Lambda(K) = (s_0, s_1, \ldots, s_N, \ldots)$. We consider the case of $s \in \Sigma \cap \Sigma_4$ first. By continuity of $f$, there exist $\delta_1, \ldots, \delta_N > 0$ such that $f^k(K[\delta_k]) \subset U_{s_k}$ for $k = 1, 2, \ldots, N$. Let $\delta$ be the minimum value of $\delta_k$. Then $f^k(K[\delta]) \subset U_{s_k}$ for $k = 1, 2, \ldots, N$. Any component $K'$ of $K_f$ with $d_H(K, K') < \delta$ satisfies $K' \subset K[\delta]$ by the definition of the Hausdorff metric. Moreover any component $K' \subset K[\delta]$ of $K_f$ satisfies $\Lambda(K') = (s_0, s_1, \ldots, s_N, t_{N+1}, \ldots)$. Therefore if any component $K'$ of $K_f$ satisfies $d_H(K, K') < \delta$, then

$$\rho(\Lambda(K), \Lambda(K')) = \sum_{k=N+1}^{\infty} \frac{\delta(s_k, t_k)}{2^k} \leq \sum_{k=N+1}^{\infty} \frac{1}{2^k} = \frac{1}{2^N} < \epsilon.$$ 

If $s_n = A$ and $s_{n-1} \neq A$ or $s_n = B$ and $s_{n-1} \neq B$, then $s$ is an isolated point in $\Sigma$. Since corresponding $K$ is also an isolated point in $\text{Comp}(K_f)$, $\Lambda$ is continuous at $K$. \hfill \Box

We define a map $\widetilde{\Lambda} : \Sigma \rightarrow \text{Comp}(K_f)$ as follows: for $s = (s_n) \in \Sigma$, if $s_n = A$ and $s_{n-1} \neq A$,

$$\widetilde{\Lambda}(s) = I_{s_0} \circ \cdots \circ I_{s_{n-1}}(K_{f_1}).$$

If $s_n = B$ and $s_{n-1} \neq B$,

$$\widetilde{\Lambda}(s) = I_{s_0} \circ \cdots \circ I_{s_{n-1}}(K_{f_2}).$$

If $s \in \Sigma_4$, there exists a subsequence $(s_{n(l)})_{l=1}^{\infty}$ such that $s_{n(l)} = 1$ or $2$ and $s_{n(l)-1} = 3$ or $4$. We set $K_{s_1}^{(l)} = I_{s_0} \circ \cdots \circ I_{s_n} (\overline{U}_A)$. Then $K_{s_1}^{(l)} \supset K_{s_1}^{(l+1)}$. We define

$$\widetilde{\Lambda}(s) = \bigcap_{l=1}^{\infty} K_{s_1}^{(l)}.$$ 

Note that $\bigcap_{l=1}^{\infty} K_{s_1}^{(l)}$ is a one-point set since each $I_k$ decreases the Poincaré distance on $V_A$ or $V_B$.

Remark 2.4. We check that $I_k$ decreases the Poincaré distance on $V_A$ or $V_B$. For $x$ and $y$ in $V_A$, let $\gamma$ be the Poincaré geodesic from $x$ to $y$ in $V_A$. Then there exists a constant $c < 1$ such that

$$\int_{I_k(\gamma)} ds_{V_A} \leq c \int_{I_k(\gamma)} ds_{U_1},$$
where $ds_{V_A}$ and $ds_{U_1}$ are the Poincaré metrics on $V_A$ and $U_1$ respectively. Let $\gamma'$ be the Poincaré geodesic from $I_1(x)$ to $I_1(y)$ in $V_A$. Then

$$\text{dist}_{V_A}(I_1(x), I_1(y)) = \int_{\gamma'} ds_{V_A} \leq \int_{I_1(\gamma)} ds_{U_1} = \text{dist}_{V_A}(x, y).$$

As mentioned above,

$$\text{dist}_{V_A}(I_1(x), I_1(y)) \leq c \cdot \text{dist}_{V_A}(x, y).$$

Therefore $I_1$ decreases the Poincaré distance on $V_A$. It is similarly proved about $I_2$, $I_3$ and $I_4$.

**Lemma 2.5.** $\tilde{\Lambda}$ is the inverse map of $\Lambda$.

**Proof.** What is necessary is just to prove that $\Lambda \circ \tilde{\Lambda}$ and $\tilde{\Lambda} \circ \Lambda$ are the identity maps. We take $s = (s_0, s_1, s_2, \ldots) \in \Sigma$ arbitrarily. If $s_n = A$ and $s_{n-1} \neq A$, $\tilde{\Lambda}(s) = I_{s_0} \circ \cdots \circ I_{s_{n-1}}(K_{f_1})$. By definition, $f^k(\tilde{\Lambda}(s)) = I_{s_k} \circ \cdots \circ I_{s_{n-1}}(K_{f_1}) \subset U_{s_k}$. Then $[\Lambda(\tilde{\Lambda}(s))]_k = s_k$. Therefore $\Lambda \circ \tilde{\Lambda}(s) = s$. We can prove similarly in the case of $s_n = B$ and $s_{n-1} \neq B$. If $s \in \Sigma_4$,

$$f^k(\tilde{\Lambda}(s)) = f^k\left(\bigcap_{l=1}^{\infty} K_s^{(l)}\right) \subset \bigcap_{l=1}^{\infty} f^k(K_s^{(l)}) \subset U_{s_k}.$$ 

Then $[\Lambda(\tilde{\Lambda}(s))]_k = s_k$. Therefore $\Lambda \circ \tilde{\Lambda}(s) = s$. As mentioned above, $\Lambda \circ \tilde{\Lambda}$ is the identity map of $\Sigma$. It is clear that $\tilde{\Lambda} \circ \Lambda$ is the identity map of $\text{Comp}(K_f)$. \hfill $\Box$

**Lemma 2.6.** $\Lambda^{-1} : \Sigma \to \text{Comp}(K_f)$ is continuous.

**Proof.** For any $s = (s_0, s_1, s_2, \ldots) \in \Sigma$, we set $K = \Lambda^{-1}(s)$. If $s_n = A$ and $s_{n-1} \neq A$, $K = I_{s_0} \circ \cdots \circ I_{s_{n-1}}(K_{f_1})$. Since $K$ is an isolated point in $\text{Comp}(K_f)$, $\Lambda^{-1}$ is continuous at $s$. Similarly, if $s_n = B$ and $s_{n-1} \neq B$, then $\Lambda^{-1}$ is continuous at $s$. We take $\epsilon > 0$ arbitrarily. If $s \in \Sigma_4$,

$$\Lambda^{-1}(s) = \bigcap_{l=1}^{\infty} K_s^{(l)}.$$
Since $K_s^{(l)} \supset K_s^{(l+1)}$ and $\Lambda^{-1}(s)$ is a one-point set, there exists $l_0 \in \mathbb{N}$ such that

$$\Lambda^{-1}(s) \subset K_s^{(l_0)} \subset \Lambda^{-1}(s)[\epsilon].$$

We set $\delta = 1/2^{n(l_0)-1}$. We consider $t \in \Sigma$ with $\rho(s, t) < \delta$. At this time, we can describe

$$t = (s_0, s_1, \ldots, s_{n(l_0)-1}, s_{n(l_0)}, t_{n(l_0)+1}, \ldots).$$

If $t \in \Sigma \setminus \Sigma_4$, by definition of $\Lambda^{-1}(t)$,

$$\Lambda^{-1}(t) \subset K_s^{(l_0)} \subset \Lambda^{-1}(s)[\epsilon].$$

When $t \in \Sigma_4$, for the definition

$$\Lambda^{-1}(t) = \bigcap_{l=1}^{\infty} K_t^{(l)}$$

of $\Lambda^{-1}(t)$, it is clear that $K_t^{(l)} = K_s^{(l)}$ for $l = 1, 2, \ldots, l_0$. Then

$$\Lambda^{-1}(t) \subset K_t^{(l_0)} \subset \Lambda^{-1}(s)[\epsilon].$$

Since $\Lambda^{-1}(s)$ is a one-point set, for $t \in \Sigma$ with $\rho(s, t) < \delta$,

$$d_H(\Lambda^{-1}(s), \Lambda^{-1}(t)) = \inf\{\epsilon' : \Lambda^{-1}(t) \subset \Lambda^{-1}(s)[\epsilon']\} < \epsilon.$$ 

Therefore $\Lambda^{-1}$ is continuous at $s$. \hfill \Box

**Lemma 2.7.** $\Lambda \circ F = \sigma \circ \Lambda$.

**Proof.** For $K \in \text{Comp}(K_f)$, we set $\Lambda(K) = (s_0, s_1, s_2, \ldots)$. Then $\sigma \circ \Lambda(K) = (s_1, s_2, \ldots)$. On the other hand, $\Lambda \circ F(K) = \Lambda(f(K)) = (s_1, s_2, \ldots)$. Therefore $\Lambda \circ F = \sigma \circ \Lambda$. \hfill \Box

We have completed the proof of Theorem 1.8.

**Remark 2.8.** Various cases of the cubic polynomial are shown by [2].

### 3 Similar Results of Theorem 1.8

For a quartic polynomial, the following two cases are also considered. Theorem 3.1 and Theorem 3.2 are shown like the proof of Theorem 1.8. Suppose that the Julia set is disconnected but not totally disconnected.
Case 1: Let $f$ be a quartic polynomial and let $c_1, c_2$ and $c_3$ be finite critical points of $f$. Suppose that $G(c_1) = 0$ and $G(c_3) \geq G(c_2) > 0$, that is, $c_1 \in K_f$ and $c_2, c_3 \in A_f(\infty)$.

Let $U$ be a bounded component of $\mathbb{C} \setminus G^{-1}(G(f(c_2)))$. Suppose that $U_A$, $U_B$ and $U_C$ be bounded components of $\mathbb{C} \setminus G^{-1}(G(c_2))$ such that $c_1 \in U_C$. Then $U_A, U_B$ and $U_C$ are proper subsets of $U$. Furthermore $(f|_{U_A}, U_A, U)$ and $(f|_{U_B}, U_B, U)$ are polynomial-like maps of degree 1 and $(f|_{U_C}, U_C, U)$ is a polynomial-like map of degree 2.

Under this situation, we define the kneading sequence $(\alpha_n)_{n \geq 0}$ of $c_1$ as

$$
\alpha_n = \begin{cases} 
  A & \text{if } f^n(c_1) \in U_A, \\
  B & \text{if } f^n(c_1) \in U_B, \\
  C & \text{if } f^n(c_1) \in U_C.
\end{cases}
$$

We assume that the kneading sequence of $c_1$ is $(CCC\cdots)$.

Let $\Sigma_5 = \{1,2,3,4, C\}^{\mathbb{N}_0}$ be the symbol space. We define a subset $\Sigma$ of $\Sigma_5$ as follows: $s = (s_n) \in \Sigma$ if and only if

1. $s_n = C \Rightarrow s_{n+1} = C$,
2. $s_n = C$ and $s_{n-1} \neq C \Rightarrow s_{n-1} = 1$ or 2,
3. if $s \in \Sigma_4 = \{1,2,3,4\}^{\mathbb{N}_0}$, then there exists a subsequence $(s_n(k))_{k=1}^{\infty}$ such that $s_{n(k)} = 1$ or 2 for all $k \in \mathbb{N}$.

**Theorem 3.1.** Let $f$ be a quartic polynomial. Suppose that its finite critical points $c_1, c_2$ and $c_3$ satisfy $G(c_1) = 0$ and $G(c_3) \geq G(c_2) > 0$ and suppose that $J_f$ is disconnected but not totally disconnected. Moreover, suppose that the kneading sequence of $c_1$ is $(CCC\cdots)$. Then there exists a homeomorphism $\Lambda : \text{Comp}(K_f) \to \Sigma$ such that $\Lambda \circ F = \sigma \circ \Lambda$.

Case 2: Let $f$ be a quartic polynomial and let $c_1, c_2$ and $c_3$ be finite critical points of $f$ such that $c_1 = c_2$ and $c_1 \neq c_3$. Suppose that $G(c_1) = 0$ and $G(c_3) > 0$, that is, $c_1 \in K_f$ and $c_3 \in A_f(\infty)$.

Let $U$ be a bounded component of $\mathbb{C} \setminus G^{-1}(G(f(c_3)))$. Suppose that $U_A$ and $U_B$ be bounded components of $\mathbb{C} \setminus G^{-1}(G(c_3))$ such that $c_1 \in U_B$. Then $U_A$ and $U_B$ are proper subsets of $U$. Furthermore $(f|_{U_A}, U_A, U)$ is a polynomial-like map of degree 1 and $(f|_{U_B}, U_B, U)$ is a polynomial-like map of degree 3. We assume that the kneading sequence of $c_1$ is $CBBB\cdots$.

Let $\Sigma_5 = \{1,2,3,4, B\}^{\mathbb{N}_0}$ be the symbol space. We define a subset $\Sigma$ of $\Sigma_5$ as follows: $s = (s_n) \in \Sigma$ if and only if

1. $s_n = B \Rightarrow s_{n+1} = B$, 

2. $s_n = B$ and $s_{n-1} \neq B \Rightarrow s_{n-1} = 1$ or 2,
3. if $s \in \Sigma_4 = \{1,2,3,4\}^{\mathbb{N}_0}$, then there exists a subsequence $(s_n(k))_{k=1}^{\infty}$ such that $s_{n(k)} = 1$ or 2 for all $k \in \mathbb{N}$.

**Theorem 3.2.** Let $f$ be a quartic polynomial. Suppose that its finite critical points $c_1, c_2$ and $c_3$ satisfy $G(c_1) = 0$ and $G(c_3) > G(c_2) > 0$ and suppose that $J_f$ is disconnected but not totally disconnected. Moreover, suppose that the kneading sequence of $c_1$ is $(CCC\cdots)$. Then there exists a homeomorphism $\Lambda : \text{Comp}(K_f) \to \Sigma$ such that $\Lambda \circ F = \sigma \circ \Lambda$. 

...
2. \( s_n = B \) and \( s_{n-1} \neq B \) \( \Rightarrow \) \( s_{n-1} = 1 \),

3. if \( s \in \Sigma_4 = \{1, 2, 3, 4\}^{\mathbb{N}_0} \), then there exists a subsequence \( (s_{n(k)})_{k=1}^{\infty} \) such that \( s_{n(k)} = 1 \) for all \( k \in \mathbb{N} \).

**Theorem 3.2.** Let \( f \) be a quartic polynomial. Suppose that its finite critical points \( c_1, c_2 \) and \( c_3 \) satisfy \( c_1 = c_2 \), \( c_1 \in K_f \) and \( c_3 \in A_f(\infty) \) and suppose that \( J_f \) is disconnected but not totally disconnected. Moreover, suppose that the kneading sequence of \( c_1 \) is \( (BBB\cdots) \). Then there exists a homeomorphism \( \Lambda : \text{Comp}(K_f) \rightarrow \Sigma \) such that \( \Lambda \circ F = \sigma \circ \Lambda \).

### 4 Relevances with Polynomial Semigroups

In this section, we explore relevances of polynomials and polynomial semigroups. The following theorem about the polynomial-like map is important.

**Theorem 4.1** ([3, 7]). Every polynomial-like map \( (f, U, V) \) of degree \( d \geq 2 \) is hybrid equivalent to a polynomial \( p \) of degree \( d \). That is to say, there exist a polynomial \( p \) of degree \( d \), a neighborhood \( W \) of \( K_f \) in \( U \) and a quasiconformal map \( h : W \rightarrow h(W) \) such that

1. \( h(K_f) = K_p \),
2. the complex dilatation \( \mu_h \) of \( h \) is zero almost everywhere on \( K_f \),
3. \( h \circ f = p \circ h \) on \( W \cap f^{-1}(W) \).

If \( K_f \) is connected, \( p \) is unique up to conjugation by affine map.

Under the assumption of Theorem 1.8, \( (f_1, U_A, U) \) and \( (f_2, U_B, U) \) are polynomial-like maps of degree 2. Furthermore \( K_{f_1} \) and \( K_{f_2} \) are connected.

By Theorem 4.1, there exist quadratic polynomials \( g_1 \) and \( g_2 \) with \( K_{g_1} \cap K_{g_2} = \emptyset \), a neighborhood \( W_1 \) of \( K_{f_1} \) in \( U_A \), a neighborhood \( W_2 \) of \( K_{f_2} \) in \( U_B \) and quasiconformal maps \( h_1 \) on \( W_1 \) and \( h_2 \) on \( W_2 \) such that \( h_1(K_{f_1}) = K_{g_1} \) and \( h_2(K_{f_2}) = K_{g_2} \).

We define branches \( \tilde{I}_1 \) and \( \tilde{I}_2 \) of \( g_1^{-1} \). Since \( K_{g_1} \) is connected, there exists a conformal map \( \Psi_1 : \mathbb{C} \setminus \overline{D} \rightarrow \mathbb{C} \setminus K_{g_1} \) such that \( \Psi_1^{-1} \circ g_1 \circ \Psi_1(z) = z^2 \).

The external ray \( R_1 = \Psi_1(\{z \in \mathbb{C} : |z| > 1 \text{ and } \arg(z) = 0\}) \) lands at a fixed point of \( g_1 \). Let \( R'_1 \) be the external ray which satisfies \( g_1(R'_1) = R_1 \) and differs from \( R_1 \). At this time, we replace \( g_2 \) so that

\[ R_1 \cap K_{g_2} = \emptyset \quad \text{and} \quad R'_1 \cap K_{g_2} = \emptyset. \]
Then we define branches $\tilde{I}_1$ and $\tilde{I}_2$ of $g_1^{-1}$ as
\[
\tilde{I}_1 : \mathbb{C} \setminus (K_{g_1} \cup R_1) \to \tilde{U}_1 \quad \text{and} \quad \tilde{I}_2 : \mathbb{C} \setminus (K_{g_1} \cup R_1) \to \tilde{U}_2,
\]
where $\tilde{U}_1$ and $\tilde{U}_2$ are components of $\mathbb{C} \setminus (K_{g_1} \cup R_1 \cup R_1')$ respectively. Similarly, we take external rays $R_2$ and $R_2'$. Then we define branches $\tilde{I}_3$ and $\tilde{I}_4$ of $g_2^{-1}$ as
\[
\tilde{I}_3 : \mathbb{C} \setminus (K_{g_2} \cup R_2) \to \tilde{U}_3 \quad \text{and} \quad \tilde{I}_4 : \mathbb{C} \setminus (K_{g_2} \cup R_2) \to \tilde{U}_4,
\]
where $\tilde{U}_3$ and $\tilde{U}_4$ are components of $\mathbb{C} \setminus (K_{g_2} \cup R_2 \cup R_2')$ respectively.

For $s \in \Sigma$, we set $K_s = \Lambda^{-1}(s)$ and $J_s = \partial K_s$. $K_s$ is a component of $K_f$ and $J_s$ is a component of $J_f$. For $s = (s_0, s_1, s_2, \ldots) \in \Sigma \setminus \Sigma_4$, we define a quasiconformal map $h_s$ on a neighborhood of $K_s$.

Let $n \in \mathbb{N}_0$ be the smallest number with $s_n = A$ and $s_{n-1} \neq A$ or $s_n = B$ and $s_{n-1} \neq B$. $h_s$ is defined on $W_s = I_{\epsilon_0} \circ \cdots \circ I_{\epsilon_{n-1}}(W_i)$ as
\[
h_s = \tilde{I}_{\epsilon_0} \circ \cdots \circ \tilde{I}_{\epsilon_{n-1}} \circ h \circ f^n,
\]
where $i = \begin{cases} 1 & \text{if } s_n = A \text{ and } s_{n-1} \neq A, \\ 2 & \text{if } s_n = B \text{ and } s_{n-1} \neq B. \end{cases}
\]

We set $\tilde{K}_s = h_s(K_s)$, $\tilde{J}_s = \partial \tilde{K}_s$ and $G = \langle g_1, g_2 \rangle$. If necessary, we replace $g_1$ and $g_2$ so that each $\tilde{K}_s$ is disjoint. Since $\tilde{J}_s = \partial \tilde{K}_s = h_s(\partial K_s) = h_s(J_s)$ and $J_G$ is backward invariant (see [4]), $h_s$ maps $J_s$ onto a component $\tilde{J}_s$ of $J_G$. By definition, we turn out that $h_{(A,A,A,\ldots)} = h_1$ and $h_{(B,B,B,\ldots)} = h_2$.

Next, we define a homeomorphism
\[
h : \bigcup_{s \in \Sigma \setminus \Sigma_4} K_s \to \bigcup_{s \in \Sigma \setminus \Sigma_4} \tilde{K}_s
\]
as $h|_{K_s} = h_s$.

Remark 4.2. For $s \in \Sigma \cap \Sigma_4$, a one-point component $K_s$ of $K_f$ is characterized using the Hausdorff topology. For $s = (s_0, s_1, s_2, \ldots) \in \Sigma \cap \Sigma_4$, we set
\[
\begin{align*}
t^{(n)} = & \begin{cases} (s_0, s_1, \ldots, s_{n-1}, A, A, \ldots) & \text{if } s_{n-1} = 3 \text{ or } 4, \\ (s_0, s_1, \ldots, s_{n-1}, B, B, \ldots) & \text{if } s_{n-1} = 1 \text{ or } 2. \end{cases}
\end{align*}
\]
Then the sequence $\{t^{(n)}\}_{n=1}^{\infty}$ is in $\Sigma \setminus \Sigma_4$ and $t^{(n)} \to s$ as $n \to \infty$. Since $\Lambda^{-1}$ is continuous,
\[
K_s = \Lambda^{-1}(s) = \lim_{n \to \infty} \Lambda^{-1}(t^{(n)}) = \lim_{n \to \infty} K_{t^{(n)}}.
\]
Finally, we extend $h$ homeomorphically on $K_f = \bigcup_{s \in \Sigma} K_s$. For $s \in \Sigma \cap \Sigma_4$, we define $\tilde{K}_s = h(K_s)$ as

$$h(K_s) = \lim_{n \to \infty} h(K_{t^{(n)}}).$$

Note that each $\tilde{T}_k$ decreases the Poincaré distance on $\mathbb{C} \setminus (K_{g_1} \cup R_1)$ or $\mathbb{C} \setminus (K_{g_2} \cup R_2)$. As mentioned above, $h$ is a homeomorphism between $K_f = \bigcup_{s \in \Sigma} K_s$ and $\bigcup_{s \in \Sigma} \tilde{K}_s$.

**Lemma 4.3.**

$$\partial \left( \bigcup_{s \in \Sigma} \tilde{K}_s \right) = J_G.$$

**Proof.** Lemma 4.3 follows from the following lemma.

**Lemma 4.4 ([4]).** If $z$ is in $J_G \setminus E_G$, then

$$\overline{O^-(z)} = J_G,$$

where $O^-(z) = \{ w \in \hat{\mathbb{C}} : \text{there exists } g \in G \text{ such that } g(w) = z \}$ is the backward orbit of $z$ and $E_G = \{ z \in \hat{\mathbb{C}} : O^-(z) \text{ contains at most two points} \}$ is the exceptional set of $G$.

By Lemma 4.4,

$$\partial \left( \bigcup_{s \in \Sigma} \tilde{K}_s \right) = \bigcup_{s \in \Sigma} \partial \tilde{K}_s = \bigcup_{s \in \Sigma} \tilde{J}_s = \bigcup_{s \in \Sigma \setminus \Sigma_4} \tilde{J}_s = J_G.$$

We have completed the proof of Theorem 1.9.

**References**


