数理解析研究所講究録
1494 巻 2006 年 1-8

Simultaneous linearization of hyperbolic and parabolic fixed points

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1 Statement of the result

This note is a summary of the preprint [8]. We will show that the Fatou coordinates (the solution to Abel equation) for a parabolic fixed point of holomorphic map of one variable can be obtained as a modified limit of the solution to Schröder equation for the perturbed hyperbolic maps. (An alternative proof is given by Kawahira [4].)

Let $\{f_\tau\}_\tau$ be a family, depending on the parameter τ, of holomorphic maps of the form

$$f_\tau(z) = \tau z + 1 + \frac{a_1(\tau)}{z} + \frac{a_2(\tau)}{z^2} + \cdots$$

defined in a neighborhood of ∞ of the Riemann sphere $\widehat{\mathbb{C}}$.

For each τ with $|\tau| > 1$, we have a unique analytic function $\chi_\tau(z)$ in a neighborhood of ∞ satisfing the Schröder equation

$$\chi_\tau(f_\tau(z)) = \tau\chi_\tau(z)$$

and normalized so that

$$\lim_{z\to\infty} \frac{\chi_\tau(z)}{z} = 1.$$

We will show that, when τ tends to 1 non-tangentially within the domain $|\tau| > 1$, the sequence

$$\chi_\tau(z) - \frac{1}{\tau - 1} - a_1(\tau)\log(\tau - 1)$$

converges to a solution to the Abel equation $\varphi(z)\ \varphi(f_1(z)) = \varphi(z) + 1$, on a half plane $\{\mathrm{Re}\, z > R\}$ with sufficiently large R.

2 A family of linear maps

We begin with studying the family $\{\ell_\tau\}_\tau$ of linear maps

$$\ell_\tau(z) = \tau z + 1 \tag{1}$$

on the Riemann sphere $\widehat{\mathbb{C}}$ with a fixed point at ∞.

We will investigate the uniformity, with respect to the parameter τ, of convergence of the sequence of the iterates $\{f_\tau^n\}_{n=1}^\infty$. Here, the parameter will be restricted in the closed sector

$$T_\alpha = \{\tau \in \mathbb{C} \mid \operatorname{Re}\tau - 1 \geq |\tau - 1| \cos\alpha\},$$

where α is a real number with $0 < \alpha < \pi/2$.

To measure the rate of convergence to ∞, we define a function $N : \widehat{\mathbb{C}} \times T_\alpha - \{(\infty, 1)\} \to \mathbb{R} \cup \{\infty\}$ as follows.

$$\begin{aligned} N_\tau(z) &= \left| z - \frac{1}{1-\tau} \right| - \left| \frac{1}{1-\tau} \right| && \text{for } (z,\tau) \in \widehat{\mathbb{C}} \times (T_\alpha - \{1\}); \\ N_1(z) &= \sup_{|\theta| \leq \alpha} \operatorname{Re}(e^{i\theta} z) && \text{for } z \in \mathbb{C}. \end{aligned}$$

We will not define $N_1(\infty)$.

As is easily shown, $N_\tau(z)$ is upper semi-continuous and

$$N_1(z) = \limsup_{T \ni \tau \to 1} N_\tau(z).$$

Further the inequality

$$|N_\tau(z) - N_\tau(w)| \leq |z - w| \quad z, w \in \mathbb{C}, \tau \in T_\alpha$$

and, in particular,

$$N_\tau(z) \leq |z|, \quad z \in \mathbb{C}, \tau \in T_\alpha.$$

hold.

For a real number R, let

$$\mathcal{V}_\alpha(R) = \{(z,\tau) \in \widehat{\mathbb{C}} \times T_\alpha - \{(\infty, 1)\} \mid N_\tau(z) > R\}.$$

We note that $\mathcal{V}_\alpha(R)$ is not open. Slices of $\mathcal{V}_\alpha(R)$ by $\tau = $ const. are open sets given by

$$\begin{aligned} V_\tau(R) &= \{z \in \widehat{\mathbb{C}} \mid N_\tau(z) > R\} \qquad (\tau \neq 1); \\ V_1(R) &= \{z \in \mathbb{C} \mid N_1(z) > R\} = \bigcup_{|\theta| \leq \alpha} \{\operatorname{Re}(e^{i\theta} z) > 0\}. \end{aligned}$$

Lemma 2.1 *For* $(z, \tau) \in \widehat{\mathbb{C}} \times T_\alpha - \{(\infty, 1)\}$, *we have*

$$N_\tau(\ell_\tau(z)) \geq |\tau| N_\tau(z) + \cos\alpha.$$

If $N_\tau(z) > 0$, we have $N_\tau(\ell_\tau(z)) \geq N_\tau(z) + \cos\alpha$. So we have the following.

Proposition 2.2 *The sequence* $\{\ell_\tau^n(z)\}_n$ *converges to* ∞ *as* $n \to \infty$ *uniformly on the set* $\mathcal{V}_\alpha(0)$.

3 Families of maps with attracting/parabolic fixed points — Domain of convergence

Now we consider a family of holomorphic maps $f_\tau : U \to \widehat{\mathbb{C}}$ of the form

$$f_\tau(z) = \tau z + 1 + \frac{a_1(\tau)}{z} + \frac{a_2(\tau)}{z^2} + \cdots . \tag{2}$$

defined on a neighborhood

$$U = \{z \in \widehat{\mathbb{C}} \mid R < |z| \leq \infty\}$$

of $\infty \in \widehat{\mathbb{C}}$. We suppose that f depends holomorphically on $\tau \in \Delta_\rho(1) = \{\tau \in \mathbb{C} \mid |\tau - 1| < \rho\}$. Let

$$A_\tau(z) = \frac{a_1(\tau)}{z} + \frac{a_2(\tau)}{z^2} + \cdots .$$

As in the previous section, we choose and fix α so that $0 < \alpha < \pi/2$ and let $\delta = \frac{1}{2}\cos\alpha$. By shrinking the neighbohoods U and W, we assume that there is a constant K_1 such

$$|A_\tau(z)| < \frac{K_1}{|z|} < \delta \tag{3}$$

for $(z, \tau) \in U \times W$. Further we assume that $f_\tau(z)$ is injective in z for every $\tau \in \Delta_\rho(1)$

Since $f_\tau(z)$ are approximated by linear maps $\ell_\tau(z)$, we have a result concerning the uniformity of convergence of $\{f_\tau^n(z)\}$. Let $T_{\alpha,\rho} = T_\alpha \cap \Delta_\rho(1)$.

Lemma 3.1 *For* $(z, \tau) \in U \times T_{\alpha,\rho}$ *we have*

$$N_\tau(f_\tau(z)) \geq |\tau| N_\tau(z) + \delta.$$

Now let $\mathcal{V} = \mathcal{V}_{\alpha,\rho}(R) = \{(z, \tau) \in \mathcal{V}_\alpha(R) \mid \tau \in T_{\alpha,\rho}\}$.

Proposition 3.2 *If* $(z, \tau) \in \mathcal{V}$, *then* $(f_\tau(z), \tau) \in \mathcal{V}$. *The sequence* $\{f_\tau^n(z)\}_n$ *converges uniformly on* $\mathcal{V}$ *to* ∞ *as* $n \to \infty$.

4 Schröder-Abel equation — special case

Here we consider the special case where the coefficient $a_1(\tau)$ in (2) vanishes identically.

Theorem 4.1 *There exists a function $\varphi_\tau(z)$ continuous on $\mathcal{V}$ such that*
(i) $\varphi_\tau(z)$ satisfies the functional equation

$$\varphi_\tau(f_\tau(z)) = \tau\varphi_\tau(z) + 1; \tag{4}$$

(ii) $\varphi_\tau(z)$ is injective in the variable z for each parameter $\tau \in T_{\alpha,r}$.
(iii) $\lim_{z\to\infty} \varphi_\tau(z)/z = 1$ as $z \to \infty$, when $|\tau| > 1$.

In fact $\varphi_\tau(z)$ is given by

$$\varphi_\tau(z) = \lim_{n\to\infty} \left\{ \frac{1}{\tau^n} f^n(z) - \sum_{k=1}^{n} \frac{1}{\tau^k} \right\} \tag{5}$$

In the case where $a_1(\tau)$ does not identically vanish, the expression in (5) is not convergent. So we have to modify (5) in order to yield convergence. For this purpose, we will introduce a function satisfying a difference equation in the next section.

5 Solution to a difference equation

We consider the difference equation

$$h_\tau(\ell_\tau(z)) - \tau h_\tau(z) = \frac{1}{z} + C_\tau. \tag{6}$$

where $\ell_\tau(z) = \tau z + 1$ with $|\tau| > 1$ or $\tau = 1$; and C_τ is a constant depending on τ, which will be given later.

A solution to this equation is given by

$$h_\tau(z) = -\frac{1}{\tau z} + \sum_{n=1}^{\infty} \frac{1}{\tau^{n+1}} \left\{ \frac{1}{\ell_\tau^n(0)} - \frac{1}{\ell_\tau^n(z)} \right\}. \tag{7}$$

Proposition 5.1 *The function $h_\tau(z)$ is continuous on $\mathcal{V}_\alpha(0)$.*

For a fixed τ with $|\tau| > 1$, the function $h_\tau(z)$ is meromorphic on $\widehat{\mathbb{C}}$ except the essential singularity at $1/(1-\tau)$, and has poles at $(1-\tau^{-n})/(1-\tau)$, $(n = 0, 1, 2, \ldots)$. This function $h_\tau(z)$ is holomorphic at ∞ and we write

$$H_\tau = h_\tau(\infty) = \sum_{n=1}^{\infty} \frac{1}{\tau^{n+1}\ell_\tau^n(0)}. \tag{8}$$

For $\tau = 1$, we have $\ell^n(z) = z + n$ and

$$h_1(z) = -\frac{1}{z} + \sum_{n=1}^{\infty} \left\{ \frac{1}{n} - \frac{1}{z+n} \right\}.$$

This function is meromorphic on $\mathbb{C}$ and has poles at $0, -1, -2, \ldots$. We note that

$$h_1(z) = \frac{\Gamma'(z)}{\Gamma(z)} + \gamma$$

where $\Gamma(z)$ denotes the gamma function and γ denotes the Euler constant

$$\gamma = \lim_{n \to \infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log n \right).$$

Now we study the dependence of $h_\tau(z)$ on the parameter τ.

Corollary 1 *The constat C_τ is a continuous function of $\tau \in T_\alpha$.*

The function $h_\tau(z)$ satisfies the equation () with

$$C_\tau = (1 - \tau) H_\tau. \tag{9}$$

for $|\tau| > 1$ and with $C_1 = 0$ for $\tau = 1$. We have $C_\tau \to C_1 = 0$ $(\tau \to 1)$, since $h_\tau(z)$ is continuous.

Proposition 5.2 *For any $\varepsilon > 0$, there is a constant M such that*

$$|h'_\tau(z)| \leq \frac{M}{N_\tau(z)} \quad \text{on } \mathcal{V}_\alpha(\varepsilon)$$

6 Behavior of H_τ

Now we look at the behavior of the function H_τ defined by (), when $\tau \to 1$ within the sector T. It is clear from the expression () that H_τ is unbounded, while $C_\tau = (1-\tau)H_\tau$ tends to 0 by the corollary to Proposition 2.4. Here we give a more precise description of its behavior.

Proposition 6.1 *We have*

$$H_\tau = -\log(\tau - 1) + \gamma - 1 + o(1)$$

as $\tau \to 1$ within the sector T Here γ denotes the Euler constant.

To show this, we write $\lambda = 1/\tau$. We have

$$H_{1/\lambda} = (1-\lambda)L(\lambda) - \lambda.$$

Here $L(\lambda)$ denotes the Lambert series defined by

$$L(\lambda) = \sum_{n=1}^{\infty} \frac{\lambda^n}{1-\lambda^n}.$$

This series $L(\lambda)$ defines a holomorphic function on $|\lambda| < 1$, and is developped into the power series

$$L(\lambda) = \sum_{n=1}^{\infty} d(n)\lambda^n = \lambda + 2\lambda^2 + 2\lambda^3 + 3\lambda^4 + \cdots,$$

where $d(n)$ denotes the number of divisors of n. Let

$$\frac{L(\lambda)}{1-\lambda} = \sum_{n=1}^{\infty} D(n)\lambda^n$$

with

$$D(n) = d(1) + \cdots + d(n).$$

The asymptotic behavior of $D(n)$ is given by a theorem of Dirichlet (see Apostol [1], Chandrasekharan [2]) :

$$D(n) = n\log n + (2\gamma - 1)n + O(\sqrt{n}) \quad (n \to \infty).$$

Using this estimate, we have

$$\frac{L(\lambda)}{1-\lambda} = \sum_{n=1}^{\infty} D(n)\lambda^n = -\frac{\lambda\log(1-\lambda)}{(1-\lambda)^2} + \frac{\gamma\lambda}{(1-\lambda)^2} + P(\lambda)$$

where $P(\lambda) = \sum_{n=1}^{\infty} p_n\lambda^n$. From the estimate of p_n we have

$$P(\lambda) = o((1-\lambda)^{-2}) \qquad \text{as } \lambda \to 1 \text{ non-tangentially}$$

Hence it follows that

$$H_\tau = -\log(\tau - 1) + \gamma - 1 + o(\tau - 1)$$

7 Schröder-Abel equation — general case

Now we treat the general case where $a_1(\tau)$ does not necessarily vanish. Let

$$B_\tau = 1 - a_(\tau)C_\tau$$

we have the following result corresponding to Theorem ?

Theorem 7.1 *There exists a function $\varphi_\tau(z)$ continuous on $\mathcal{V}$ such that*
(i) $\varphi_\tau(z)$ satisfies the functional equation

$$\varphi_\tau(f_\tau(z)) = \tau\varphi_\tau(z) + B_\tau; \tag{10}$$

(ii) $\varphi_\tau(z)$ is injective in the variable z for each parameter $\tau \in T_{\alpha,\tau}$.
(iii) $\lim_{z\to\infty} \varphi_\tau(z)/z = 1$ as $z \to \infty$, when $|\tau| > 1$.

To define $\varphi_\tau(z)$, we let

$$\Phi_\tau(z) = z - a_1(\tau)h_\tau(z).$$

Then

$$\Phi_\tau(f_\tau(z)) = \tau\Phi(z) + B_\tau + \tilde{A}(z).$$

From this we can define

$$\varphi_\tau(z) = \lim_{n\to\infty}\left\{\frac{1}{\tau^n}\Phi_\tau(f_\tau^n(z)) - B_\tau\sum_{k=1}^{n}\frac{1}{\tau^k}\right\} \tag{11}$$

8 Relation with the Schröder equation

When $|\tau| > 1$, the Schröder equation

$$\chi_\tau(f_\tau(z)) = \tau\chi_\tau(z).$$

has a unique solution $\chi_\tau(z)$ of the form

$$\chi_\tau(z) = z + c_0 + \frac{c_1}{z} + \cdots$$

in a neighbouhood of ∞.

Theorem 8.1 *For $\tau \in T_{\alpha,\rho} - \{1\}$ we have*

$$\varphi_\tau(z) = \chi_\tau(z) - \frac{B_\tau}{\tau - 1}.$$

Proof We can easily verify that $\varphi(z) + B_\tau/(\tau - 1)$ satisfies the Schröder equation. The assertion follows from the uniqueness of the solution. $\square$

Now recall that

$$\begin{aligned}\frac{B_\tau}{\tau - 1} &= \frac{1 - a_1 C_\tau}{\tau - 1} \\ &= \frac{1}{\tau - 1} - a_1 H_\tau \\ &= \frac{1}{\tau - 1} + a_1 \log(\tau - 1) + a_1(1 - \gamma) + o(1)\end{aligned}$$

Using this fact the theorem is reformulated as follows:

Theorem 8.2 *Let*

$$\varphi(z) = \chi(z) - \frac{1}{\tau - 1} - a_1 \log(\tau - 1)$$

for $\tau \in T - \{1\}$. *Then* $\varphi(z)$ *converges to a solution to the Abel equation.*

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