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Simultaneous linearization of hyperbolic and parabolic fixed points

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1 Statement of the result

This note is a summary of the preprint [8]. We will show that the Fatou coordinates (the solution to Abel equation) for a parabolic fixed point of holomorphic map of one variable can be obtained as a modified limit of the solution to Schröder equation for the perturbed hyperbolic maps. (An alternative proof is given by Kawahira [4].)

Let \( \{f_\tau\}_\tau \) be a family, depending on the parameter \( \tau \), of holomorphic maps of the form

\[
f_\tau(z) = \tau z + 1 + \frac{a_1(\tau)}{z} + \frac{a_2(\tau)}{z^2} + \cdots
\]

defined in a neighborhood of \( \infty \) of the Riemann sphere \( \hat{\mathbb{C}} \).

For each \( \tau \) with \( |\tau| > 1 \), we have a unique analytic function \( \chi_\tau(z) \) in a neighborhood of \( \infty \) satisfying the Schröder equation

\[
\chi_\tau(f_\tau(z)) = \tau \chi_\tau(z)
\]

and normalized so that

\[
\lim_{z \to \infty} \frac{\chi_\tau(z)}{z} = 1.
\]

We will show that, when \( \tau \) tends to 1 non-tangentially within the domain \( |\tau| > 1 \), the sequence

\[
\chi_\tau(z) - \frac{1}{\tau - 1} - a_1(\tau) \log(\tau - 1)
\]

converges to a solution to the Abel equation \( \varphi(z) \varphi(f_1(z)) = \varphi(z) + 1 \), on a half plane \( \{\text{Re} \ z > R\} \) with sufficiently large \( R \).
2 A family of linear maps

We begin with studying the family \( \{\ell_{\tau}\}_{\tau} \) of linear maps
\[
\ell_{\tau}(z) = \tau z + 1
\] (1)
on the Riemann sphere \( \hat{\mathbb{C}} \) with a fixed point at \( \infty \).

We will investigate the uniformity, with respect to the parameter \( \tau \), of convergence of the sequence of the iterates \( \{f_{\tau}^{n}\}_{n=1}^{\infty} \). Here, the parameter will be restricted in the closed sector
\[
T_{\alpha} = \{\tau \in \mathbb{C} \mid \text{Re } \tau - 1 \geq |\tau - 1| \cos \alpha\}
\]
where \( \alpha \) is a real number with \( 0 < \alpha < \pi/2 \).

To measure the rate of convergence to \( \infty \), we define a function \( N : \hat{\mathbb{C}} \times T_{\alpha} - \{(\infty, 1)\} \rightarrow \mathbb{R} \cup \{\infty\} \) as follows.
\[
N_{r}(z) = \left| z - \frac{1}{1 - \tau} \right| - \left| \frac{1}{1 - \tau} \right|
\]
for \( (z, \tau) \in \hat{\mathbb{C}} \times (T_{\alpha} - \{1\}) \); \( N_{1}(z) = \sup_{|\theta| \leq \alpha} \text{Re}(e^{i\theta}z) \) for \( z \in \mathbb{C} \).

We will not define \( N_{1}(\infty) \).

As is easily shown, \( N_{r}(z) \) is upper semi-continuous and
\[
N_{1}(z) = \lim_{T \ni \tau \rightarrow 1} \sup_{\tau} N_{r}(z).
\]
Further the inequality
\[
|N_{r}(z) - N_{r}(w)| \leq |z - w|\quad z, w \in \mathbb{C}, \tau \in T_{\alpha}
\]
and, in particular,
\[
N_{r}(z) \leq |z|\quad z \in \mathbb{C}, \tau \in T_{\alpha}
\]
hold.

For a real number \( R \), let
\[
\mathcal{V}_{\alpha}(R) = \{(z, \tau) \in \hat{\mathbb{C}} \times T_{\alpha} - \{(\infty, 1)\} \mid N_{r}(z) > R\}.
\]
We note that \( \mathcal{V}_{\alpha}(R) \) is not open. Slices of \( \mathcal{V}_{\alpha}(R) \) by \( \tau = \text{const.} \) are open sets given by
\[
\mathcal{V}_{r}(R) = \{z \in \hat{\mathbb{C}} \mid N_{r}(z) > R\} \quad (\tau \neq 1);
\]
\[
\mathcal{V}_{1}(R) = \{z \in \mathbb{C} \mid N_{1}(z) > R\} = \bigcup_{|\theta| \leq \alpha} \{\text{Re}(e^{i\theta}z) > 0\}.
\]
Lemma 2.1 For \((z, \tau) \in \hat{\mathbb{C}} \times T_{\alpha} - \{ (\infty, 1) \},\) we have

\[ N_{\tau}(\ell_{\tau}(z)) \geq |\tau| N_{\tau}(z) + \cos \alpha. \]

If \(N_{\tau}(z) > 0,\) we have \(N_{\tau}(\ell_{\tau}(z)) \geq N_{\tau}(z) + \cos \alpha.\) So we have the following.

Proposition 2.2 The sequence \(\{ \ell_{\tau}^{n}(z) \}_{n}\) converges to \(\infty\) as \(n \to \infty\) uniformly on the set \(\mathcal{V}_{\alpha}(0)\).

3 Families of maps with attracting/parabolic fixed points
— Domain of convergence

Now we consider a family of holomorphic maps \(f_{\tau} : U \to \hat{\mathbb{C}}\) of the form

\[ f_{\tau}(z) = \tau z + 1 + \frac{a_{1}(\tau)}{z} + \frac{a_{2}(\tau)}{z^{2}} + \cdots. \]

(2)

defined on a neighborhood

\[ U = \{ z \in \hat{\mathbb{C}} \mid R < |z| \leq \infty \} \]

of \(\infty \in \hat{\mathbb{C}}.\) We suppose that \(f\) depends holomorphically on \(\tau \in \Delta_{\rho}(1) = \{ \tau \in \mathbb{C} \mid |\tau - 1| < \rho \}.\) Let

\[ A_{\tau}(z) = \frac{a_{1}(\tau)}{z} + \frac{a_{2}(\tau)}{z^{2}} + \cdots. \]

As in the previous section, we choose and fix \(\alpha\) so that \(0 < \alpha < \pi/2\) and let \(\delta = \frac{1}{2} \cos \alpha.\) By shrinking the neighborhoods \(U\) and \(W,\) we assume that there is a constant \(K_{1}\) such

\[ |A_{\tau}(z)| < \frac{K_{1}}{|z|} < \delta \]

(3)

for \((z, \tau) \in U \times W.\) Further we assume that \(f_{\tau}(z)\) is injective in \(z\) for every \(\tau \in \Delta_{\rho}(1)\)

Since \(f_{\tau}(z)\) are approximated by linear maps \(\ell_{\tau}(z),\) we have a result concerning the uniformity of convergence of \(\{ f_{\tau}^{n}(z) \}.\) Let \(T_{\alpha,\rho} = T_{\alpha} \cap \Delta_{\rho}(1).\)

Lemma 3.1 For \((z, \tau) \in U \times T_{\alpha,\rho}\) we have

\[ N_{\tau}(f_{\tau}(z)) \geq |\tau| N_{\tau}(z) + \delta. \]

Now let \(\mathcal{V} = \mathcal{V}_{\alpha,\rho}(R) = \{ (z, \tau) \in \mathcal{V}_{\alpha}(R) \mid \tau \in T_{\alpha,\rho} \}.\)

Proposition 3.2 If \((z, \tau) \in \mathcal{V},\) then \((f_{\tau}(z), \tau) \in \mathcal{V}.\) The sequence \(\{ f_{\tau}^{n}(z) \}_{n}\) converges uniformly on \(\mathcal{V}\) to \(\infty\) as \(n \to \infty.\)
4 Schröder-Abel equation — special case

Here we consider the special case where the coefficient \( a_1(\tau) \) in (2) vanishes identically.

**Theorem 4.1** There exists a function \( \varphi_\tau(z) \) continuous on \( \mathcal{V} \) such that
(i) \( \varphi_\tau(z) \) satisfies the functional equation
\[
\varphi_\tau(f_\tau(z)) = \tau \varphi_\tau(z) + 1;
\]
(ii) \( \varphi_\tau(z) \) is injective in the variable \( z \) for each parameter \( \tau \in T_{\alpha,r} \).
(iii) \( \lim_{z \to \infty} \varphi_\tau(z)/z = 1 \) as \( z \to \infty \), when \( |\tau| > 1 \).

In fact \( \varphi_\tau(z) \) is given by
\[
\varphi_\tau(z) = \lim_{n \to \infty} \left\{ \frac{1}{\tau^n} f^n(z) - \sum_{k=1}^{n} \frac{1}{\tau^k} \right\}.
\]  

In the case where \( a_1(\tau) \) does not identically vanish, the expression in (5) is not convergent. So we have to modify (5) in order to yield convergence. For this purpose, we will introduce a function satisfying a difference equation in the next section.

5 Solution to a difference equation

We consider the difference equation
\[
h_\tau(\ell_\tau(z)) - \tau h_\tau(z) = \frac{1}{z} + C_\tau.
\]
where \( \ell_\tau(z) = \tau z + 1 \) with \( |\tau| > 1 \) or \( \tau = 1 \); and \( C_\tau \) is a constant depending on \( \tau \), which will be given later.

A solution to this equation is given by
\[
h_\tau(z) = -\frac{1}{\tau z} + \sum_{n=1}^{\infty} \frac{1}{\tau^{n+1}} \left\{ \frac{1}{\ell^n_\tau(0)} - \frac{1}{\ell^n_\tau(z)} \right\}.
\]  

**Proposition 5.1** The function \( h_\tau(z) \) is continuous on \( \mathcal{V}_\alpha(0) \).

For a fixed \( \tau \) with \( |\tau| > 1 \), the function \( h_\tau(z) \) is meromorphic on \( \hat{\mathbb{C}} \) except the essential singularity at \( 1/(1-\tau) \), and has poles at \( (1-\tau^{-n})/(1-\tau) \), \( (n = 0, 1, 2, \ldots) \). This function \( h_\tau(z) \) is holomorphic at \( \infty \) and we write
\[
H_\tau = h_\tau(\infty) = \sum_{n=1}^{\infty} \frac{1}{\tau^{n+1} \ell^n_\tau(0)}.
\]
For \( \tau = 1 \), we have \( \ell^n(z) = z + n \) and

\[
  h_1(z) = -\frac{1}{z} + \sum_{n=1}^{\infty} \left\{ \frac{1}{n} - \frac{1}{z+n} \right\}.
\]

This function is meromorphic on \( \mathbb{C} \) and has poles at 0, -1, -2, \ldots. We note that

\[
  h_1(z) = \frac{\Gamma'(z)}{\Gamma(z)} + \gamma
\]

where \( \Gamma(z) \) denotes the gamma function and \( \gamma \) denotes the Euler constant

\[
  \gamma = \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{k} - \log n \right).
\]

Now we study the dependence of \( h_\tau(z) \) on the parameter \( \tau \).

**Corollary 1** The constat \( C_\tau \) is a continuous function of \( \tau \in T_\alpha \).

The function \( h_\tau(z) \) satisfies the equation (9) with

\[
  C_\tau = (1 - \tau)H_\tau.
\]

for \( |\tau| > 1 \) and with \( C_1 = 0 \) for \( \tau = 1 \). We have \( C_\tau \to C_1 = 0 \) \((\tau \to 1)\), since \( h_\tau(z) \) is continuous.

**Proposition 5.2** For any \( \varepsilon > 0 \), there is a constant \( M \) such that

\[
  |h'_\tau(z)| \leq \frac{M}{N_\tau(z)} \quad \text{on} \ V_\alpha(\varepsilon)
\]

**6 Behavior of \( H_\tau \)**

Now we look at the behavior of the function \( H_\tau \) defined by (9), when \( \tau \to 1 \) within the sector \( T \). It is clear from the expression (9) that \( H_\tau \) is unbounded, while \( C_\tau = (1 - \tau)H_\tau \) tends to 0 by the corollary to Proposition 2.4. Here we give a more precise description of its behavior.

**Proposition 6.1** We have

\[
  H_\tau = -\log(\tau - 1) + \gamma - 1 + o(1)
\]

as \( \tau \to 1 \) within the sector \( T \). Here \( \gamma \) denotes the Euler constant.
To show this, we write $\lambda = 1/\tau$. We have

$$H_{1/\lambda} = (1 - \lambda)L(\lambda) - \lambda.$$

Here $L(\lambda)$ denotes the Lambert series defined by

$$L(\lambda) = \sum_{n=1}^{\infty} \frac{\lambda^n}{1 - \lambda^n}.$$

This series $L(\lambda)$ defines a holomorphic function on $|\lambda| < 1$, and is developped into the power series

$$L(\lambda) = \sum_{n=1}^{\infty} d(n)\lambda^n = \lambda + 2\lambda^2 + 2\lambda^3 + 3\lambda^4 + \cdots,$$

where $d(n)$ denotes the number of divisors of $n$. Let

$$\frac{L(\lambda)}{1 - \lambda} = \sum_{n=1}^{\infty} D(n)\lambda^n$$

with

$$D(n) = d(1) + \cdots + d(n).$$

The asymptotic behavior of $D(n)$ is given by a theorem of Dirichlet (see Apostol [1], Chandrasekharan [2]):

$$D(n) = n \log n + (2\gamma - 1)n + O(\sqrt{n}) \quad (n \to \infty).$$

Using this estimate, we have

$$\frac{L(\lambda)}{1 - \lambda} = \sum_{n=1}^{\infty} D(n)\lambda^n = -\frac{\lambda \log(1 - \lambda)}{(1 - \lambda)^2} + \frac{\gamma \lambda}{(1 - \lambda)^2} + P(\lambda)$$

where $P(\lambda) = \sum_{n=1}^{\infty} p_n \lambda^n$. From the estimate of $p_n$ we have

$$P(\lambda) = o((1 - \lambda)^{-2}) \quad \text{as } \lambda \to 1 \text{ non-tangentially}$$

Hence it follows that

$$H_{\tau} = -\log(\tau - 1) + \gamma - 1 + o(\tau - 1)$$
7 Schröder-Abel equation — general case

Now we treat the general case where \( a_1(\tau) \) does not necessarily vanish. Let

\[ B_\tau = 1 - a_1(\tau)C_\tau \]

we have the following result corresponding to Theorem 7.

**Theorem 7.1** There exists a function \( \varphi_\tau(z) \) continuous on \( \mathcal{V} \) such that

(i) \( \varphi_\tau(z) \) satisfies the functional equation

\[ \varphi_\tau(f_\tau(z)) = \tau \varphi_\tau(z) + B_\tau; \]  

(ii) \( \varphi_\tau(z) \) is injective in the variable \( z \) for each parameter \( \tau \in T_{\alpha,\rho} \).

(iii) \( \lim_{z \to \infty} \varphi_\tau(z)/z = 1 \) as \( z \to \infty \), when \( |\tau| > 1 \).

To define \( \varphi_\tau(z) \), we let

\[ \Phi_\tau(z) = z - a_1(\tau)h_\tau(z). \]

Then

\[ \Phi_\tau(f_\tau(z)) = \tau \Phi(z) + B_\tau + \tilde{A}(z). \]

From this we can define

\[ \varphi_\tau(z) = \lim_{n \to \infty} \left\{ \frac{1}{\tau^n} \Phi_\tau(f_\tau^n(z)) - B_\tau \sum_{k=1}^{n} \frac{1}{\tau^k} \right\} \]  

8 Relation with the Schröder equation

When \( |\tau| > 1 \), the Schröder equation

\[ \chi_\tau(f_\tau(z)) = \tau \chi_\tau(z). \]

has a unique solution \( \chi_\tau(z) \) of the form

\[ \chi_\tau(z) = z + c_0 + \frac{c_1}{z} + \cdots \]

in a neighbourhood of \( \infty \).

**Theorem 8.1** For \( \tau \in T_{\alpha,\rho} - \{1\} \) we have

\[ \varphi_\tau(z) = \chi_\tau(z) - \frac{B_\tau}{\tau - 1}. \]
Proof We can easily verify that $\varphi(z) + B_{\tau}/(\tau - 1)$ satisfies the Schröder equation. The assertion follows from the uniqueness of the solution. \(\square\)

Now recall that

$$
\frac{B_{\tau}}{\tau - 1} = \frac{1 - a_1 C_{\tau}}{\tau - 1} = \frac{1}{\tau - 1} - a_1 H_{\tau} = \frac{1}{\tau - 1} + a_1 \log(\tau - 1) + a_1 (1 - \gamma) + o(1)
$$

Using this fact the theorem is reformulated as follows:

**Theorem 8.2** Let

$$
\varphi(z) = \chi(z) - \frac{1}{\tau - 1} - a_1 \log(\tau - 1)
$$

for $\tau \in T - \{1\}$. Then $\varphi(z)$ converges to a solution to the Abel equation.

**References**


