<table>
<thead>
<tr>
<th>Title</th>
<th>Simultaneous linearization of hyperbolic and parabolic fixed points (Complex Dynamics and its Related Fields)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Ueda, Tetsuo</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2006), 1494: 1-8</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2006-05</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/58304">http://hdl.handle.net/2433/58304</a></td>
</tr>
<tr>
<td>Right</td>
<td></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
Simultaneous linearization of hyperbolic and parabolic fixed points

Tetsuo Ueda (上田 哲生)
Kyoto University

1 Statement of the result

This note is a summary of the preprint [8]. We will show that the Fatou coordinates (the solution to Abel equation) for a parabolic fixed point of holomorphic map of one variable can be obtained as a modified limit of the solution to Schröder equation for the perturbed hyperbolic maps. (An alternative proof is given by Kawahira [4].)

Let \( \{f_\tau\}_\tau \) be a family, depending on the parameter \( \tau \), of holomorphic maps of the form

\[
f_\tau(z) = \tau z + 1 + \frac{a_1(\tau)}{z} + \frac{a_2(\tau)}{z^2} + \ldots
\]

defined in a neighborhood of \( \infty \) of the Riemann sphere \( \hat{\mathbb{C}} \).

For each \( \tau \) with \( |\tau| > 1 \), we have a unique analytic function \( \chi_\tau(z) \) in a neighborhood of \( \infty \) satisfying the Schröder equation

\[
\chi_\tau(f_\tau(z)) = \tau \chi_\tau(z)
\]

and normalized so that

\[
\lim_{z \to \infty} \frac{\chi_\tau(z)}{z} = 1.
\]

We will show that, when \( \tau \) tends to \( 1 \) non-tangentially within the domain \( |\tau| > 1 \), the sequence

\[
\chi_\tau(z) - \frac{1}{\tau - 1} - a_1(\tau) \log(\tau - 1)
\]

converges to a solution to the Abel equation \( \varphi(z) \varphi(f_1(z)) = \varphi(z) + 1 \), on a half plane \( \{\text{Re } z > R\} \) with sufficiently large \( R \).
2 A family of linear maps

We begin with studying the family \( \{ \ell_{\tau} \} \) of linear maps

\[
\ell_{\tau}(z) = \tau z + 1 \tag{1}
\]
on the Riemann sphere \( \hat{\mathbb{C}} \) with a fixed point at \( \infty \).

We will investigate the uniformity, with respect to the parameter \( \tau \), of convergence of the sequence of the iterates \( \{ f^n \}_{n=1}^{\infty} \). Here, the parameter will be restricted in the closed sector

\[
T_\alpha = \{ \tau \in \mathbb{C} \mid \text{Re} \tau - 1 \geq |\tau - 1| \cos \alpha \},
\]
where \( \alpha \) is a real number with \( 0 < \alpha < \pi/2 \).

To measure the rate of convergence to \( \infty \), we define a function \( N : \hat{\mathbb{C}} \times T_\alpha - \{ (\infty, 1) \} \rightarrow \mathbb{R} \cup \{ \infty \} \) as follows.

\[
N_{\tau}(z) = \left| z - \frac{1}{1 - \tau} \right| - \left| \frac{1}{1 - \tau} \right| \quad \text{for } (z, \tau) \in \hat{\mathbb{C}} \times (T_\alpha - \{ 1 \});
\]

\[
N_1(z) = \sup_{|\theta| \leq \alpha} \text{Re}(e^{i\theta}z) \quad \text{for } z \in \mathbb{C}.
\]

We will not define \( N_1(\infty) \).

As is easily shown, \( N_{\tau}(z) \) is upper semi-continuous and

\[
N_1(z) = \limsup_{T \ni \tau \rightarrow 1} N_{\tau}(z).
\]

Further the inequality

\[
|N_{\tau}(z) - N_{\tau}(w)| \leq |z - w| \quad z, w \in \mathbb{C}, \tau \in T_\alpha
\]

and, in particular,

\[
N_{\tau}(z) \leq |z|, \quad z \in \mathbb{C}, \tau \in T_\alpha.
\]

hold.

For a real number \( R \), let

\[
\mathcal{V}_\alpha(R) = \{(z, \tau) \in \hat{\mathbb{C}} \times T_\alpha - \{ (\infty, 1) \} \mid N_{\tau}(z) > R \}.
\]

We note that \( \mathcal{V}_\alpha(R) \) is not open. Slices of \( \mathcal{V}_\alpha(R) \) by \( \tau = \text{const.} \) are open sets given by

\[
\mathcal{V}_{\tau}(R) = \{ z \in \hat{\mathbb{C}} \mid N_{\tau}(z) > R \} \quad (\tau \neq 1);
\]

\[
\mathcal{V}_1(R) = \{ z \in \mathbb{C} \mid N_1(z) > R \} = \bigcup_{|\theta| \leq \alpha} \{ \text{Re}(e^{i\theta}z) > 0 \}.
\]
Lemma 2.1 For $(z, \tau) \in \hat{\mathbb{C}} \times T_{\alpha} - \{(\infty, 1)\}$, we have

$$N_{\tau}(\ell_{\tau}(z)) \geq |\tau|N_{\tau}(z) + \cos \alpha.$$ 

If $N_{\tau}(z) > 0$, we have $N_{\tau}(\ell_{\tau}(z)) \geq N_{\tau}(z) + \cos \alpha$. So we have the following.

Proposition 2.2 The sequence $\{\ell_{\tau}^{n}(z)\}_{n}$ converges to $\infty$ as $n \to \infty$ uniformly on the set $\mathcal{V}_{\alpha}(0)$.

3 Families of maps with attracting/parabolic fixed points — Domain of convergence

Now we consider a family of holomorphic maps $f_{\tau} : U \to \hat{\mathbb{C}}$ of the form

$$f_{\tau}(z) = \tau z + 1 + \frac{a_{1}(\tau)}{z} + \frac{a_{2}(\tau)}{z^{2}} + \cdots$$ \hspace{1cm} (2)

defined on a neighborhood

$$U = \{z \in \hat{\mathbb{C}} \mid R < |z| \leq \infty\}$$

of $\infty \in \hat{\mathbb{C}}$. We suppose that $f$ depends holomorphically on $\tau \in \Delta_{\rho}(1) = \{\tau \in \mathbb{C} \mid |	au - 1| < \rho\}$. Let

$$A_{\tau}(z) = \frac{a_{1}(\tau)}{z} + \frac{a_{2}(\tau)}{z^{2}} + \cdots.$$ 

As in the previous section, we choose and fix $\alpha$ so that $0 < \alpha < \pi/2$ and let $\delta = \frac{1}{2} \cos \alpha$. By shrinking the neighborhoods $U$ and $W$, we assume that there is a constant $K_{1}$ such

$$|A_{\tau}(z)| < \frac{K_{1}}{|z|} < \delta$$ \hspace{1cm} (3)

for $(z, \tau) \in U \times W$. Further we assume that $f_{\tau}(z)$ is injective in $z$ for every $\tau \in \Delta_{\rho}(1)$.

Since $f_{\tau}(z)$ are approximated by linear maps $\ell_{\tau}(z)$, we have a result concerning the uniformity of convergence of $\{f_{\tau}^{n}(z)\}$. Let $T_{\alpha,\rho} = T_{\alpha} \cap \Delta_{\rho}(1)$.

Lemma 3.1 For $(z, \tau) \in U \times T_{\alpha,\rho}$ we have

$$N_{\tau}(f_{\tau}(z)) \geq |\tau|N_{\tau}(z) + \delta.$$ 

Now let $\mathcal{V} = \mathcal{V}_{\alpha,\rho}(R) = \{(z, \tau) \in \mathcal{V}_{\alpha}(R) \mid \tau \in T_{\alpha,\rho}\}$.

Proposition 3.2 If $(z, \tau) \in \mathcal{V}$, then $(f_{\tau}(z), \tau) \in \mathcal{V}$. The sequence $\{f_{\tau}^{n}(z)\}_{n}$ converges uniformly on $\mathcal{V}$ to $\infty$ as $n \to \infty$. 

4 Schröder-Abel equation — special case

Here we consider the special case where the coefficient $a_1(\tau)$ in (2) vanishes identically.

**Theorem 4.1** There exists a function $\varphi_\tau(z)$ continuous on $\mathcal{V}$ such that

(i) $\varphi_\tau(f_\tau(z)) = \tau \varphi_\tau(z) + 1$;

(ii) $\varphi_\tau(z)$ is injective in the variable $z$ for each parameter $\tau \in T_{\alpha,r}$.

(iii) $\lim_{z \to \infty} \varphi_\tau(z)/z = 1$ as $z \to \infty$, when $|\tau| > 1$.

In fact $\varphi_\tau(z)$ is given by

$$\varphi_\tau(z) = \lim_{n \to \infty} \left\{ \frac{1}{\tau^n f^n(z)} - \sum_{k=1}^{n} \frac{1}{\tau^k} \right\}$$

In the case where $a_1(\tau)$ does not identically vanish, the expression in (5) is not convergent. So we have to modify (5) in order to yield convergence. For this purpose, we will introduce a function satisfying a difference equation in the next section.

5 Solution to a difference equation

We consider the difference equation

$$h_\tau(\ell_\tau(z)) - \tau h_\tau(z) = \frac{1}{z} + C_\tau,$$

where $\ell_\tau(z) = \tau z + 1$ with $|\tau| > 1$ or $\tau = 1$; and $C_\tau$ is a constant depending on $\tau$, which will be given later.

A solution to this equation is given by

$$h_\tau(z) = -\frac{1}{\tau z} + \sum_{n=1}^{\infty} \frac{1}{\tau^{n+1}} \left\{ \frac{1}{\ell_\tau^n(0)} - \frac{1}{\ell_\tau^n(z)} \right\}.$$  

**Proposition 5.1** The function $h_\tau(z)$ is continuous on $\mathcal{V}_\alpha(0)$.

For a fixed $\tau$ with $|\tau| > 1$, the function $h_\tau(z)$ is meromorphic on $\hat{\mathbb{C}}$ except the essential singularity at $1/(1-\tau)$, and has poles at $(1-\tau^{-n})/(1-\tau)$, $(n = 0, 1, 2, \ldots)$. This function $h_\tau(z)$ is holomorphic at $\infty$ and we write

$$H_\tau = h_\tau(\infty) = \sum_{n=1}^{\infty} \frac{1}{\tau^{n+1} \ell_\tau^n(0)}.$$
For $\tau = 1$, we have $\ell^n(z) = z + n$ and

$$h_1(z) = -\frac{1}{z} + \sum_{n=1}^{\infty} \left\{ \frac{1}{n} - \frac{1}{z+n} \right\}.$$  

This function is meromorphic on $\mathbb{C}$ and has poles at $0, -1, -2, \ldots$. We note that

$$h_1(z) = \frac{\Gamma'(z)}{\Gamma(z)} + \gamma$$

where $\Gamma(z)$ denotes the gamma function and $\gamma$ denotes the Euler constant

$$\gamma = \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{k} - \log n \right).$$

Now we study the dependence of $h_\tau(z)$ on the parameter $\tau$.

**Corollary 1** The constant $C_\tau$ is a continuous function of $\tau \in T_\alpha$.

The function $h_\tau(z)$ satisfies the equation (8) with

$$C_\tau = (1-\tau)H_\tau.$$  \hspace{1cm} (9)

for $|\tau| > 1$ and with $C_1 = 0$ for $\tau = 1$. We have $C_\tau \to C_1 = 0$ ($\tau \to 1$), since $h_\tau(z)$ is continuous.

**Proposition 5.2** For any $\varepsilon > 0$, there is a constant $M$ such that

$$|h'_\tau(z)| \leq \frac{M}{N_\tau(z)} \text{ on } V_\alpha(\varepsilon)$$

### 6 Behavior of $H_\tau$

Now we look at the behavior of the function $H_\tau$ defined by (6), when $\tau \to 1$ within the sector $T$. It is clear from the expression (6) that $H_\tau$ is unbounded, while $C_\tau = (1-\tau)H_\tau$ tends to 0 by the corollary to Proposition 2.4. Here we give a more precise description of its behavior.

**Proposition 6.1** We have

$$H_\tau = -\log(\tau - 1) + \gamma - 1 + o(1)$$

as $\tau \to 1$ within the sector $T$. Here $\gamma$ denotes the Euler constant.
To show this, we write $\lambda = 1/\tau$. We have

$$H_{1/\lambda} = (1 - \lambda)L(\lambda) - \lambda.$$

Here $L(\lambda)$ denotes the Lambert series defined by

$$L(\lambda) = \sum_{n=1}^{\infty} \frac{\lambda^n}{1 - \lambda^n}.$$

This series $L(\lambda)$ defines a holomorphic function on $|\lambda| < 1$, and is developed into the power series

$$L(\lambda) = \sum_{n=1}^{\infty} d(n)\lambda^n = \lambda + 2\lambda^2 + 2\lambda^3 + 3\lambda^4 + \cdots,$$

where $d(n)$ denotes the number of divisors of $n$. Let

$$\frac{L(\lambda)}{1 - \lambda} = \sum_{n=1}^{\infty} D(n)\lambda^n$$

with

$$D(n) = d(1) + \cdots + d(n).$$

The asymptotic behavior of $D(n)$ is given by a theorem of Dirichlet (see Apostol [1], Chandrasekharan [2]):

$$D(n) = n \log n + (2\gamma - 1)n + O(\sqrt{n}) \quad (n \to \infty).$$

Using this estimate, we have

$$\frac{L(\lambda)}{1 - \lambda} = \sum_{n=1}^{\infty} D(n)\lambda^n = -\frac{\lambda \log(1 - \lambda)}{(1 - \lambda)^2} + \frac{\gamma \lambda}{(1 - \lambda)^2} + P(\lambda)$$

where $P(\lambda) = \sum_{n=1}^{\infty} p_n\lambda^n$. From the estimate of $p_n$ we have

$$P(\lambda) = o((1 - \lambda)^{-2}) \quad \text{as} \quad \lambda \to 1 \text{ non-tangentially}$$

Hence it follows that

$$H_\tau = -\log(\tau - 1) + \gamma - 1 + o(\tau - 1)$$
7 Schröder-Abel equation — general case

Now we treat the general case where \( a_1(\tau) \) does not necessarily vanish. Let

\[
B_\tau = 1 - a(\tau)C_\tau
\]

we have the following result corresponding to Theorem?

**Theorem 7.1** There exists a function \( \varphi_\tau(z) \) continuous on \( \mathcal{V} \) such that

(i) \( \varphi_\tau(z) \) satisfies the functional equation

\[
\varphi_\tau(f_\tau(z)) = \tau \varphi_\tau(z) + B_\tau;
\]

(ii) \( \varphi_\tau(z) \) is injective in the variable \( z \) for each parameter \( \tau \in T_{\alpha,\tau} \).

(iii) \( \lim_{z \to \infty} \varphi_\tau(z)/z = 1 \) as \( z \to \infty \), when \( |\tau| > 1 \).

To define \( \varphi_\tau(z) \), we let

\[
\Phi_\tau(z) = z - a_1(\tau)h_\tau(z).
\]

Then

\[
\Phi_\tau(f_\tau(z)) = \tau \Phi(z) + B_\tau + \tilde{A}(z).
\]

From this we can define

\[
\varphi_\tau(z) = \lim_{n \to \infty} \left\{ \frac{1}{\tau^n} \Phi_\tau(f_\tau^n(z)) - B_\tau \sum_{k=1}^{n} \frac{1}{\tau^k} \right\}
\]

8 Relation with the Schröder equation

When \( |\tau| > 1 \), the Schröder equation

\[
\chi_\tau(f_\tau(z)) = \tau \chi_\tau(z)
\]

has a unique solution \( \chi_\tau(z) \) of the form

\[
\chi_\tau(z) = z + c_0 + \frac{c_1}{z} + \cdots
\]

in a neighbourhood of \( \infty \).

**Theorem 8.1** For \( \tau \in T_{\alpha,\rho} - \{1\} \) we have

\[
\varphi_\tau(z) = \chi_\tau(z) - \frac{B_\tau}{\tau - 1}.
\]
Proof We can easily verify that $\varphi(z) + B_\tau/\tau - 1$ satisfies the Schröder equation. The assertion follows from the uniqueness of the solution. \qed

Now recall that

$$\frac{B_\tau}{\tau - 1} = \frac{1 - a_1 C_\tau}{\tau - 1} = \frac{1}{\tau - 1} - a_1 H_\tau = \frac{1}{\tau - 1} + a_1 \log(\tau - 1) + a_1 (1 - \gamma) + o(1)$$

Using this fact the theorem is reformulated as follows:

Theorem 8.2 Let

$$\varphi(z) = \chi(z) - \frac{1}{\tau - 1} - a_1 \log(\tau - 1)$$

for $\tau \in T - \{1\}$. Then $\varphi(z)$ converges to a solution to the Abel equation.

References


