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| Title | On Escaping Sets of Entire Functions(Complex Dynamics and its Related Fields) |
| Author(s) | Singh, Anand P. |
| Citation | 数理解析研究所講究録 (2006), 1494: 24-33 |
| Issue Date | 2006-05 |
| URL | http://hdl.handle.net/2433/58301 |
| Right | |
| Type | Departmental Bulletin Paper |
| Textversion | publisher |

数理解析研究所講究録
1494 巻 2006 年 24-33



# On Escaping Sets of Entire Functions

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**Abstract**

We give a brief survey of results on the escaping sets of entire functions, and mention some of the results obtained in this direction jointly by the author and M. Taniguchi.

Let $f$ be a transcendental entire function. For $n \in \mathbb{N}$, let $f^n$ denote the n-th iterate of $f$. Thus $f^o(z) = z$, $f^n(z) = f(f^{n-1}(z))$ for all $n = 1, 2, \ldots$. The set

$F(f) = \{z \in \mathbb{C} : \{f^n\}_{n\in\mathbb{N}}$ is normal in some neighbourhood of $z\}$

is called the Fatou set of $f$ or the set of normality of $f$ and its complement $J(f)$ is the Julia set of $f$.

Fatou set is open and completely invariant: $z \in F(f)$ if and only if $f(z) \in F(f)$. If $U$ is a component of $F(f)$, then $f(U)$ lies in some component $V$ of $F(f)$. If $U_n \cap U_m = \phi$ for $n \neq m$ where $U_n$ denotes the component of $F(f)$ which contains $f^n(U)$, then $U$ is called a wandering domain, else $U$ is either a pre-periodic domain or a periodic domain. If $U_n = U$ for some $n \in \mathbb{N}$, then $U$ is called periodic domain. For details, we refer the reader for instance, to [**8, 9**].

Consider the functions

$$f(z) = z + 1 + \frac{1}{e^z} \text{ and } f(z) = a\sin(z) \text{ ( where } 0 < a < 1 \text{ ).}$$

*The author is thankful to JSPS, and to Kyoto university for the nice hospitality.

Then clearly there are infinitely many curves $\gamma_k$, $(k \in \mathbb{Z})$ such that for $z \in \gamma_k$, $f^n(z) \to \infty$ as $n \to \infty$. This was observed by Fatou [7], and he posed whether this is always true. It is known to be true for certain families of entire functions (see for instance [13]).

The problem of Fatou in a more general setting was studied by Eremenko [6] who defined the escaping set:

$$I(f) := \{z : f^n(z) \to \infty \text{ as } n \to \infty\}$$

and proved that
i) $I(f) \neq \phi$
ii) $J(f) = \partial I(f)$
iii) $\underline{I(f)} \cap J(f) \neq \phi$
iv) $\overline{I(f)}$ has no bounded components.
Eremenko further conjectured that $I(f)$ itself has no bounded components.

Several subsets of $I(f)$ and their properties and their applications have been obtained by various authors. We mention a few of them. A subset of $I(f)$ in which the iterates of a transcendental entire function tend to infinity relatively fast was considered by Bergweiler [3] who defined

$$I_o(f) := \{z \in I(f) : \frac{\log |f^{n+1}(z)|}{\log |f^n(z)|} \to \infty \text{ as } n \to \infty\}$$

and showed that $I_o(f) \neq \phi$ and also $J(f) = \partial I_o(f)$, and used it to prove that if $f$ is non constant and non linear entire function and $g$ is analytic self map of $\mathbb{C} \setminus \{0\}$ and if $e^{f(z)} = g(e^z)$, then $\exp^{-1} J(g) = J(f)$.

Rippon and Stallard [11] considered the set

$$T(f) = \{z \in I(f) : \frac{\log\log |f^n(z)|}{n} \to \infty \text{ as } n \to \infty\}$$

which is described as the set of points which are "zipping towards infinity". They [11] showed that for a transcendental entire function $f$, both $\overline{I_o(f)}$ and $\overline{T(f)}$ have no bounded components.

Another subset of $I(f)$ in which the iterates of a transcendental entire function tend to infinity arbitrarily fast was considered by Bergweiler and Hinkkanen [4] who defined the set

$$A(f) := \{z : \text{ there exists } L \in \mathbb{N} \text{ such that } |f^n(z)| > M(R, f^{n-L}) \text{ for } n > L\}$$

where $M(R, f) = \max_{|z|=R} |f(z)|$ and $R$ is any value such that $R > \min_{z \in J(f)} |z|$, and proved that

i) $A(f) \neq \phi$
ii) $J(f) = \partial A(f)$
iii) $\underline{A(f)} \cap J(f) \neq \phi$
iv) $\overline{A(f)} = J(f)$ if $f$ does not have wandering domains.

The properties of $A(f)$ was also utilized by Bergweiler and Hinkkanen [**4**] to prove that if $f$ and $g$ are two transcendental entire functions with $f \circ g = g \circ f$, and further if both $f$ and $g$ have no wandering domains, then $J(f) = J(g)$. Thus this gave a partial answer in affirmative to the following open question.

**Question.** Let $f$ and $g$ be two permutable transcendental entire functions, i.e. transcendental entire functions satisfying $f \circ g = g \circ f$. Then is $J(f) = J(g)$?

Note that if $f$ and $g$ are rational functions with $f \circ g = g \circ f$, then $J(f) = J(g)$ is a well known result.

Properties of escaping sets of permutable transcendental entire functions have also been studied by Wang and Yang [**16**]. They proved the following.

**Theorem.** ([16]) Let $f$ and $g$ be two distinct permutable transcendental entire functions and $q(z)$ be a nonconstant polynomial. Suppose that $q(g) = aq(f) + b, a(\neq 0), b \in \mathbb{C}$. Then the following conclusions hold:

i) $I_o(f) = I_o(g)$
ii) $T(f) = T(g)$
iii) $J(f) = J(g)$.

**Theorem.** ([16]) Let $f$ and $g$ be two distinct permutable transcendental entire functions and $q(z)$ be a non-constant polynomial. Suppose $q(g) = aq(f) + b, a(\neq 0), b \in \mathbb{C}$. Then

i) if $g(z)$ has atleast one fix-point, then $A(f) \subset A(g)$
ii) $A(f^2) = A(g^2)$
iii) $A(f) \subset A(g)$ or $A(g) \subset A(f)$.

They further conjectured that if $f$ and $g$ satisfy the conditions of the above theorem then $A(f) = A(g)$.

Wang and Yang [**17**] also investigated when a Fatou component $D$ is contained in $I(f), I_o(f), T(f)$ or $A(f)$. They showed that if $D$ is a Baker domain of $F(f)$, then $D \subset I(f), D \cap I_o(f) = \phi, D \cap A(f) = \phi$ and also $D \cap T(f) = \phi$. And if $D$ is a multiply connected wandering domain of $F(f)$ then $D \subset I(f), D \subset I_o(f), D \subset A(f)$, and $D \subset T(f)$. Regarding simply connected wandering domain they showed that there exists a transcendental entire function $g$ such that $F(g)$ contains a simply connected infinitely wandering domain ( i.e., a wandering domain $D$ in which $f^n(z) \to \infty$ as $n \to \infty$ for any $z \in D$ ) such that $D \subset I_o(g)$.

An alternate definition for $A(f)$ was given by Rippon and Stallard [**12**] who defined

$$B(f) := \{z : \text{ there exists } L \in \mathbb{N} \text{ such that} f^{n+L}(z) \notin \widetilde{f^n(D)}, n \in \mathbb{N}\}$$

where $D$ is an open disk meeting Julia set of $f$ and $\widetilde{U}$ denotes the union of $U$ and its bounded complementary components, and proved the following theorem.

**Theorem.** ([12]) Let $f$ be a transcendental entire function. Let $B(f)$ be as defined above. Then
(i) $B(f)$ is independent of $D$
(ii) $B(f)$ is completely invariant
(iii) $B(f^p) = B(f), p = 0, 1, ...$
(iv) if $g = h^{-1} \circ f \circ h$ where $h(z) = az + b, a \neq 0$, then $B(f) = h(B(g))$
(v) $B(f) = A(f)$.

Note that the results (iii) and (v) immediately give an affirmative answer to the conjecture of Yang and Wang mentioned above. The concept of $B(f)$ was utilized by Rippon and Stallard to prove also the following:

**Theorem.** ([12]) Let $f$ be a transcendental entire function. Then each $z_o \in A(f)$ lies in an unbounded closed connected subset of $A(f)$. In particular $A(f)$ has no bounded components.

Rippon and Stallard [**12**] also gave a positive answer to the conjecture of Eremenko atleast when $F(f)$ has a multiply connected Fatou component by proving:

**Theorem.** ([12]) Let $f$ be a transcendental entire function and suppose that $F(f)$ has a multiply connected component. Then

(i) $A(f)$ is connected and unbounded, and contains the closure of every multiply connected component of $F(f)$

(ii) $I(f)$ is connected and unbounded.

If $f$ and $g$ are transcendental entire functions, then so are $f \circ g$ and $g \circ f$, and the dynamics of $f \circ g$ many times help in understanding the dynamics of $g \circ f$ and vice-versa. For instance Bergweiler and Wang [**5**] and also independently Poon and Yang [**10**] proved that if $f$ and $g$ are transcendental entire functions then $f \circ g$ has no wandering domains if and only if $g \circ f$ has no wandering domains. Bergweiler and Wang [**5**] also proved that if $f$ and $g$ are non linear entire functions, then $z \in J(g \circ f)$ if and only if $f(z) \in J(f \circ g)$. This gives an immediate solution to the following proposition.

**Proposition 1.** Let $f$ and $g$ be nonlinear entire functions such that $J(f \circ g) = \mathbb{C}$. Then $J(g \circ f) = \mathbb{C}$.

**Proof.** If $J(g \circ f) \neq \mathbb{C}$ then there exists a $w \in F(g \circ f)$ which implies from the above, that $f(w) \in F(f \circ g)$, hence $J(f \circ g) \neq \mathbb{C}$ which proves the proposition.

One would also be interested to know whether similar results hold for escaping sets also. Also what other results can one get regarding the escaping sets of composition of entire functions. The author and Taniguchi have worked on this aspect in [**14**]. We start with an elementary observation.

Note that there exist transcendental entire functions whose Julia set is $\mathbb{C}$. However there does not exist any transcendental entire function $f$ such that $B(f) = \mathbb{C} \setminus A$, where $A$ is empty set or a finite set. For if such $f$ exists then $\partial B(f) = \phi$ or $\partial B(f)$ is a finite set, contradicting $J(f)$ is non empty and has infinitely many points. If $B(h) = \mathbb{C} \setminus A$ where $A$ is an infinite set and $h$ is a composite transcendental entire function, we have the following result.

**Theorem 1.** ([14]) Let $f$ and $g$ be transcendental entire functions. Let $B(g \circ f) = \mathbb{C} \setminus A$ and $B(f \circ g) = \mathbb{C} \setminus E$ where $A$ and $E$ are infinite sets. Then

$$E \subset f(A) \cup \{\alpha\} \text{ and } A \subset g(E) \cup \{\beta\}$$

where $\alpha$ and $\beta$ are Picard exceptional values of $f$ and $g$ respectively.

**Corollary.** Let $f$ and $g$ satisfy the conditions of Theorem 1. Then $A$ and $g(E)$ differ by atmost two points.

**Proof of Corollary.** By Theorem 1,

$$E \subset f(A) \cup \{\alpha\} \text{ and } A \subset g(E) \cup \{\beta\}$$

and so by complete invariance of $B(g \circ f)$ it follows that

$$g(E) \subset g(f(A)) \cup \{g(\alpha)\} \subset A \cup \{g(\alpha)\} \subset g(E) \cup \{\beta\} \cup \{g(\alpha)\}$$

And so

$$g(E) \cup \{\beta\} \cup \{g(\alpha)\} = A \cup \{\beta\} \cup \{g(\alpha)\}.$$

A general relation between $B(g \circ f)$ and $B(f \circ g)$ is the following.

**Theorem 2.** ([14]) Let $f$ and $g$ be transcendental entire functions. Then

$$g(B(f \circ g)) = B(g \circ f)$$

except possibly for two points.

With regard to the composition and its factors, we have:

**Theorem 3.** ([14]) Let $f$ and $g$ be transcendental entire functions with $f \circ g = g \circ f$. Then

$$B(f \circ g) \subset B(f) \cap B(g).$$

For our next result, we need the concept of order and lower order. An entire function $f$ is said to be of order $\rho$ and lower order $\mu$ respectively if

$$\rho = \limsup_{r \to \infty} \frac{\log \log M(r, f)}{\log r}$$

and

$$\mu = \liminf_{r \to \infty} \frac{\log \log M(r, f)}{\log r}.$$

Baker [**1**] proved that if $\rho = 0$ with sufficiently small growth condition then $F(f)$ has no unbounded Fatou component. Infact he showed that if $f$ is an entire function with

$$\log M(r, f) = O\{(\log r)^t\}$$

as $r \to \infty$ where $1 < t < 3$, then every component of $F(f)$ is bounded, and further conjectured that every entire function of order $\rho < \frac{1}{2}$ will have only bounded Fatou components.

Stallard [**15**] improved the result of Baker and proved that for a transcendental entire function $f$, if for some $\epsilon \in (0, 1)$,

$$\log\log M(r, f) < \frac{(\log r)^{\frac{1}{2}}}{(\log\log r)^{\epsilon}}$$

for large values of $r$, then every component of $F(f)$ is bounded.

By imposing a condition on the regularity of the growth, Stallard [**15**] also proved that if a transcendental entire function $f$ of order $\rho < \frac{1}{2}$ is such that

$$\frac{\log M(2r, f)}{\log M(r, f)} \to c$$

as $r \to \infty$ where $c$ is a finite constant that depends only on $f$, then every component of $F(f)$ is bounded.

Results on the boundedness of the Fatou components was also obtained by Wang [**18**] who proved that if $f$ is an entire function of order $\rho < \frac{1}{2}$ and if its lower order $\mu > 0$, then every component of $F(f)$ is bounded.

Thus the case with $\mu = 0$ remains open. Recently the author and Taniguchi [**14**] have further improved the result of Wang. We defined a new class of entire functions as follows. For $k \geq 1$, let $\mathcal{F}_k$ be the set of transcendental entire functions $f$ such that $\log\log M(r, f) \geq (\log r)^{1/k}$ for every sufficiently large $r$. Let $\mathcal{F} = \cup_{k\geq 1}\mathcal{F}_k$. We proved the following.

**Theorem 4.** ([14]) Let $f \in \mathcal{F}$ have an order $\rho < \frac{1}{2}$. Then every component of $F(f)$ is bounded.

Thus we observe that the result mentioned in Wang hold even for functions of zero lower order, however with some extra conditions. The general case however is still open.

As an application of the above theorem we have proved

**Theorem 5.** ([14]) Let $f$ be a transcendental entire function in $\mathcal{F}$. Let $C$ be any component of $B(f)$. Then

(i) $C \cap J(f) \neq \phi$

(ii) every $z_o$ in $C \cap F(f)$ lies in some wandering component of the Fatou set of $f$.

We have also shown the following .

**Theorem 6.** ([14]) Let $f$ be a transcendental entire function with

$$\limsup_{r\to\infty} \frac{m(r,f)}{r} = \infty$$

where $m(r,f) = \min_{|z|=r} |f(z)|$. Let C be any component of $B(f)$. Then

(i) $C \cap J(f) \neq \phi$

(ii) every $z_o$ in $C \cap F(f)$ lies in some wandering component of the Fatou set of $f$.

The proof Theorem 6 uses the theorem of Zheng [**19**] which states that if $f$ is a transcendental entire function with $\limsup_{r\to\infty} \frac{m(r,f)}{r} = \infty$, then $F(f)$ has no unbounded pre-periodic or periodic component.

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