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Large-time behavior of solutions for the damped wave equation

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Dedicated to Professor Tai-Ping Liu on his 60th birthday

1 Introduction

We consider the Cauchy problem for the semilinear damped wave equation

$$(D)\qquad \begin{cases} u_{tt} - \Delta u + u_t = f(u), & (t,x) \in \mathbf{R}_+ \times \mathbf{R}^N \\ (u, u_t)(0,x) = (u_0, u_1)(x), & x \in \mathbf{R}^N, \end{cases}$$

corresponding to the semilinear heat equation

$$(H)\qquad \begin{cases} \phi_t - \Delta\phi = f(\phi), & (t,x) \in \mathbf{R}_+ \times \mathbf{R}^N \\ \phi(0,x) = \phi_0(x), & x \in \mathbf{R}^N. \end{cases}$$

Many mathematicians have recognized that the damped wave equation approaches to the heat equation in some sense as $t \to \infty$. Therefore, in the first part we treat

(I) the linear damped wave equation and heat equation, that is, $f(u) \equiv 0$,

and show precisely how the solutions to those equations behave as $t \to \infty$. Based on these results we consider the Cauchy problem (D) for the semilinear damped wave equation related to (H). The semilinear term $f(u)$ is typically $\pm|u|^{\rho-1}u, \pm|u|^{\rho}$ etc. with the exponent $\rho > 1$. When $f(u) = |u|^{\rho-1}u$ or $\pm|u|^{\rho}$, it works as a sourcing term. On the other hand, when $f(u) = -|u|^{\rho-1}u$, it does as an absorbing term. So, in the second and third parts we respectively treat

(II) the semilinear problem (D) with a sourcing term, that is, $f(u) = |u|^{\rho-1}u$ or $\pm|u|^{\rho}$,

(III) the semilinear problem (D) with an absorbing term, that is, $f(u) = -|u|^{\rho-1}u$,

with relation to (H).

Original motivation to investigate (D) with (H) is coming from the results in the model system of 1-dimensional compressible flow through porous media

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = -\alpha u, \end{cases} \quad \text{with} \quad \begin{pmatrix} v \\ u \end{pmatrix}(0,x) = \begin{pmatrix} v_0 \\ u_0 \end{pmatrix}(x) \to \begin{pmatrix} v_\pm \\ u_\pm \end{pmatrix}, \quad x \to \pm\infty,$$

with $v_\pm > 0$, where $v(\geq 0)$, u and p are, respectively, the specific volume, the velocity and the pressure with $p'(v) < 0\,(v > 0)$, and α is a positive constant. The system was first considered by Nishida [24]. In Hsiao and Liu [14] they reformulated the problem to the second order wave equation with damping and asserted that the solution (v, u) behaves as $(\tilde{v}, \tilde{u})$ which is a solution to the parabolic system

$$\begin{cases} \tilde{v}_t - \tilde{u}_x = 0, \\ \quad p(\tilde{v})_x = -\alpha \tilde{u}, \end{cases}$$

due to the Darcy law. This, roughly speaking, implies that the damped wave equation is near to the corresponding parabolic equation, in other words, the damped wave equation has the diffusive structure as $t \to \infty$. So, we want to obtain the details how near as $t \to \infty$.

2 Linear damped wave equation and heat equation

By $S_N(t)g$ and $P_N(t)\phi_0$, we respectively denote the solution $v(t,x)$ to

$$\begin{cases} v_{tt} - \Delta v + v_t = 0, & (t,x) \in \mathbf{R}_+ \times \mathbf{R}^N, \\ (v, v_t)(0,x) = (0, g)(x), & x \in \mathbf{R}^N, \end{cases} \tag{2.1}$$

and the solution $\phi(t,x)$ to

$$\begin{cases} \phi_t - \Delta \phi = 0, & (t,x) \in \mathbf{R}_+ \times \mathbf{R}^N, \\ \phi(0,x) = \phi_0(x), & x \in \mathbf{R}^N. \end{cases} \tag{2.2}$$

Then the solution u to

$$\begin{cases} u_{tt} - \Delta u + u_t = 0, & (t,x) \in \mathbf{R}_+ \times \mathbf{R}^N, \\ (u, u_t)(0,x) = (u_0, u_1)(x), & x \in \mathbf{R}^N \end{cases} \tag{2.3}$$

is given by

$$u(t, \cdot) = S_N(t)(u_0 + u_1) + \partial_t(S_N(t)u_0). \tag{2.4}$$

Our aim in this section is to show the precise L^p-L^q estimate on the difference of $S_N(t)g$ and $P_N(t)g$, and so, the difference of solutions $u(t,x)$ and $\phi(t,x)$ to (2.3) and (2.2). We treat the cases $N = 1, 2, 3$, mainly $N = 3$.

The solution $S_N(t)g$ has the explicit formula, which is found in Courant and Hilbert [3]. We decompose the formula to the following form:

$$S_N(t)g = e^{-t/2} W_N(t)g + J_N(t)g, \tag{2.5}$$

where $W_N(t)g$ is a solution to the linear wave equation without dissipation

$$\begin{cases} w_{tt} - \Delta w = 0, & (t,x) \in \mathbf{R}_+ \times \mathbf{R}^N, \\ (w, w_t)(0,x) = (0, g)(x), & x \in \mathbf{R}^N. \end{cases} \tag{2.6}$$

Hence, when $N = 1, 2, 3$, we have the following form

$$S_1(t)g = \frac{e^{-t/2}}{2}\int_{|z|\le t} g(x+z)\,dz + \frac{e^{-t/2}}{2}\int_{|z|\le t}\left(I_0\Big(\frac{\sqrt{t^2-|z|^2}}{2}\Big)-1\right)g(x+z)\,dz,$$
$$=: e^{-t/2}W_1(t)g + J_1(t)g$$

$$S_2(t)g = \frac{e^{-t/2}}{2\pi}\int_{|z|\le t}\frac{g(x+z)}{\sqrt{t^2-|z|^2}}dz + \frac{e^{-t/2}}{2\pi}\int_{|z|\le t}\frac{\cosh\big(\frac{\sqrt{t^2-|z|^2}}{2}\big)-1}{\sqrt{t^2-|z|^2}}g(x+z)\,dz,$$
$$=: e^{-t/2}W_2(t)g + J_2(t)g$$

$$S_3(t)g = e^{-t/2}\frac{t}{4\pi}\int_{S^2} g(x+t\omega)\,d\omega + \frac{e^{-t/2}}{4\pi}\int_{|z|\le t}\frac{I_1\big(\frac{\sqrt{t^2-|z|^2}}{2}\big)}{2\sqrt{t^2-|z|^2}}g(x+z)\,dz,$$
$$=: e^{-t/2}W_2(t)g + J_2(t)g,$$

by the D'Alembert, Poisson and Kirchhoff formulas for the wave equation in each dimension. Then we have the following estimates.

Proposition 2.1 (cf. [21, 16, 25]) *For $1 \le q \le p \le \infty$ and $1 \le N \le 3$, it holds that*

$$\|(J_N(t) - P_N(t))g\|_{L^p} \le Ct^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})-1}\|g\|_{L^q}, \quad t \ge t_0 > 0,$$

$$\|\partial_t(J_N(t)g)\|_{L^p} \le C(1+t)^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})-1}\|g\|_{L^q}, \quad t \ge 0.$$

Since it is well-known that

$$\|\partial_t(P_N(t)g)\|_{L^p} \le Ct^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})}\|g\|_{L^q}, \quad t \ge t_0 > 0, \tag{2.7}$$

Proposition 2.1 means that $J_N(t)g$ behave as $P_N(t)g$ as $t \to \infty$. By (2.5) and (2.4) the solution $u(t,x)$ to (2.3) is given by

$$u(t,\cdot) = J_N(t)(u_0+u_1) + \partial_t(J_N(t)u_0) + e^{-t/2}\mathbf{W}(t; u_0, u_1), \tag{2.8}$$

where

$$\begin{aligned} e^{-t/2}\mathbf{W}(t;u_0,u_1) &= e^{-t/2}W_N(t)(u_0+u_1) + \partial_t(e^{-t/2}W_N(t)u_0) \\ &= e^{-t/2}\left\{W_N(t)(\frac{1}{2}u_0+u_1) + \partial_t(W_N(t)u_0)\right\}. \end{aligned} \tag{2.9}$$

Hence we have the following L^p-L^q estimate.

Theorem 2.1 (cf. [21, 16, 25]) *Let $u(t,x)$ be a solution to (2.3) and $\phi(t,x)$ be a solution to (2.2) with $\phi_0 = u_0 + u_1$. Then, for $1 \le q \le p \le \infty$, it holds that*

$$\|u(t,\cdot) - \phi(t,\cdot) - e^{-t/2}\mathbf{W}_0(t;u_0,u_1)\|_{L^p} \le Ct^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})-1}\|u_0,u_1\|_{L^q}, \quad t \ge t_0 > 0.$$

For this kind of estimate, see also Hosono and Ogawa [13] in case of $N=2$, Narazaki [23] in case of general dimension, Ikehata and Nishihara [15], Chill and Haraux [2] in the abstract setting.

By Proposition 2.1 $J_N(t)g$ behaves as $P_N(t)g$ as $t \to \infty$, and hence we call $J_N(t)g$, $\partial_t(J_N(t)g)$ "the parabolic part" of $S_N(t)g$, while we call $e^{-t/2}\mathbf{W}_0(t;u_0,u_1)$ "the wave part". Therefore, the solution $u(t,x)$ to the damped wave equation decomposed to the sum of "the wave part" and "the parabolic part". Since we assume $u_0, u_1 \in L^q$ only, the wave part may include the singularity in case of $N \geq 2$. Thus, by Theorem 2.1 we can say that

- *if we remove the singularity, though its strength decays exponentially, then the solution to the damped wave equation behaves as that to the heat equation,*

in other words,

- *if we resolve the regularity problem from the solution to the damped wave equation, then we may show that the solution behaves as that to the heat equation.*

Basic L^p-L^q estimate was given by Matsumura [22] as

$$\|S_N(t)g\|_{L^p} \leq C(1+t)^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})}\|g\|_{L^q} + Ce^{-\alpha t}\|g\|_{H^{[\frac{N}{2}]}}$$

for some constant $\alpha > 0$, which is also interpreted on the same line.

Thus, when we consider the semilinear problems (D) in the next two sections, it becomes important to resolve the regularity problem. The parabolic problem (H) has a smoothing effect and a maximum principle and hence the nonnegativity of the solution for the nonnegative data, roughly speaking. Our aim is to investigate the damped wave equation, which generally has not the nonnegativity of the solution nor the smoothing effect. These raise up the difficulties in (D).

3 Semilinear problem with a sourcing term

In this section we consider (D)

$$(3.1) \qquad \begin{cases} u_{tt} - \Delta u + u_t = f(u), & (t,x) \in \mathbf{R}_+ \times \mathbf{R}^N \\ (u,u_t)(0,x) = (u_0,u_1)(x), & x \in \mathbf{R}^N, \end{cases}$$

related to

$$(3.2) \qquad \begin{cases} \phi_t - \Delta\phi = f(\phi), & (t,x) \in \mathbf{R}_+ \times \mathbf{R}^N \\ \phi(0,x) = \phi_0(x), & x \in \mathbf{R}^N, \end{cases}$$

where

$$(3.3) \qquad f(u) = |u|^{\rho-1}u \quad \text{or} \quad \pm |u|^\rho.$$

For (3.2) with $f(\phi) = |\phi|^{\rho-1}\phi$ there are long literatures. Summing these up, we know that, if $\rho > \rho_c(N) := 1 + \frac{2}{N}$, then the solution $\phi(t,x)$ globally exists for small data, and that the positive solution $\phi(t,x)$ blows up within a finite time if $\rho \leq \rho_c(N)$. For these results refer Fujita [8], Hayakawa [11], Weissler [29] etc. and the servey papers Levine [19], Deng and Levine [4]. The critical exponent $\rho_c(N)$ is called the Fujita exponent named after his work [8].

From the discussion in the preceding section we expect that the solution $u(t,x)$ to (3.1) behaves as ϕ to (3.2). In fact, Todorova and Yordanov [28] have shown that, if $\rho_c(N) < \rho < \frac{N+2}{N-2}$, then the solution u to (3.1) with $f(u) = |u|^\rho$ globally exists for small data u_0, u_1 with compact support, and that, if $\rho < \rho_c(N)$, then the local solution u blows up within a finite time for suitable data. In the critical case $\rho = \rho_c(N)$ Zhang [30] has shown the solution to blow up. Note that the small data global existence of solutions is available for $f(u) = \pm|u|^{\rho-1}u,\ \pm|u|^\rho$. But, for the blow-up result they have treated $f(u) = |u|^\rho$ to get positivity.

For the related topics, when $N = 1$, Gallay and Raugel [10] has shown the asymptotic profile in the supercritical case, applying the scaling variables. When $N = 1, 2$, Li and Zhou [20] obtained the estimate of the blow-up time in the critical and subcritical case, using the explicit formula of solutions.

We present two theorems in case of $N = 3$, based on the preceding discussions. The weak solution $u(t,x)$ to (3.1) is defined by the solution to

$$
\begin{aligned}
u(t,\cdot) &= S_N(t)(u_0+u_1) + \partial_t(S_N(t)u_0) + \int_0^t S_N(t-\tau)f(u)(\tau)\,d\tau \\
&= S_N(t)(u_0+u_1) + \partial_t(S_N(t)u_0) \\
&\quad + \int_0^t e^{-\frac{t-\tau}{2}} W_N(t-\tau)f(u)(\tau)\,d\tau + \int_0^t J_N(t-\tau)f(u)(\tau)\,d\tau.
\end{aligned}
\tag{3.4}
$$

Also, denoting the Sobolev space by $W^{m,p} = W^{m,p}(\mathbf{R}^N) = \{f;\ \partial_x^i f \in L^p\ (i = 0, 1, \cdots, m)\}$ for $1 \leq p \leq \infty$, we have the followings.

Theorem 3.1 (**[25]**) *Let $N = 3$ and $(u_0, u_1) \in (W^{1,\infty} \cap W^{1,1}) \times (L^\infty \cap L^1) =: Z_0$ be small. Then, when $\rho > \rho_c(3) = 5/3$, there exists a unique weak solution $u \in C([0,\infty);\ L^1 \cap L^\infty)$ to (3.1) satisfying*

$$
\|u(t,\cdot)\|_{L^p} \leq C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})}\|u_0, u_1\|_{Z_0}, \quad (1 \leq p \leq \infty).
$$

Remark 3.1. Since we assume suitable regularity, i.e. $u_0, u_1 \in Z_0$, only smallness of the data yields the global existence theorem even for the exponent bigger than the Sobolev critical exponent $1 + \frac{4}{N-2}$.

Theorem 3.2 (**[26]**) *Let $f(u) = |u|^\rho$ (resp. $f(u) = |u|^{\rho-1}u$) and (u_0, u_1) be replaced by $(\varepsilon u_0, \varepsilon u_1) \in Z_0$, $0 < \varepsilon \ll 1$ with $\int_{\mathbf{R}^3}(u_0+u_1)(x)\,dx > 0$ (resp. $(0, \varepsilon u_1) \in Z_0$, $0 < \varepsilon \ll 1$ with $u_1(x) \geq 0$, $\int_{\mathbf{R}^3} u_1(x)\,dx > 0$). Then, when $\rho \leq \rho_c(3)$, the time local solution u to (3.1) blows up within a finite time, and the blow-up time T_ε of u is estimated as*

$$
T_\varepsilon \leq \begin{cases} \exp(C\varepsilon^{-\alpha}) & \rho =: 1+\alpha = \rho_c(N), \\ C\varepsilon^{2\alpha/(2-N\alpha)} & \rho = 1+\alpha < \rho_c(N), \end{cases} \quad (N = 3).
\tag{3.5}
$$

Remark 3.2. When the data is $(0, \varepsilon u_1)$ with $u_1(x) \geq 0$, $\int_{\mathbf{R}^3} u_1(x)\,dx > 0$, the local solution u to (3.1) with $f(u) = |u|^{p-1}u$ becomes positive by the solution formula

$$u(t,\cdot) = S_N(t)u_1 + \int_0^t S_N(t-\tau)|u|^{p-1}u(\tau,\cdot)\,d\tau.$$

For $f(u) = -|u|^p$ we don't know the blow-up result.

The estimate (3.5) of T_ε is available even in $N = 1, 2$, which was given by Li and Zhou [20]. In fact, to prove Theorem 3.2 we use the explicit formula (3.4) and apply the following lemma given in [20].

Lemma 3.1 *If $I(t)$ satisfies*

$$\begin{cases} I''(t) + I'(t) \geq c_0 \dfrac{I^{1+\alpha}(t)}{(1+t)^\beta}, & t > 0 \\ I(0) \geq \varepsilon > 0, \quad I'(0) \geq 0. \end{cases}$$

with $\alpha > 0$, $0 \leq \beta \leq 1$, then $I(t)$ blows up in a finite time. More precisely, the life span T_ε is estimated from above as

$$T_\varepsilon \leq \begin{cases} \exp(C\varepsilon^{-\alpha}) & \beta = 1 \\ C\varepsilon^{-\frac{\alpha}{1-\beta}} & 0 \leq \beta < 1. \end{cases}$$

4 Semilinear problem with an absorbing term

In this section we consider

$$\text{(4.1)} \qquad \begin{cases} u_{tt} - \Delta u + u_t + |u|^{p-1}u = 0, & (t,x) \in \mathbf{R}_+ \times \mathbf{R}^N \\ (u, u_t)(0,x) = (u_0, u_1)(x), & x \in \mathbf{R}^N, \end{cases}$$

related to

$$\text{(4.2)} \qquad \begin{cases} \phi_t - \Delta\phi + |\phi|^{p-1}\phi = 0, & (t,x) \in \mathbf{R}_+ \times \mathbf{R}^N \\ \phi(0,x) = \phi_0(x), & x \in \mathbf{R}^N. \end{cases}$$

Again we remind the results on (4.2). For big data ϕ_0, in the supercritical case the solution $\phi(t,x)$ approaches to $\theta_0 G(t,x)$ as $t \to \infty$, where $G(t,x) = (4\pi t)^{-N/2}\exp(-|x|^2/4t)$ is the Gauss kernel and the constant θ_0 is given by

$$\theta_0 = \int_{\mathbf{R}^N} \phi_0(x)\,dx - \int_0^\infty \int_{\mathbf{R}^N} |\phi|^{p-1}\phi(t,x)\,dx\,dt.$$

In the critical case $\phi(t,x)$ approaches to $\theta_0 G(t,x)(\log t)^{-N/2}$ for some constant θ_0. In the subcritical case there is a unique *positive* similarity solution $w_0(t,x) = t^{-1/(p-1)}f(|x|/\sqrt{t})$ to (4.2), where $f(r)$, $r = |x|$, is a positive solution to

$$\text{(4.3)} \qquad \begin{cases} -f'' - (\frac{r}{2} + \frac{N-1}{r})f' + |f|^{p-1}f = \dfrac{1}{p-1}f \\ f'(0) = 0, \quad \lim_{r\to\infty} r^{\frac{2}{p-1}} f(r) = 0. \end{cases}$$

Then the positive solution $\phi(t,x)$ to (4.2) with $\phi_0 \in L^1$, $\lim_{|x|\to\infty}|x|^{2/(\rho-1)}\phi_0(x) = 0$, $\phi_0(x) \geq 0$ tends to the similarity solution $w_0(t,x)$ in the sense that

$$\lim_{t\to\infty} t^{\frac{1}{\rho-1}}\|\phi(t,x) - w_0(t,x)\|_{L^\infty} = 0. \tag{4.4}$$

If we impose $\lim_{r\to\infty} r^{\frac{2}{\rho-1}}f(r) = b > 0$ instead of $\lim_{r\to\infty} r^{\frac{2}{\rho-1}}f(r) = 0$ in (4.3), then we still have a solution f and a similarity solution $w_b(t,x) = t^{-1/(\rho-1)}f(|x|/\sqrt{t})$. Then for the data $\phi_0 \in L^1$, not necessarily positive, satisfying $\lim_{|x|\to\infty}|x|^{2/(\rho-1)}\phi_0(x) = b$, the solution $\phi(t,x)$ tends to $w_b(t,x)$ as $t \to \infty$, in the same sense in (4.4). However, if we do not impose the positivity in (4.3), then we have at least 4 non-trivial solutions when $1 < \rho < 1+2/(N+1)$, so that we don't know until now how the solution $\phi(t,x)$ to (4.2), which may change its sign, behaves as $t \to \infty$. For these results, refer Brezis, Peletier and Terman [1], Escobedo and Kavian [5, 6], Escobedo, Kavian and Matano [7], Galaktionov, Kurdyumov and Samarskii [9] and the references therein.

We now go back to (4.1) and remember some known results. First, Kawashima, Nakao and Ono [18] showed the existence of a unique and global solution $u(t,x)$ to (4.1) in $C([0,\infty); H^1)\cap C([0,\infty); L^2)$ for any big data $(u_0,u_1) \in H^1\times L^2$ if $1 < \rho < 1+4/(N-2)$ ($1 < \rho < \infty$ when $N = 1, 2$). Moreover, the decay property

$$\|u(t)\|_{L^2} \leq C(1+t)^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{2})} \tag{4.5}$$

was proved provided that $(u_0,u_1) \in (H^2\cap L^q)\times(H^1\cap L^q)$ $(1 \leq q \leq 2)$ and $1+\frac{4}{N} < \rho < 1+\frac{4}{N-2}$ $(N \leq 4)$. The decay rate (4.5) seems to be best possible, because we expect the Gauss kernel to be an asymptotic profile. Based on [18], Karch [17] showed that, when $N = 1, 2, 3$,

$$\|u(t,\cdot) - \theta_0 G(t,\cdot)\|_{L^p} = o(t^{-\frac{N}{2}(1-\frac{1}{p})}) \quad \text{as} \quad t \to \infty \tag{4.6}$$

for $2 \leq p \begin{cases} \leq \infty & N=1 \\ < \infty & N=2 \\ \leq \frac{2N}{N-2} & N=3 \end{cases}$ and

$$\theta_0 = \int_{\mathbf{R}^N}(u_0+u_1)(x)\,dx - \int_0^\infty\int_{\mathbf{R}^N}|u|^{\rho-1}u(t,x)\,dx\,dt. \tag{4.7}$$

Recently, Hayashi, Kaikina and Naumkin [12] have shown that, when $N = 1$, the solution $u(t,x)$ behaves as $t \to \infty$

$$u(t,x) \sim \begin{cases} w_0(t,x) & \rho_c(1) - \varepsilon < \rho < \rho_c(1) \\ \theta_0 G(t,x)(\log t)^{-1/2} & \rho = \rho_c(1) \\ \theta_0 G(t,x) & \rho > \rho_c(1), \end{cases}$$

for suitably small $\varepsilon > 0$.

In these situations we, roughly speaking, show the decay properties in the subcritical case and the asymptotic profile in the supercritical case. In the critical case we cannot have any sharp estimate. First theorem is the following.

Theorem 4.1 (**[16]**) *Let* $1 < \rho \le 1 + \frac{4}{N} (< 1 + \frac{4}{N-2})$. *Suppose that*

$$(1 + |x|)^m (u_0, \nabla u_0, u_1, |u_0|^{(\rho+1)/2}) \in L^2, \tag{4.8}$$

where for any small fixed constant $\delta > 0$

$$\begin{cases} m = \frac{2}{\rho-1} - \frac{N-\delta}{2} & \text{when } \rho < \rho_c(N) \\ m > N/2 & \text{when } \rho \ge \rho_c(N). \end{cases} \tag{4.9}$$

Then the solution $u(t,x)$ *to (4.1) satisfies the decay properties*

$$\|u(t,\cdot)\|_{L^p} \le C(1+t)^{-\frac{1}{\rho-1}+\frac{N}{2p}} \quad \text{for } 1 \le p \begin{cases} \le \infty & N = 1 \\ < \infty & N = 2 \\ \le \frac{2N}{N-2} & N \ge 3. \end{cases} \tag{4.10}$$

We give some remarks. In the subcritical case the similarity solution $w_0(t,x)$ satisfies for $1 \le p \le \infty$

$$\|w_0(t,\cdot)\|_{L^p} = t^{-\frac{1}{\rho-1}} \left(\int_{\mathbf{R}^N} t^{N/2} |f(\frac{|x|}{\sqrt{t}})|^p \frac{dx}{t^{N/2}} \right)^{1/p} = Ct^{-\frac{1}{\rho-1}+\frac{N}{2p}}.$$

Hence the decay rate (4.10) is best possible, though there are restrictions on p. Our final goal is to obtain the asymptotic profile. As we stated above, even in the parabolic problem (4.2) the asymptotic profile is not known for the solution $\phi(t,x)$ which may change its sign. In our problem (4.1) the solution $u(t,x)$ generally changes its sign. Therefore, to get our goal seems to be difficult in the subcritical case.

While in the supercritical case we cannot have L^1-boundedness from (4.10) since $-\frac{1}{\rho-1} + \frac{N}{2} > 0$. We expect that the asymptotic profile is the Gauss kernel and hence the rate (4.10) is less sharp. However, standing on this less sharp rate we will improve to get the asymptotic profile in Theorem 4.2 below.

The rates of the weight in (4.8)-(4.9) seem to be reasonable. In fact, in the subcritical case we imposed $\lim_{|x|\to\infty} |x|^{\frac{2}{\rho-1}} f(|x|) = 0$ in (4.3), which corresponds to (4.9) in the L^2-framework. In the supercritical case the solution should be in L^1 to get the asymptotic profle $G(t,x)$. Clearly (4.8)-(4.9) mean $u_0, u_1, etc. \in L^1$.

Theorem 4.2 (**[16, 27]**) *Let*

$$\rho_c(N) < \rho \begin{cases} \le 1 + \frac{4}{N} & (N = 1,2,3) \\ < 1 + \frac{4}{N} & (N = 4). \end{cases} \tag{4.11}$$

Suppose that $(u_0, u_1) \in H^2 \times H^1$ $(N \le 3)$ *[resp.* $H^3 \times H^2$ $(N = 4)$*] and*

$$(1 + |x|)^m (u_0, \nabla u_0, \Delta u_0, |u_0|^{\frac{\rho+1}{2}}, u_1, \nabla u_1) \in L^2(\mathbf{R}^N) \tag{4.12}$$

with $m > N/2$. *Then it holds that*

$$\|u(t,\cdot) - \theta_0 G(t,\cdot)\|_{L^p} = o(t^{-\frac{N}{2}(1-\frac{1}{p})}) \quad \text{as } t \to \infty, \quad 1 \le p \le \infty \tag{4.13}$$

with θ_0 *in (4.7).*

Theorem 4.2 covers the gap remained open in the supercritical case. For the proof of Theorem 4.1 we employ the weighted energy method, originally developed by Todorova and Yordanov [28]. By applying the Gagliardo-Nirenberg inequality to the L^2-decay results obtained, L^p-decay rate (4.10) is derived. For Theorem 4.2 we combine the weighted energy method with the explicit formula

$$\begin{aligned} u(t,\cdot) =& S_N(t)(u_0+u_1) + \partial_t(S_N(t)u_0) \\ & - \int_0^t e^{-\frac{t-\tau}{2}} W_N(t-\tau)|u|^{\rho-1}u(\tau,\cdot)\,d\tau - \int_0^t J_N(t-\tau)|u|^{\rho-1}u(\tau,\cdot)\,d\tau \end{aligned} \tag{4.14}$$

(cf. (3.4)) developed in Section 2. It is a key point how to treat the wave part.

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