Semi-hyperbolicity of entire functions

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Abstract

In this paper, we investigate a condition for semi-hyperbolicity of (transcendental) entire functions (Theorem A). As an application of the main theorem, we show a result on a measure theoretical property for the dynamics of entire functions (Theorem B). In particular, we give a sufficient condition which guarantees that $\{\infty\}$ is a metric global attractor (Corollary C).

1 Preliminaries

Let $f$ be an entire function and $f^n$ denote the $n$-th iterate of $f$. Recall that the Fatou set $F_f$ and the Julia set $J_f$ of $f$ are defined as follows:

\[
F_f := \{z \in \mathbb{C} | \{f^n\}_{n=1}^{\infty} \text{ is a normal family in a neighborhood of } z\},
\]

\[
J_f := \mathbb{C} \setminus F_f.
\]

By definition, $F_f$ is open and $J_f$ is closed in $\mathbb{C}$. Also $J_f$ is compact if $f$ is a polynomial, while it is non-compact if $f$ is transcendental. This is due to the fact that $\infty$ is an essential singularity of $f$. A connected component $U$ of $F_f$ is called a Fatou component of $f$. $U$ is called a wandering domain if $f^m(U) \cap f^n(U) = \emptyset$ for every $m, n \in \mathbb{N} (m \neq n)$. If there exists an $n_0 \in \mathbb{N}$ with $f^{n_0}(U) \subseteq U$, $U$ is called a periodic component of period $n_0$ and it is well known that there are four possibilities, namely, an attracting basin, a parabolic basin, a Siegel disk and a Baker domain.
A critical value is a point $p := f(c)$ for a point $c$ with $f'(c) = 0$. This is a singularity of $f^{-1}$. For polynomials we have only to consider this type of singularities but there can be another type of singularities called an asymptotic value for transcendental entire functions. A point $p$ is called an asymptotic value if there exists a continuous curve $L(t) (0 \leq t < 1)$ (which is called an asymptotic path) with
\[
\lim_{t \to 1} L(t) = \infty \quad \text{and} \quad \lim_{t \to 1} f(L(t)) = p.
\]
A point $p$ is called a singular value if it is either a critical or an asymptotic value and we denote the set of all singular values by $\text{sing}(f^{-1})$. Also we define
\[
P(f) := \bigcup_{n=0}^{\infty} f^n(\text{sing}(f^{-1}))
\]
and call it the post-singular set of $f$.

The following are some basic concepts from dynamical system theory:

**Definition 1.1.** Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function and $z \in \mathbb{C}$.

1. The **forward orbit** of a point $z$ is the set
   \[
   O^+(z) := \{z, f(z), \cdots, f^n(z), \cdots\}.
   \]

2. We define
   \[
   \omega(z) := \{w \mid w = \lim_{n_i \to \infty} f^{n_i}(z), \exists n_1 < n_2 < \cdots\}
   \]
   and call it the $\omega$-limit set of $z$.

3. A point $z$ is called **recurrent** if $z \in \omega(z)$, that is, the forward orbit of $z$ passes through an arbitrary small neighborhood of $z$ infinitely often. Otherwise, it is called non-recurrent.

4. $f$ is called **ergodic** if any measurable set $A$ satisfying $f^{-1}(A) = A$ has zero or full measure in $\mathbb{C}$.

### 2 The Mañé's Theorem — Semi-hyperbolicity —

The following is a part of the Mañé's theorem, which was proved in 1993.
Theorem 2.1 (Mañé, [M]). Let $f$ be a rational function and $x \in J_f$. Suppose that

(i) $x$ is not a parabolic periodic point and

(ii) $x \notin \bigcup_{c \in \text{Rec} \cap J_f} \omega(c)$,

where \(\text{Rec} = \{\text{recurrent critical points of } f\}\).

Then for every $\epsilon > 0$, there exists a neighborhood $U$ of $x$ which satisfies the following:

(1) For every $n \in \mathbb{N}$ and every connected component $V$ of $f^{-n}(U)$,

$$\text{diam}_{\text{sph}}(V) \leq \epsilon$$

holds, where \(\text{diam}_{\text{sph}}\) denotes the spherical diameter on $\hat{\mathbb{C}}$.

(2) There exists an $N \in \mathbb{N}$ such that for any connected component $V$ of $f^{-n}(U)$ ($\forall n$), $f^n|_V : V \to U$ satisfies

$$\deg(f^n|_V : V \to U) \leq N.$$ 

Taking this result into account, we define the semi-hyperbolicity of $f$ at a point $x_0 \in J_f$ as follows:

**Definition 2.2.** $f$ is semi-hyperbolic at $x \in J_f$ if there exists a neighborhood $U$ of $x$ such that the condition (2) in Theorem 2.1 holds. In the case that $f$ is transcendental, we add the following property:

$$f^n|_V : V \to U$$

is proper for every $V$.

Recall that $f : X \to Y$ is called proper if $f^{-1}(K) \subset X$ is compact for every compact subset $K \subset Y$. Note that this property is automatically satisfied when $f$ is a polynomial or rational. We say $f$ is semi-hyperbolic if $f$ is semi-hyperbolic at any point $x_0 \in J_f$.

The converse of Theorem 2.1 is also true. That is, if $x$ is a parabolic periodic point or $x \notin \bigcup_{c \in \text{Rec} \cap J_f} \omega(c)$, then $f$ is not semi-hyperbolic at $x \in J_f$. In this paper we investigate a condition for semi-hyperbolicity for transcendental entire functions. In transcendental case, a new phenomena can occur. For example, Bergweiler and Morosawa ([BM]) constructed an example of $f$ with no parabolic periodic point and no recurrent critical point, but has a point $x_0 \in J_f$ at which $f$ is not semi-hyperbolic.
3 Main Result

Define the sets $\text{Rec}$, $\text{Non-Rec}$ and $\text{AV}$ as follows:

\[
\text{Rec} := \{c \mid c \text{ is a recurrent critical point of } f\}
\]
\[
\text{Non-Rec} := \{c \mid c \text{ is a non-recurrent critical point of } f\}
\]
\[
\text{AV} := \{c \mid c \text{ is an asymptotic value of } f\}.
\]

Then the main result of this paper is the following:

**Theorem A (Mañe's Theorem for entire functions).** Let $f$ be a (transcendental) entire function and $z_0 \in J_f$. Then $f$ is semi-hyperbolic at $z_0$ if and only if $z_0 \notin Z$, where the set $Z$ is defined as follows:

\[
Z = \left( \bigcup_{i=1}^{3} X_i \right) \cup \left( \bigcup_{j=1}^{5} Y_j \right),
\]

where

\[
X_1 = \{p \mid p \text{ is a parabolic periodic point of } f\},
\]
\[
X_2 = \text{derived set of } \{p \mid p \text{ is a attracting periodic point of } f\},
\]
\[
X_3 = \{p \mid f^n|_W \to p \ (n_i \to \infty) \text{ for some wandering domain } W\},
\]
\[
Y_1 = \bigcup_{c \in \text{Rec} \cap J_f} \omega(c),
\]
\[
Y_2 = \bigcup_{n=0}^{\infty} f^n(\text{AV}) \cap J_f,
\]
\[
Y_3 = \{p \mid p = \lim_{i \to \infty} f^{n_i}(c_i), \ c_i \in \text{Non-Rec} \cap J_f \ (i \in \mathbb{N}) \text{ are mutually different and order of } c_i \to \infty \ (i \to \infty)\},
\]
\[
Y_4 = \{p \mid p = \lim_{i \to \infty} f^{n_i}(c_i), \ c_i \in \text{Non-Rec} \cap J_f \ (i \in \mathbb{N}) \text{ are mutually different with } \sup_i \text{ (order of } c_i) < \infty \text{ and for any } \epsilon > 0 \text{ let } N_i(\epsilon) := \#\{c \mid c : \text{critical point, } O^+(c_i) \cap U_\epsilon(c) \neq \emptyset\},
\]
\[
\text{then } \sup_i N_i(\epsilon) = \infty \},
\]
\[
Y_5 = \{p \mid p = \lim_{i \to \infty} f^{n_i}(c_i), \ c_i \in \text{Non-Rec} \cap J_f \ (i \in \mathbb{N}) \text{ are mutually different with } \sup_i \text{ (order of } c_i) < \infty \text{ and let } \delta_i(n) := \sup\{\delta \mid \#\{O^+(c_i) \cap (U_\delta(c_i) \setminus \{c_i\}) \leq n\},
\]
\[
\text{then } \inf_i \delta_i(n) = 0 \text{ for } \forall n\}.\]
4 Outline of the proof of Theorem A

Suppose \( z_0 \in J_f \), \( z_0 \not\in Z \), then take a neighborhood \( U \) of \( z_0 \) with \( \overline{U} \cap Z = \emptyset \).

**Definition 4.1.** For \( z \in U \) let \( S(z, \varepsilon) \) be a square centered at \( z \) with side length \( 2\varepsilon \) and with sides parallel to coordinate axes. We say \( S(z, \varepsilon) \) is admissible if \( S(z, 3\varepsilon) \subset U \).

**Lemma 4.2.** For a given \( \varepsilon > 0 \) and an \( N \in \mathbb{N} \), there exists a \( \delta > 0 \) which satisfies the following: If \( S(z, \delta) \) is an admissible square and \( S_n \) is a connected component of \( f^{-n}(S(z, \delta)) \) such that \( \deg(f^n|_{S_n}) \leq N \), then

\[
\text{diam}(f^{-n}(S(z, \frac{\delta}{2}))) \leq \varepsilon
\]

holds for the same branch of \( f^{-n} \).

(Proof of Lemma 4.2) : Suppose not, then there exist a \( z_l \in U \) and admissible squares \( S_l := S(z_l, 2^{-l}) \) such that for some component \( V_l \) of \( f^{-n_l}(S(z_l, 2^{-(l+1)}) \) it holds that \( \text{diam} V_l \geq \varepsilon > 0 \) and \( \deg(f^{n_l}|_{S(z_l, 2^{-l})}) \leq N \).

Now suppose there exist a subsequence \( l_k \uparrow \infty \) and a disk \( D_{l_k} \subset V_{l_k} \) with (spherical) radius \( r > 0 \) which is independent of \( l_k \). Taking subsequence, if necessary, we have

\[
D_{l_k} \rightarrow \exists \text{D} \quad (k \rightarrow \infty).
\]

Then \( \{f^{n_k}|_D\}_{k=1}^{\infty} \) is bounded, since \( f^{n_k}(D) \subset U \). Hence \( \{f^{n_k}|_D\}_{k=1}^{\infty} \) is normal. So we have \( D \subset F_f \) and let \( D_{F_f} \supset D \) be the Fatou component containing \( D \). On the other hand, taking subsequence, if necessary, we have

\[
S_{l_k} \rightarrow \exists z_{\infty} \in U \quad (k \rightarrow \infty).
\]

Then

\[
f^{n_k}|_D \rightarrow z_{\infty}.
\]

Such a \( z_{\infty} \) is either one of the following:

(i) attracting periodic point,
(ii) parabolic periodic point,
(iii) finite constant limit function on a wandering domain.

In other words, \( D_{F_f} \) is not a Siegel disk or a Baker domain. This is a contradiction by the assumption. Hence let \( D_l \) be the maximal disk in \( V_l \), then it follows that \( \text{diam}(D_l) \rightarrow 0 \). This again contradicts the following
Lemma 4.3 (cf. Carleson-Jones-Yoccoz, [CJY]). Let $W \subset \mathbb{C}$ be a simply connected domain and let $g : W \to \mathbb{D}$, $g(\partial W) \subset \partial \mathbb{D}$ be degree $N$. Then there exists a constant $C > 0$ depending only on $N$ such that

$$B_{\mathbb{D}}(g(z), Cr) \subset g(B_{W}(z, r)) \subset B_{\mathbb{D}}(g(z), r).$$

$\square$

Now since $z_0 \notin Z$, there is a neighborhood $U$ of $z_0$ satisfying

(0) $U$ does not contain attracting periodic points, parabolic periodic points, wandering domains, points in orbits of recurrent critical points or asymptotic values.

Moreover, $U$ satisfies either one of the following:

(1) The number of critical points with $O^{+}(c) \cap U \neq \emptyset$ is finite (let us denote them by $c_1, c_2, \cdots, c_{N_0}$) and all of them are non-recurrent. Then for some $\epsilon_0 > 0$ we have

$$(O^{+}(c_i) \setminus \{c_i\}) \cap U_{\epsilon_0}(c_i) = \emptyset.$$ 

(2) The number of critical points with $O^{+}(c) \cap U \neq \emptyset$ is infinite (let us denote them by $c_1, c_2, \cdots$) and all of them are non-recurrent. There exists an $M_0 > 0$ such that

order of $c_i \leq M_0$, for $\forall i \in \mathbb{N}$.

Also there exists an $\epsilon_1 > 0$ and an $N_0 \in \mathbb{N}$ such that

$$\#\{c \mid c : \text{critical point, } O^{+}(c_i) \cap U_{\epsilon_1}(c) \neq \emptyset\} \leq N_0 < \infty$$

holds for every $i \in \mathbb{N}$. Furthermore there exists a $\delta_1 > 0$ and an $n_1 \in \mathbb{N}$ such that

$$\#\{O^{+}(c_i) \cap (U_{\delta_1}(c_i) \setminus \{c_i\})\} \leq n_1, \forall i \in \mathbb{N}.$$ 

In this case, we put $\epsilon_0 := \min(\epsilon_1, \delta_1)$

Now let $N := (M_0 + 1)^{N_0(n_1+1)}$ and take $\epsilon > 0$ with $\epsilon < \epsilon_0/36N$. Then there is a $\delta > 0$ which is determined by the previous Lemma 4.2.

Lemma 4.4. For any $\eta$ with $0 < \eta \leq \delta$ and $n \in \mathbb{N}$, we have

$$\text{diam}(f^{-n}(S(z_0, \frac{1}{2}\eta))) \leq \epsilon.$$
That is, the conclusion of Lemma 4.2 holds without the assumption on degree. 

Hence for any $\varepsilon > 0$ with $\varepsilon < \varepsilon_0/36N$ by taking $\sigma > 0$ sufficiently small, we have
\[
\text{diam}(f^{-n}(S(z_0, \sigma))) \leq \varepsilon, \quad \forall n.
\]
With a little more argument, we can conclude
\[
\deg(f^n|_{S(z_0, \sigma)}) < N = (M_0 + 1)^{N_0(n_1 + 1)}.
\]
For the opposite implication, it is rather easy to check that $z_0 \in Z$ implies that $f$ is not semi-hyperbolic at $z_0$. 

Remark. (1) Comparing Theorem A with the original Mañé's Theorem, in the case that $f$ is rational, we have
\[
Z = X_1 \cup Y_1
\]
i.e. $X_2, X_3, Y_2, Y_3, Y_4, Y_5$ are all empty.

(2) Theorem A includes the following result:

Theorem 4.5 (Bergweiler-Morosawa (2002)). Let $f$ be entire. If $f$ is semi-hyperbolic at $a \in \mathbb{C}$, then $a$ is not a limit function of $\{f^n\}_{n=1}^{\infty}$ in any component of $F_f$.

(3) Consider the following question:

Question : For each $X_i$ ($i = 1 \sim 3$) and $Y_j$ ($j = 1 \sim 5$), is there an $f$ with $X_i \neq \emptyset$ or $Y_j \neq \emptyset$ ?

First, there are a lot of $f$ with $X_1 \neq \emptyset$. But I do not know whether parabolic periodic points can accumulate to a finite point in $\mathbb{C}$. It is somehow surprising that there is an $f$ with $X_2 \neq \emptyset$. We can construct such an example by using the similar method in [KS]. We omit the details. For $X_3$, Éremenko and Lyubich ([EL]) constructed an $f$ with $X_3 \neq \emptyset$, that is, $f$ has a wandering domain with (infinitely many) finite constant limit functions.

There are a lot of $f$ with $Y_1 \neq \emptyset$ or $Y_2 \neq \emptyset$. It is not difficult to construct an $f$ with $Y_3 \neq \emptyset$. For $Y_4$, Bergweiler and Morosawa ([BM]) showed the
following example: Consider
\[ f(z) = \frac{z}{2} - \frac{1}{2\pi} \sin \pi z + c(\cos \pi z - 1), \]
where \( c = 0.467763 \cdots \) is a solution of
\[ \pi + 2 \cos 2c\pi - 4c\pi \sin 2c\pi = 0. \]
Then, \( f \) has no asymptotic values, no parabolic periodic point and no recurrent critical point, but \( f \) is not semi-hyperbolic at \( 1 \in J_f \). This \( f \) has a sequence of critical points \( \{c_i\}_{i=1}^{\infty} \) with
\[ f(c_i) = c_{i-1} \quad (i = 2, 3, \cdots), \quad f(c_1) = 1 \]
and \( f(1) \) is a repelling fixed point of \( f \) so \( 1 \in J_f \). Hence \( 1 \in Y_4 \) in this case. Finally we do not know an example of \( f \) with \( Y_5 \neq \emptyset \).

5 Some applications of the main theorem

As an application of Theorem A, we can show the following result on a measure theoretical property for the dynamics of entire functions. This is a refinement of the result by Bock ([B]).

**Theorem B.** Either one of the following (AT\( \hat{\mathcal{Z}} \)) or (ERG) holds for an entire function \( f \):

(\text{AT}\( \hat{\mathcal{Z}} \)) Almost every point \( z \in J_f \) is attracted to the set \( \hat{\mathcal{Z}} \), that is,
\[ \lim_{n \to \infty} \text{dist}_{\text{sph}}(f^n(z), \hat{\mathcal{Z}}) = 0, \quad \text{(i.e. } \omega(z) \subset \hat{\mathcal{Z}}) \]
holds for a.e. \( z \in J_f \), where \( \hat{\mathcal{Z}} := \mathcal{Z} \cup \{\infty\} \).

\text{(ERG)} \( J_f = \mathbb{C} \) and \( f \) is ergodic.

Furthermore, (ERG) can be replaced by the following (IR) or (FOD):

\text{(IR)} \( J_f = \mathbb{C} \) and \( f \) is infinitely recurrent, i.e. for every \( X \subset \mathbb{C} \) with \( \text{Leb}(X) > 0 \) and every \( z \in \mathbb{C} \),
\[ \#\{n \in \mathbb{N} \mid f^n(z) \in X\} = \infty \]
holds, where \( \text{Leb}(\cdot) \) denotes the Lebesgue measure on \( \mathbb{C} \).

\text{(FOD)} \( J_f = \mathbb{C} \) and for a.e. \( z \in \mathbb{C} \), the forward orbit \( O^+(z) \subset \mathbb{C} \) is dense.
Corollary C. Let $f$ be an entire function with the following properties:

(i) Every critical point $c$ of $f$ is either preperiodic or satisfies $f^n(c) \to \infty$ ($n \to \infty$).

(ii) Every asymptotic value is eventually periodic.

(iii) The post-singular set $P(f)$ is discrete in $\mathbb{C}$.

Then either one of the following holds:

(MGA) $\{\infty\}$ is a metric global attractor, that is, $f^n(z) \to \infty$ ($n \to \infty$) for a.e. $z \in \mathbb{C}$ (i.e. $\omega(z) = \{\infty\}$).

(FOD) $J_f = \mathbb{C}$ and $O^+(z) \subset \mathbb{C}$ is dense for a.e. $z \in \mathbb{C}$ (i.e. $\omega(z) = \hat{\mathbb{C}}$).

In particular, if $f$ satisfies the conditions (i) $\sim$ (iii) and $J_f \neq \mathbb{C}$, then $\{\infty\}$ is a metric global attractor for $f$.

(Proof): It follows from the assumptions (i) $\sim$ (iii) that every singular value $p$ satisfies either $f^n(p) \to \infty$ or eventually lands on a repelling periodic point. If $F_f \neq \emptyset$, then only possible Fatou components are either Baker domains (or their preimages) or wandering domains. If there is a wandering domain $U$, then we have $f^n|_U \to \infty$, because in general a finite limit function on a wandering domain is a constant which belongs to the derived set of $P(f)$ (see [BHKMT]), which is empty by (iii) in our case.

Then either (ATZ) or (FOD) holds by Theorem A. In the case of (ATZ), it follows that

$$\omega(z) \subset \hat{Z} = Y_2 \cup \{\infty\}, \text{ for a.e. } z \in J_f.$$ 

On the other hand, $Y_2$ consists of repelling periodic points only and hence $O^+(z)$ cannot accumulate on $Y_2$. Therefore

$$\omega(z) = \hat{Z} = \{\infty\}, \text{ i.e. } f^n(z) \to \infty \text{ for a.e. } z \in J_f,$$ 

which implies that $\{\infty\}$ is a metric global attractor.

In the case of (FOD), it follows that $J_f = \mathbb{C}$ and $O^+(z) \subset \mathbb{C}$ is dense for a.e. $z \in \mathbb{C}$, which means that $\omega(z) = \hat{\mathbb{C}}$. This completes the proof of Corollary C. \qed

Corollary D. Let $f$ be a semi-hyperbolic (transcendental) entire function with $J_f \neq \mathbb{C}$. Then,
(1) $\text{Leb}(J_f) = 0 \iff \text{Leb}(J_f \cap I_f) = 0$, where $I_f := \{z \mid f^n(z) \to \infty\}$.

(2) $\text{Leb}(J_f) > 0 \implies f^n(z) \to \infty (n \to \infty)$ for a.e. $z \in J_f$

(Proof): Since $f$ is semi-hyperbolic, we have $Z = \emptyset$ by Theorem A. Also $(\text{ATZ})$ holds from Theorem B, because we assume that $J_f \neq \mathbb{C}$. This means that $f^n(z) \to \infty$ for a.e. $z \in J_f$. Now it is obvious to see that (1) and (2) hold.

References


