Simultaneous linearization of hyperbolic and parabolic fixed points

Tetsuo Ueda (上田 哲生)
Kyoto University

1 Statement of the result

This note is a summary of the preprint [8]. We will show that the Fatou coordinates (the solution to Abel equation) for a parabolic fixed point of holomorphic map of one variable can be obtained as a modified limit of the solution to Schröder equation for the perturbed hyperbolic maps. (An alternative proof is given by Kawahira [4].)

Let \( \{f_\tau\}_\tau \) be a family, depending on the parameter \( \tau \), of holomorphic maps of the form

\[
f_\tau(z) = \tau z + 1 + \frac{a_1(\tau)}{z} + \frac{a_2(\tau)}{z^2} + \cdots
\]
defined in a neighborhood of \( \infty \) of the Riemann sphere \( \hat{\mathbb{C}} \).

For each \( \tau \) with \( |\tau| > 1 \), we have a unique analytic function \( \chi_\tau(z) \) in a neighborhood of \( \infty \) satisfying the Schröder equation

\[
\chi_\tau(f_\tau(z)) = \tau \chi_\tau(z)
\]
and normalized so that

\[
\lim_{z \to \infty} \frac{\chi_\tau(z)}{z} = 1.
\]

We will show that, when \( \tau \) tends to 1 non-tangentially within the domain \( |\tau| > 1 \), the sequence

\[
\chi_\tau(z) - \frac{1}{\tau - 1} - a_1(\tau) \log(\tau - 1)
\]
converges to a solution to the Abel equation \( \varphi(z) \varphi(f_1(z)) = \varphi(z) + 1 \), on a half plane \( \{\Re z > R\} \) with sufficiently large \( R \).
2 A family of linear maps

We begin with studying the family $\{\ell_{\tau}\}_{\tau}$ of linear maps
\[ \ell_{\tau}(z) = \tau z + 1 \quad (1) \]
on the Riemann sphere $\hat{\mathbb{C}}$ with a fixed point at $\infty$.

We will investigate the uniformity, with respect to the parameter $\tau$, of convergence of the sequence of the iterates $\{f_{\tau}^{n}\}_{n=1}^{\infty}$. Here, the parameter will be restricted in the closed sector
\[ T_{\alpha} = \{\tau \in \mathbb{C} \mid \text{Re}\tau - 1 \geq |\tau - 1| \cos \alpha\}, \]
where $\alpha$ is a real number with $0 < \alpha < \pi/2$.

To measure the rate of convergence to $\infty$, we define a function $N : \hat{\mathbb{C}} \times T_{\alpha} - \{(\infty, 1)\} \rightarrow \mathbb{R} \cup \{\infty\}$ as follows.
\[
N_{\tau}(z) = \left| z - \frac{1}{1 - \tau} \right| - \left| \frac{1}{1 - \tau} \right| \quad \text{for} \ (z, \tau) \in \hat{\mathbb{C}} \times (T_{\alpha} - \{1\});
\]
\[
N_{1}(z) = \sup_{|\theta| \leq \alpha} \text{Re}(e^{*\theta}z) \quad \text{for} \ z \in \mathbb{C}.
\]

We will not define $N_{1}(\infty)$.

As is easily shown, $N_{\tau}(z)$ is upper semi-continuous and
\[ N_{1}(z) = \lim_{T \ni \tau \to 1} \sup \ N_{\tau}(z). \]

Further the inequality
\[ |N_{\tau}(z) - N_{\tau}(w)| \leq |z - w| \quad z, w \in \mathbb{C}, \tau \in T_{\alpha} \]
and, in particular,
\[ N_{\tau}(z) \leq |z|, \quad z \in \mathbb{C}, \tau \in T_{\alpha}. \]

hold.

For a real number $R$, let
\[ V_{\alpha}(R) = \{(z, \tau) \in \hat{\mathbb{C}} \times T_{\alpha} - \{(\infty, 1)\} \mid N_{\tau}(z) > R\}. \]

We note that $V_{\alpha}(R)$ is not open. Slices of $V_{\alpha}(R)$ by $\tau = \text{const.}$ are open sets given by
\[
V_{\tau}(R) = \{z \in \hat{\mathbb{C}} \mid N_{\tau}(z) > R\} \quad (\tau \neq 1);
\]
\[
V_{1}(R) = \{z \in \mathbb{C} \mid N_{1}(z) > R\} = \bigcup_{|\theta| \leq \alpha} \{\text{Re}(e^{\theta}z) > 0\}. \]
Lemma 2.1 For \((z, \tau) \in \mathbb{C} \times T_{\alpha} - \{(\infty, 1)\}\), we have
\[
N_{\tau}(\ell_{\tau}(z)) \geq |\tau|N_{\tau}(z) + \cos \alpha.
\]

If \(N_{\tau}(z) > 0\), we have \(N_{\tau}(\ell_{\tau}(z)) \geq N_{\tau}(z) + \cos \alpha\). So we have the following.

Proposition 2.2 The sequence \(\{\ell_{\tau}(z)\}_n\) converges to \(\infty\) as \(n \to \infty\) uniformly on the set \(V_{\alpha}(0)\).

3 Families of maps with attracting/parabolic fixed points — Domain of convergence

Now we consider a family of holomorphic maps \(f_{\tau} : U \to \mathbb{C}\) of the form
\[
f_{\tau}(z) = \tau z + 1 + \frac{a_{1}(\tau)}{z} + \frac{a_{2}(\tau)}{z^{2}} + \cdots \tag{2}
\]
defined on a neighborhood
\[U = \{z \in \mathbb{C} | R < |z| \leq \infty\}\]
of \(\infty \in \mathbb{C}\). We suppose that \(f\) depends holomorphically on \(\tau \in \Delta_{\rho}(1) = \{\tau \in \mathbb{C} | |\tau - 1| < \rho\}\). Let
\[A_{\tau}(z) = \frac{a_{1}(\tau)}{z} + \frac{a_{2}(\tau)}{z^{2}} + \cdots \]

As in the previous section, we choose and fix \(\alpha\) so that \(0 < \alpha < \pi/2\) and let \(\delta = \frac{1}{2} \cos \alpha\). By shrinking the neighborhoods \(U\) and \(W\), we assume that there is a constant \(K_{1}\) such
\[
|A_{\tau}(z)| < \frac{K_{1}}{|z|} < \delta \tag{3}
\]
for \((z, \tau) \in U \times W\). Further we assume that \(f_{\tau}(z)\) is injective in \(z\) for every \(\tau \in \Delta_{\rho}(1)\).

Since \(f_{\tau}(z)\) are approximated by linear maps \(\ell_{\tau}(z)\), we have a result concerning the uniformity of convergence of \(\{f_{\tau}^{n}(z)\}\). Let \(T_{\alpha, \rho} = T_{\alpha} \cap \Delta_{\rho}(1)\).

Lemma 3.1 For \((z, \tau) \in U \times T_{\alpha, \rho}\) we have
\[
N_{\tau}(f_{\tau}(z)) \geq |\tau|N_{\tau}(z) + \delta.
\]

Now let \(V = V_{\alpha, \rho}(R) = \{(z, \tau) \in V_{\alpha}(R) | \tau \in T_{\alpha, \rho}\}\).

Proposition 3.2 If \((z, \tau) \in V\), then \((f_{\tau}(z), \tau) \in V\). The sequence \(\{f_{\tau}^{n}(z)\}_n\) converges uniformly on \(V\) to \(\infty\) as \(n \to \infty\).
4 Schröder-Abel equation — special case

Here we consider the special case where the coefficient $a_1(\tau)$ in (2) vanishes identically.

**Theorem 4.1** There exists a function $\varphi_\tau(z)$ continuous on $\mathcal{V}$ such that

(i) $\varphi_\tau(f_\tau(z)) = \tau\varphi_\tau(z) + 1$; \hspace{1cm} (4)

(ii) $\varphi_\tau(z)$ is injective in the variable $z$ for each parameter $\tau \in T_{a,r}$.

(iii) $\lim_{z \to \infty} \varphi_\tau(z)/z = 1$ as $z \to \infty$, when $|\tau| > 1$.

In fact $\varphi_\tau(z)$ is given by

$$\varphi_\tau(z) = \lim_{n \to \infty} \left\{ \frac{1}{\tau^n} f^n(z) - \sum_{k=1}^{n} \frac{1}{\tau^k} \right\}$$ \hspace{1cm} (5)

In the case where $a_1(\tau)$ does not identically vanish, the expression in (5) is not convergent. So we have to modify (5) in order to yield convergence. For this purpose, we will introduce a function satisfying a difference equation in the next section.

5 Solution to a difference equation

We consider the difference equation

$$h_\tau(\ell_\tau(z)) - \tau h_\tau(z) = \frac{1}{z} + C_\tau.$$ \hspace{1cm} (6)

where $\ell_\tau(z) = \tau z + 1$ with $|\tau| > 1$ or $\tau = 1$; and $C_\tau$ is a constant depending on $\tau$, which will be given later.

A solution to this equation is given by

$$h_\tau(z) = -\frac{1}{\tau z} + \sum_{n=1}^{\infty} \frac{1}{\tau^{n+1}} \left\{ \frac{1}{\ell_\tau^n(0)} - \frac{1}{\ell_\tau^n(z)} \right\}.$$ \hspace{1cm} (7)

**Proposition 5.1** The function $h_\tau(z)$ is continuous on $\mathcal{V}_\alpha(0)$.

For a fixed $\tau$ with $|\tau| > 1$, the function $h_\tau(z)$ is meromorphic on $\hat{\mathbb{C}}$ except the essential singularity at $1/(1-\tau)$, and has poles at $(1-\tau^{-n})/(1-\tau)$, ($n = 0, 1, 2, \ldots$). This function $h_\tau(z)$ is holomorphic at $\infty$ and we write

$$H_\tau = h_\tau(\infty) = \sum_{n=1}^{\infty} \frac{1}{\tau^{n+1} \ell_\tau^n(0)}.$$ \hspace{1cm} (8)
For $\tau = 1$, we have $\ell^n(z) = z + n$ and

$$h_1(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \left\{ \frac{1}{n} - \frac{1}{z+n} \right\}.$$  

This function is meromorphic on $\mathbb{C}$ and has poles at $0, -1, -2, \ldots$. We note that

$$h_1(z) = \frac{\Gamma'(z)}{\Gamma(z)} + \gamma$$

where $\Gamma(z)$ denotes the gamma function and $\gamma$ denotes the Euler constant

$$\gamma = \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{k} - \log n \right).$$

Now we study the dependence of $h_\tau(z)$ on the parameter $\tau$.

**Corollary 1** The constat $C_\tau$ is a continuous function of $\tau \in T_\alpha$.

The function $h_\tau(z)$ satisfies the equation (9) with

$$C_\tau = (1-\tau)H_\tau.$$  

(9)

for $|\tau| > 1$ and with $C_1 = 0$ for $\tau = 1$. We have $C_\tau \to C_1 = 0 (\tau \to 1)$, since $h_\tau(z)$ is continuous.

**Proposition 5.2** For any $\varepsilon > 0$, there is a constant $M$ such that

$$|h'_\tau(z)| \leq \frac{M}{N_\tau(z)} \text{ on } V_\alpha(\varepsilon).$$

6 Behavior of $H_\tau$

Now we look at the behavior of the function $H_\tau$ defined by (9), when $\tau \to 1$ within the sector $T$. It is clear from the expression (9) that $H_\tau$ is unbounded, while $C_\tau = (1-\tau)H_\tau$ tends to 0 by the corollary to Proposition 2.4. Here we give a more precise description of its behavior.

**Proposition 6.1** We have

$$H_\tau = -\log(\tau - 1) + \gamma - 1 + o(1)$$

as $\tau \to 1$ within the sector $T$. Here $\gamma$ denotes the Euler constant.
To show this, we write $\lambda = 1/\tau$. We have

$$H_{1/\lambda} = (1 - \lambda)L(\lambda) - \lambda.$$  

Here $L(\lambda)$ denotes the Lambert series defined by

$$L(\lambda) = \sum_{n=1}^{\infty} \frac{\lambda^n}{1 - \lambda^n}.$$  

This series $L(\lambda)$ defines a holomorphic function on $|\lambda| < 1$, and is developed into the power series

$$L(\lambda) = \sum_{n=1}^{\infty} d(n)\lambda^n = \lambda + 2\lambda^2 + 2\lambda^3 + 3\lambda^4 + \cdots,$$

where $d(n)$ denotes the number of divisors of $n$. Let

$$\frac{L(\lambda)}{1 - \lambda} = \sum_{n=1}^{\infty} D(n)\lambda^n$$

with

$$D(n) = d(1) + \cdots + d(n).$$

The asymptotic behavior of $D(n)$ is given by a theorem of Dirichlet (see Apostol [1], Chandrasekharan [2]):

$$D(n) = n \log n + (2\gamma - 1)n + O(\sqrt{n}) \quad (n \to \infty).$$

Using this estimate, we have

$$\frac{L(\lambda)}{1 - \lambda} = \sum_{n=1}^{\infty} D(n)\lambda^n = -\frac{\lambda \log(1 - \lambda)}{(1 - \lambda)^2} + \frac{\gamma\lambda}{(1 - \lambda)^2} + P(\lambda)$$

where $P(\lambda) = \sum_{n=1}^{\infty} p_n\lambda^n$. From the estimate of $p_n$ we have

$$P(\lambda) = o((1 - \lambda)^{-2}) \quad \text{as} \quad \lambda \to 1 \text{ non-tangentially}$$

Hence it follows that

$$H_\tau = -\log(\tau - 1) + \gamma - 1 + o(\tau - 1).$$
7 Schröder-Abel equation — general case

Now we treat the general case where $a_1(\tau)$ does not necessarily vanish. Let

$$B_\tau = 1 - a_1(\tau)C_\tau$$

we have the following result corresponding to Theorem 7.1

**Theorem 7.1** There exists a function $\varphi_\tau(z)$ continuous on $\mathcal{V}$ such that

(i) $\varphi_\tau(z)$ satisfies the functional equation

$$\varphi_\tau(f_\tau(z)) = \tau \varphi_\tau(z) + B_\tau; \quad (10)$$

(ii) $\varphi_\tau(z)$ is injective in the variable $z$ for each parameter $\tau \in T_{\alpha, \rho}$.

(iii) $\lim_{z \to \infty} \varphi_\tau(z)/z = 1$ as $z \to \infty$, when $|\tau| > 1$.

To define $\varphi_\tau(z)$, we let

$$\Phi_\tau(z) = z - a_1(\tau)h_\tau(z).$$

Then

$$\Phi_\tau(f_\tau(z)) = \tau \Phi(z) + B_\tau + A(z).$$

From this we can define

$$\varphi_\tau(z) = \lim_{n \to \infty} \left\{ \frac{1}{\tau^n} \Phi_\tau(f_\tau^n(z)) - B_\tau \sum_{k=1}^{n} \frac{1}{\tau^k} \right\} \quad (11)$$

8 Relation with the Schröder equation

When $|\tau| > 1$, the Schröder equation

$$\chi_\tau(f_\tau(z)) = \tau \chi_\tau(z).$$

has a unique solution $\chi_\tau(z)$ of the form

$$\chi_\tau(z) = z + c_0 + \frac{c_1}{z} + \cdots$$

in a neighbourhood of $\infty$.

**Theorem 8.1** For $\tau \in T_{\alpha, \rho} - \{1\}$ we have

$$\varphi_\tau(z) = \chi_\tau(z) - \frac{B_\tau}{\tau - 1}.$$
Proof We can easily verify that $\varphi(z) + B_\tau/(\tau - 1)$ satisfies the Schröder equation. The assertion follows from the uniqueness of the solution. □

Now recall that

$$\frac{B_\tau}{\tau - 1} = \frac{1 - a_1 C_\tau}{\tau - 1}$$

$$= \frac{1}{\tau - 1} - a_1 H_\tau$$

$$= \frac{1}{\tau - 1} + a_1 \log(\tau - 1) + a_1 (1 - \gamma) + o(1)$$

Using this fact the theorem is reformulated as follows:

Theorem 8.2 Let

$$\varphi(z) = \chi(z) - \frac{1}{\tau - 1} - a_1 \log(\tau - 1)$$

for $\tau \in T - \{1\}$. Then $\varphi(z)$ converges to a solution to the Abel equation.

References


