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# Local theory in critical spaces for the dissipative quasi-geostrophic equation

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*Dedicated to Professor Tai-Ping Liu on his 60th birthday*

**Abstract**

We consider the two dimensional critical and super-critical dissipative quasi-geostrophic equations. We prove the local existence of a unique regular solution for arbitrary initial data in $B^{2-2\alpha}_{2,1}$ which is corresponding to the scaling invariant space of the equation. We also investigate the behavior of the solution near $t=0$ in the Besov space.

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## 1 Introduction

Let us consider the dissipative quasi-geostrophic equation in $\mathbb{R}^2$:

$$\begin{cases} \dfrac{\partial\theta}{\partial t}+(-\Delta)^{\alpha}\theta+u\cdot\nabla\theta=0 \quad \text{in} \quad \mathbb{R}^2\times(0,\infty), \\ u=(-R_2\theta,R_1\theta) \quad \text{in} \quad \mathbb{R}^2\times(0,\infty), \\ \theta|_{t=0}=\theta_0 \quad \text{in} \quad \mathbb{R}^2, \end{cases} \tag{DQG$_\alpha$}$$

where the scalar $\theta$ and the vector $u$ denote the potential temperature and the fluid velocity, respectively, and $\alpha$ is non-negative constant. $R_i=\frac{\partial}{\partial x_i}(-\Delta)^{-1/2}$ $(i=1,2)$ represents the Riesz transform. We are concerned with the initial value problem for this equation. It is known that (DQG$_\alpha$) is an important model in geophysical fluid dynamics. Indeed, it is derived from general quasi-geostrophic equations in the special case of constant potential vorticity and buoyancy frequency. Since there are a number of applications to the theory of oceanography and meteology, a lot of mathematical researches are devoted to the equation.

The case $\alpha = 1/2$ is called critical since its structure is quite similar to that of the 3-dimensinal Navier-Stokes equations. The case $\alpha > 1/2$ is called sub-critical and $\alpha < 1/2$ is called super-critical, respectively. In the sub-critical cases, Constantin and Wu [4] proved global existence of the unique regular solution. However, in the critical and super-critical cases, global well-posedness for *large* initial data is still open. In the critical case, Constantin, Cordoba and Wu [3] constructed a global regular solution for the initial data in $H^1$ with small $L^\infty$ norm. In the critical and super-critical cases, Chae and Lee [2] proved the global well-posedness for the initial data in the Besov space $B^{2-2\alpha}_{2,1}$ with small homogeneous norm. Later on, Ju [8] improved their results on the space of initial data. Indeed, he proved the global existence of a unique regular solution for the initial data in $H^{2-2\alpha}$ with small homogeneous norm. For large initial data, Cordoba-Cordoba [5] proved the local existence of a regular solution for the initial data in $H^s$ with $s > 2 - \alpha$. Ju [8], [9] improved the admissible exponent up to $s > 2 - 2\alpha$. Here the exponent $s_c \equiv 2 - 2\alpha$ is important, because this is the borderline exponent with respect to the scaling. We observe that if $\theta(x,t)$ is the solution of (DQG$_\alpha$), then $\theta_\lambda(x,t) \equiv \lambda^{2\alpha-1}\theta(\lambda x, \lambda^{2\alpha}t)$ is also a solution of (DQG$_\alpha$). Then the homogeneous spaces $\dot{H}^{2-2\alpha}$ and $\dot{B}^{2-2\alpha}_{2,q}$ are called scaling invariant, since $\|\theta_\lambda(\cdot,0)\|_{\dot{H}^{2-2\alpha}} = \|\theta(\cdot,0)\|_{\dot{H}^{2-2\alpha}}$ and $\|\theta_\lambda(\cdot,0)\|_{\dot{B}^{2-2\alpha}_{2,q}} = \|\theta(\cdot,0)\|_{\dot{B}^{2-2\alpha}_{2,q}}$ hold for all $\lambda > 0$. The scaling invariant spaces play an important role for the theory of nonlinear partial differential equations. If the equation has a class of scaling invariance, then it coincides with the most suitable space to construct the solution which is expected unique and regular. (See e.g. Danchin [6], Koch-Tataru [10].)

In this paper we establish the local well-posedness for (DQG$_\alpha$) with the initial data in $B^{2-2\alpha}_{2,1}$ in the critical and super-critical cases. In fact, we can extend the class of initial data $B^{2-2\alpha}_{2,1}$ to the larger class $\dot{B}^{1}_{2,1} \cap \dot{B}^{2-2\alpha}_{2,1}$. Compered with Chae-Lee [2], we can construct a local solution for arbitrary large initial data. On the other hand, we improve the local well-posedness result with respect to the space of initial data. Indeed, $\dot{B}^{2-2\alpha}_{2,1}$ contains the space such as $H^s$ ($s > 2 - 2\alpha$). See remark on Theorem 2.2 below.

We now sketch the idea of the proof. In contrast with other equations, it seems to be difficult to prove the local existence of regular solutions by the classical approach such as Fujita-Kato method [7]. As pointed out in [2], we have difficulty to find an approproate space $X$

which yields the following bilinear estimate of the Duhamel term

$$\|B(u,\theta)\|_X \leq C\|\theta\|_X^2,$$

where $B(u,\theta) \equiv \int_0^t e^{-(t-s)(-\Delta)^\alpha}(u\cdot\nabla\theta)(s)ds$ in the appropriate function space $X$. For $\alpha \leq 1/2$, we see the linear part $(-\Delta)^\alpha\theta$ is too weak to control the nonlinear term $u\cdot\nabla\theta$. In fact, the smoothing property of the semigroup $e^{-t(-\Delta)^\alpha}$ is not enough to overcome the *loss of derivatives* in the nonlinear term. To avoid this difficulty, in [2] and [8] they applied the cancelation property of the equation to construct the *small* global solution. However, their method seems to be not suitable to deal with the *large* initial data. So, in this paper we introduce the modified version of Fujita-Kato method. To be precise, we derive the family of integral inequalities on the Littlewood-Paley decomposition of the solution, which makes it possible to apply the cancelation property of the equation. In the usual Fujita-Kato method, such cancelation property seems to be not available. On the other hand, in order to treat the nonlinear equation by the perturbation argument, we establish smoothing estimates for the linear dissipative equations in the Besov spaces. Combining with these observations, we construct the local solution for large initial data in $B^{2-2\alpha}_{2,1}$. As a byproduct of our method, we obtain the precise behavior of the solution near $t = 0$ in higher order Besov spaces.

The paper is organized as follows. In Section 2, we define some function spaces and precise statements of theorems. Section 3 is devoted to establish some useful estimates such as the commutator estimate. Finally in Section 4 we prove main theorems.

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# 2 Definitions and the statements of the theorems

In this section we define some function spaces and then state main theorems. Let us first recall the definition of the Besov space. Let $\{\phi_j\}_{j=-\infty}^{\infty}$ be the Littlewood-Paley decomposition of unity i.e. $\hat{\phi} \in C_0^\infty(\mathbb{R}^n\setminus\{0\})$, $\operatorname{supp}\hat{\phi} \subset \{\xi \in \mathbb{R}^n; 3/4 \leq |\xi| \leq 8/3\}$ and $\sum_{j=-\infty}^{\infty}\hat{\phi}(2^{-j}\xi) \equiv 1$ except $\xi = 0$. We define the convolution operator $\Delta_j$ as $\Delta_j = \phi_j *$ where $\mathcal{F}(\phi_j)(\xi) = \hat{\phi}(2^{-j}\xi)$. We denote by $\mathcal{S}'$ the topological dual space of that of tempered distributions $\mathcal{S}$. Moreover, we denote by $\mathcal{Z}'$ defined as the topological dual space of $\mathcal{Z}$ defined by

$$\mathcal{Z} \equiv \{f \in \mathcal{S}; \int x^\alpha f(x)\, dx = 0 \quad \text{for all } \alpha \in \mathbb{N}^n\}.$$

**Definition 2.1** *For $s \in \mathbb{R}$, $1 \le p \le \infty$ and $1 \le q < \infty$, we write the $\dot{B}^s_{p,q}$-(quasi) norm by*

$$\|f\|_{\dot{B}^s_{p,q}} \equiv \left( \sum_{j=-\infty}^{\infty} 2^{jsq} \|\Delta_j f\|_p^q \right)^{1/q}.$$

*For $s > 0$, $1 \le p \le \infty$ and $1 \le q < \infty$ we also write the $B^s_{p,q}$-norm by*

$$\|f\|_{B^s_{p,q}} \equiv \|f\|_{L^p} + \|f\|_{\dot{B}^s_{p,q}}.$$

*We define function spaces as follows:*

$$\dot{B}^s_{p,q} \equiv \left\{ f \in \mathcal{Z}'; \|f\|_{\dot{B}^s_{p,q}} < \infty \right\},$$
$$B^s_{p,q} \equiv \left\{ f \in \mathcal{S}'; \|f\|_{B^s_{p,q}} < \infty \right\}.$$

**Remark** i) While the inhomogeneous space $B^s_{p,q}$ is a subspace of $\mathcal{S}'$, the homogeneous counterpart $\dot{B}^s_{p,q}$ is that of $\mathcal{Z}' \simeq \mathcal{S}'/\mathcal{P}$. Here we denote $\mathcal{P}$ as the set of all polynomials. Since we cannot distinguish zero from other polynomial in $\mathcal{S}'/\mathcal{P}$, they seems not to be appropriate as funstion spaces where equations are treated. Fortunately, if the exponents satisfy the following condition:

$$\text{either} \quad s < n/p \quad \text{or} \quad s = n/p \text{ and } q = 1,$$

then $\dot{B}^s_{p,q}$ can be regarded as a subspace of $\mathcal{S}'$. Indeed, if $s$, $p$ and $q$ satisfy the above condition, we have

$$\dot{B}^s_{p,q} \simeq \left\{ f \in \mathcal{S}'; \|f\|_{\dot{B}^s_{p,q}} < \infty \text{ and } f = \sum_{j=-\infty}^{\infty} \Delta_j f \text{ in } \mathcal{S}' \right\}.$$

For the details one can see, e.g. Kozono-Yamazaki [11].

ii) Roughly speaking, the exponent $s$ represents the differentiability of functions and $p$ represents the integrability. $q$ is less important since their differences are at most logarithmic. These spaces are considered as generalizations of $L^p$ space and Sobolev space. For example, we have the following embeddings:

$$\dot{B}^0_{p,1} \subset L^p \subset \dot{B}^0_{p,\infty}, \qquad \dot{B}^s_{p,1} \subset \dot{W}^{s,p} \subset \dot{B}^s_{p,\infty}.$$

We will also mention some facts on the Besov space in the remark of Theorem 2.2 below.

Now we state the main theorem of this paper.

**Theorem 2.2** *Let* $0 \leq \alpha \leq 1/2$. *Suppose that the initial data* $\theta_0 \in \dot{B}^1_{2,1} \cap \dot{B}^{2-2\alpha}_{2,1}$. *Then there exist a positive constant* $T_1$ *and a unique solution of* $(\mathrm{DQG}_\alpha)$ *in* $C([0,T_1);\dot{B}^1_{2,1}) \cap L^1(0,T_1;\dot{B}^2_{2,1})$.

**Remark** i) The assumption that the initial data belongs to the scaling invariant space $\dot{B}^{2-2\alpha}_{2,1}$ plays an crucial role in the theorem. In the critical case $\alpha = 1/2$, one can take the class of initial data as $\dot{B}^1_{2,1}$. On the other hand, in the super-critical case $\alpha < 1/2$, we must assume that the initial data belongs to $\dot{B}^1_{2,1}$ in addition to $\dot{B}^{2-2\alpha}_{2,1}$. One of the reason is that $\dot{B}^{2-2\alpha}_{2,1}$ is only the subspace of $\mathcal{S}'/\mathcal{P}$, so $\dot{B}^{2-2\alpha}_{2,1}$ is no longer appropriate to treat equation $(\mathrm{DQG}_\alpha)$.

ii) Ju [8], [9] proved local existence of a unique solution for the initial data in $H^s$ $(s > 2-2\alpha)$. Theorem 2.2 improves his result on the class of initial data. In fact, the following inclusion relation holds:

$$H^s \hookrightarrow B^{2-2\alpha}_{2,1} \hookrightarrow \dot{B}^1_{2,1} \cap \dot{B}^{2-2\alpha}_{2,1} \qquad \text{for} \quad s > 2-2\alpha.$$

iii) Chae-Lee [2] proved the global existence of a unique solution for the initial data in $B^{2-2\alpha}_{2,1}$ with small homogeneous norm. Theorem 2.2 is regarded as the local version of their result. In fact, by the argument of our proof, one can also cover their global existence theorem:

**Corollary 2.3** *There exists a positive constant* $\varepsilon$ *such that for the initial data* $\theta_0 \in \dot{B}^1_{2,1} \cap \dot{B}^{2-2\alpha}_{2,1}$ *satisfying* $\|\theta_0\|_{\dot{B}^{2-2\alpha}_{2,1}} < \varepsilon$, *there exists a unique global solution in* $C([0,\infty);\dot{B}^1_{2,1}) \cap L^1(0,\infty;\dot{B}^2_{2,1})$.

In contrast with [2] [8], we make use of Fujita-Kato type method to construct the solution. This approach also tell us the behavior of the solution in higher order Besov spaces:

**Theorem 2.4** *Suppose that* $\theta_0$ *belongs to* $\dot{B}^{2-2\alpha}_{2,1} \cap \dot{B}^1_{2,1}$ *and* $\theta$ *is the solution of* $(\mathrm{DQG}_\alpha)$ *in* $L^\infty(0,T_1;\dot{B}^1_{2,1}) \cap L^1(0,T_1;\dot{B}^2_{2,1})$. *Then for all* $\beta \in [0,2\alpha)$, *there exist constant* $T_2 \in (0,T_1)$ *such that*

$$\sup_{0<t<T_2} t^{\frac{\beta}{2\alpha}} \|\theta(t)\|_{\dot{B}^{2-2\alpha+\beta}_{2,1}} < \infty.$$

*Moreover, the solution satisfies*

$$\lim_{t\to 0} t^{\frac{\beta}{2\alpha}} \|\theta(t)\|_{\dot{B}^{2-2\alpha+\beta}_{2,1}} = 0.$$

**Notations**

Throughout this paper we denote a positive constant by $C$ (or $C'$ etc) the value of which may differ from one occasion to another. On the other hand, we denote $C_i (i = 1, 2, \cdots)$ as the certain constants. Moreover we write the space $L^p(0,T;dt)$ as $L^p_T$.

# 3 Preliminaries

In this section we prepare some estimates in the Besov space. First, we recall Bernstein's inequality.

**Lemma 3.1** (i) *For any $k \in \mathbb{R}$, $1 \leq p \leq \infty$, there exist constants $C = C(k,p)$ such that*

$$C^{-1} 2^{jk} \|f\|_{L^p} \leq \|D^k f\|_{L^p} \leq C 2^{jk} \|f\|_{L^p},$$

*holds for all $f \in \mathcal{S}'$ with* $\mathrm{supp}\hat{f} \subset \{2^{j-2} \leq |\xi| \leq 2^j\}$ *and $j \in \mathbb{Z}$.*
(ii) *We have the equivalence of norms*

$$\|D^k f\|_{\dot{B}^s_{p,q}} \sim \|f\|_{\dot{B}^{s+k}_{p,q}}.$$

Next we prepare various product estimates in the Besov space.

**Proposition 3.2** *For $s$, $t \leq n/p$ with $s + t > 0$, we have*

$$\|uv\|_{\dot{B}^{s+t-n/p}_{p,1}} \leq C \|u\|_{\dot{B}^s_{p,1}} \|v\|_{\dot{B}^t_{p,1}}.$$

Finally we state the commutator estimate associated with the operator $\Delta_j$, which plays an important role in the estimate of nonlinear term.

**Proposition 3.3** *Suppose that $1 \leq p < \infty$, $n/p \leq s \leq 1 + n/p$, $t \leq n/p$ and $s + t \geq n/p$. Then there exists a constant $C = C(s,t)$ such that*

$$2^{j(s+t-n/p)} \|[u, \Delta_j]w\|_{L^p} \leq C c_j \|u\|_{\dot{B}^s_{p,1}} \|w\|_{\dot{B}^t_{p,1}}$$

*for all $u \in \dot{B}^s_{p,1}$ and $w \in \dot{B}^t_{p,1}$ with $\sum_{j\in\mathbb{Z}} c_j = 1$. Here we denote*

$$[u, \Delta_j]w = u\Delta_j w - \Delta_j(uw).$$

# 4 Proof of Theorems

## 4.1 Linear Estimates

Let consider the following linear dissipative equation:

$$\begin{cases} \dfrac{\partial \eta}{\partial t} + (-\Delta)^\alpha \eta = 0 & \text{in} \quad \mathbb{R}^2 \times (0,\infty), \\ \eta|_{t=0} = \eta_0 & \text{in} \quad \mathbb{R}^2. \end{cases} \tag{$\mathrm{L}_\alpha$}$$

The following is the useful characterization on the Besov norm of the solution and its application to the smoothing estimate.

**Proposition 4.1** *Suppose that the initial data $\eta_0$ belongs to $\dot{B}^s_{2,1}$ for some $s \in \mathbb{R}$ and let $\eta(t) \equiv e^{-t(-\Delta)^\alpha}\eta_0$ be the solution of $(\mathrm{L}_\alpha)$ for $\alpha > 0$. Then there exist positive constants $c$ and $c'$ $(c < c')$ depending only on $\alpha > 0$ such that*

$$\sum_{j\in\mathbb{Z}} 2^{sj} e^{-2^{2\alpha j}c't}\|\eta_j(0)\|_{L^2} \le \|e^{-t(-\Delta)^\alpha}\eta_0\|_{\dot{B}^s_{2,1}} \le \sum_{j\in\mathbb{Z}} 2^{sj} e^{-2^{2\alpha j}ct}\|\eta_j(0)\|_{L^2} \tag{4.1}$$

*for all $t > 0$, where $\eta_j(0) = \Delta_j \eta_0$.*

*Moreover we have*

$$\sup_{0<t<T} t^{1/p}\|e^{-t(-\Delta)^\alpha}\eta_0\|_{\dot{B}^s_{2,1}} \le C\|e^{-t(-\Delta)^\alpha}\eta_0\|_{L^p_T\dot{B}^s_{2,1}}, \tag{4.2}$$

*and*

$$\|\partial_x^\gamma e^{-t(-\Delta)^\alpha}\eta_0\|_{L^{2\alpha/\gamma}_T\dot{B}^s_{2,1}} \le C\|\eta_0\|_{\dot{B}^s_{2,1}}. \tag{4.3}$$

**Proof** Firstly we prove (4.1). Applying the operator $\Delta_j$ to $(\mathrm{L}_\alpha)$, we have

$$\partial_t \eta_j + (-\Delta)^\alpha \eta_j = 0,$$

where we denote $\eta_j \equiv \Delta_j \eta$.

Taking inner product with $\eta_j$, we have

$$\frac{1}{2}\frac{d}{dt}\|\eta_j\|^2_{L^2} + \|(-\Delta)^\alpha \eta_j\|^2_{L^2} = 0.$$

By Lemma 3.1, there exist positive constants $c$ and $c'$ $(c < c')$ such that

$$\frac{1}{2}\frac{d}{dt}\|\eta_j\|^2_{L^2} + c2^{2\alpha j}\|\eta_j\|^2_{L^2} \le 0,$$

and

$$\frac{1}{2}\frac{d}{dt}\|\eta_j\|_{L^2}^2 + c'2^{2\alpha j}\|\eta_j\|_{L^2}^2 \geq 0.$$

Dividing by $\|\eta_j\|_{L^2}$ and solving the differential inequalities, we have

$$e^{-2^{2\alpha j}c't}\|\eta_j(0)\|_{L^2} \leq \|\eta_j(t)\|_{L^2} \leq e^{-2^{2\alpha j}ct}\|\eta_j(0)\|_{L^2}.$$

Multiplying $2^{sj}$ and summing over $j \in \mathbb{Z}$, we have (4.1).

Secondly we will prove (4.2). By (4.1), we see that it suffices to show

$$\sup_{0<t<T} t^{1/p} \sum_{j\in\mathbb{Z}} 2^{sj}e^{-2^{2\alpha j}ct}\|\eta_j(0)\|_{L^2} \leq C\Big\| \sum_{j\in\mathbb{Z}} 2^{sj}e^{-2^{2\alpha j}c't}\|\eta_j(0)\|_{L^2}\Big\|_{L^p_T}. \tag{4.4}$$

Since $e^{-2^{2\alpha j}ct}$ is monotone decreasing for $t > 0$, we have

$$\sum_{j\in\mathbb{Z}} 2^{sj}e^{-2^{2\alpha j}ct}\|\eta_j(0)\|_{L^2} \leq \sum_{j\in\mathbb{Z}} 2^{sj}e^{-2^{2\alpha j}c\tau}\|\eta_j(0)\|_{L^2} \qquad \text{for} \quad 0 < \tau < t.$$

Taking $L^p(0,t;d\tau)$ norm on the both side, we have

$$t^{1/p} \sum_{j\in\mathbb{Z}} 2^{sj}e^{-2^{2\alpha j}ct}\|\eta_j(0)\|_{L^2} \leq \Big\| \sum_{j\in\mathbb{Z}} 2^{sj}e^{-2^{2\alpha j}c\tau}\|\eta_j(0)\|_{L^2}\Big\|_{L^p(0,t;d\tau)}.$$

By change of variables, we observe that

$$\begin{aligned}
&\Big\| \sum_{j\in\mathbb{Z}} 2^{sj}e^{-2^{2\alpha j}c\tau}\|\eta_j(0)\|_{L^2}\Big\|_{L^p(0,t;d\tau)} \\
&\qquad \leq \Big(\frac{c'}{c}\Big)^{1/p} \Big\| \sum_{j\in\mathbb{Z}} 2^{sj}e^{-2^{2\alpha j}c'\tau}\|\eta_j(0)\|_{L^2}\Big\|_{L^p(0,t;d\tau)},
\end{aligned}$$

which yields (4.4).

Finally we will prove (4.3). Applying (4.1), we have

$$\|\partial_x^\gamma \eta\|_{L_T^{2\alpha/\gamma}\dot{B}^s_{2,1}} \leq C\Big\| \sum_{j\in\mathbb{Z}} 2^{(\gamma+s)j}e^{-2^{2\alpha j}ct}\|\eta_j(0)\|_{L^2}\Big\|_{L^{2\alpha/\gamma}(0,T;dt)} \tag{4.5}$$

Let $U_j(t) \equiv 2^{sj}e^{-2^{2\alpha j}ct}\|\eta_j(0)\|_{L^2}$, then $U_j$ satisfies

$$\partial_t U_j + c2^{2\alpha j}U_j = 0 \qquad \text{for} \quad t > 0 \quad \text{and} \quad j \in \mathbb{Z}.$$

Multiplying $U_j^{2\alpha/\gamma-1}$ and integrating on $(0,T)$, we have

$$U_j(T)^{2\alpha/\gamma} + \int_0^T c2^{2\alpha j} U_j(s)^{2\alpha/\gamma} dt = U_j(0)^{2\alpha/\gamma}.$$

In particular

$$\|2^{\gamma j} U_j\|_{L_T^{2\alpha/\gamma}} \leq C U_j(0).$$

Taking sum over $j \in \mathbb{Z}$ and applying Minkowski's inequality for the left hand side, we have

$$\Big\| \sum_{j\in\mathbb{Z}} 2^{\gamma j} U_j \Big\|_{L_T^{2\alpha/\gamma}} \leq C \sum_{j\in\mathbb{Z}} U_j(0).$$

By the definition of $U_j$, the above inequality shows

$$\Big\| \sum_{j\in\mathbb{Z}} 2^{(\gamma+s)j} e^{-2^{2\alpha j} ct} \|\eta_j(0)\|_{L^2} \Big\|_{L_T^{2\alpha/\gamma}} \leq C \|\eta_0\|_{\dot{B}^s_{2,1}}.$$

Combining this estimate with (4.5), we obtain (4.3).

$\square$

## 4.2 Proof of Theorem 2.2

**Step 1:** Firstly we will show a priori estimates in $L^2_T \dot{B}^{2-\alpha}_{2,1}$. Precisely we will prove that there exist a positive constant $C_1$ and a bounded function$I(T)$ with $\lim_{T\to 0} I(T) = 0$ such that

$$\|\theta\|_{L^2_T \dot{B}^{2-\alpha}_{2,1}} \leq I(T) + C_1 \|\theta\|^2_{L^2_T \dot{B}^{2-\alpha}_{2,1}}. \tag{4.6}$$

Applying the operator $\Delta_j$ to (DQG$_\alpha$), we obtain

$$\partial_t \theta_j + (-\Delta)^\alpha \theta_j = -\Delta_j (u \cdot \nabla\theta),$$

where we denote $\theta_j \equiv \Delta_j \theta$. Adding $u \cdot \nabla \Delta_j \theta$ on both sides, we have

$$\partial_t \theta_j + (-\Delta)^\alpha \theta_j + u \cdot \nabla \Delta_j \theta = [u, \Delta_j] \nabla\theta.$$

Taking inner products with $\theta_j$, it follows from the divergence free condition that

$$\frac{1}{2}\frac{d}{dt} \|\theta_j\|^2_{L^2} + c2^{2\alpha j} \|\theta_j\|^2_{L^2} \leq \|[u, \Delta_j]\nabla\theta\|_{L^2} \|\theta_j\|_{L^2}.$$

Dividing both side by $||\theta_j||_{L^2}$, we have

$$\frac{d}{dt}||\theta_j||_{L^2} + c2^{2\alpha j}||\theta_j||_{L^2} \le ||[u, \Delta_j]\nabla\theta||_{L^2}.$$

Applying Proposition 3.3 with $s = 2 - \alpha$ and $t = 1 - \alpha$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||\theta_j||_{L^2} + c2^{2\alpha j}||\theta_j||_{L^2} &\le ||[u, \Delta_j]\nabla\theta||_{L^2} \\ &\le Cc_j 2^{-(2-2\alpha)j}||u||_{\dot{B}^{2-\alpha}_{2,1}}||\nabla\theta||_{\dot{B}^{1-\alpha}_{2,1}} \\ &\le Cc_j 2^{-(2-2\alpha)j}||\theta||^2_{\dot{B}^{2-\alpha}_{2,1}}.\end{aligned}$$

Solving the differential inequality, we have

$$||\theta_j(t)||_{L^2} \le e^{-2^{2\alpha j}ct}||\theta_j(0)||_{L^2} + Cc_j 2^{-(2-2\alpha)j}\int_0^t e^{-2^{2\alpha j}c(t-s)}||\theta(s)||^2_{\dot{B}^{2-\alpha}_{2,1}}ds. \tag{4.7}$$

Multiplying $2^{(2-\alpha)j}$ and summing over $j \in \mathbb{Z}$, we obtain

$$\begin{aligned}||\theta_j(t)||_{\dot{B}^{2-\alpha}_{2,1}} \le &\sum_{j\in\mathbb{Z}} 2^{(2-\alpha)j}e^{-2^{2\alpha j}ct}||\theta_j(0)||_{L^2} \\ &+ C\sum_{j\in\mathbb{Z}} c_j 2^{\alpha j}\int_0^t e^{-2^{2\alpha j}c(t-s)}||\theta(s)||^2_{\dot{B}^{2-\alpha}_{2,1}}ds.\end{aligned} \tag{4.8}$$

In order to show (4.6), we take $L^2_T$ norm on the both sides of (4.8).

By Proposition 4.1 , the first term is estimated as follows

$$\left\|\sum_{j\in\mathbb{Z}} 2^{(2-\alpha)j}e^{-2^{2\alpha j}ct}||\theta_j(0)||_{L^2}\right\|_{L^2_T} \le C||\theta_0||_{\dot{B}^{2-2\alpha}_{2,1}}.$$

Let

$$I(T) \equiv \left\|\sum_{j\in\mathbb{Z}} 2^{(2-\alpha)j}e^{-2^{2\alpha j}ct}||\theta_j(0)||_{L^2}\right\|_{L^2_T}.$$

Then we have $I(T) \le C||\theta_0||_{\dot{B}^{2-2\alpha}_{2,1}}$ and $\lim_{T\to 0} I(T) = 0$ by the absolutely continuity of the integral.

Concerning $L_T^2$ estimate for the second term of (4.8), we have

$$\begin{aligned}
&\Big\| \sum_{j\in\mathbb{Z}} c_j 2^{\alpha j} \int_0^t e^{-2^{2\alpha j}c(t-s)} \|\theta(s)\|^2_{\dot{B}^{2-\alpha}_{2,1}} ds \Big\|_{L_T^2} \\
\le& \sum_{j\in\mathbb{Z}} c_j 2^{\alpha j} \Big\| \int_0^t e^{-2^{2\alpha j}c(t-s)} \|\theta(s)\|^2_{\dot{B}^{2-\alpha}_{2,1}} ds \Big\|_{L_T^2} \\
\le& \sum_{j\in\mathbb{Z}} c_j 2^{\alpha j} \Big( \int_0^T e^{-2^{2\alpha j+1}ct} dt \Big)^{1/2} \|\theta\|^2_{L_T^2 \dot{B}^{2-\alpha}_{2,1}} \\
\le& C\|\theta\|^2_{L_T^2 \dot{B}^{2-\alpha}_{2,1}}.
\end{aligned}$$

Therefore we have obtain a priori estimate (4.6).

Secondly we will show the following estimate:

$$\|\theta\|_{L_T^1 \dot{B}^2_{2,1}} \le I'(T) + C_2 \|\theta\|^2_{L_T^2 \dot{B}^{2-\alpha}_{2,1}}. \tag{4.9}$$

with $\lim_{T\to 0} I'(T) = 0$.

In (4.7), multiplying $2^{2j}$ and taking sum over $j \in \mathbb{Z}$, we obtain

$$\begin{aligned}
\|\theta_j(t)\|_{\dot{B}^2_{2,1}} \le& \sum_{j\in\mathbb{Z}} 2^{2j} e^{-2^{2\alpha j}ct} \|\theta_j(0)\|_{L^2} \\
&+ C\sum_{j\in\mathbb{Z}} c_j 2^{2\alpha j} \int_0^t e^{-2^{2\alpha j}c(t-s)} \|\theta(s)\|^2_{\dot{B}^{2-\alpha}_{2,1}} ds.
\end{aligned}$$

By Proposition 4.1, we have $L_T^1$ estimate for the first term as follows:

$$\Big\| \sum_{j\in\mathbb{Z}} 2^{2j} e^{-2^{2\alpha j}ct} \|\theta_j(0)\|_{L^2} \Big\|_{L_T^1} \le C\|\theta_0\|_{\dot{B}^{2-2\alpha}_{2,1}}.$$

Let $I'(T) \equiv \Big\| \sum_{j\in\mathbb{Z}} 2^{2j} e^{-2^{2\alpha j}ct} \|\theta_j(0)\|_{L^2} \Big\|_{L_T^1}$. Then we have $\lim_{T\to 0} I'(T) = 0$.

On the other hand, applying Young's inequality, we have

$$\Big\| \sum_{j\in\mathbb{Z}} c_j 2^{2\alpha j} \int_0^t e^{-2^{2\alpha j}c(t-s)} \|\theta(s)\|^2_{\dot{B}^{2-\alpha}_{2,1}} ds \Big\|_{L_T^1} \le C\|\theta\|^2_{L_T^2 \dot{B}^{2-\alpha}_{2,1}}.$$

Thus we obtain the a priori estimate (4.9).

Similary to the previous argument, we can also obtain

$$\|\theta\|_{L_T^\infty \dot{B}^{2-2\alpha}_{2,1}} \le \|\theta_0\|_{\dot{B}^{2-2\alpha}_{2,1}} + C_3 \|\theta\|^2_{L_T^2 \dot{B}^{2-\alpha}_{2,1}}.$$

**Step 2:** To construct the solution, we define the following approximation sequences:

$$\begin{cases} \partial_t\theta^0 + (-\Delta)^\alpha\theta^0 = 0 \quad \text{in} \quad \mathbb{R}^2 \times \mathbb{R}_+, \\ \theta^0|_{t=0} = \theta_0 \quad \text{in} \quad \mathbb{R}^2 \end{cases}$$

and

$$\begin{cases} \partial_t\theta^{n+1} + (-\Delta)^\alpha\theta^{n+1} + u^n \cdot \nabla\theta^{n+1} = 0 \quad \text{in} \quad \mathbb{R}^2 \times \mathbb{R}_+, \\ u^n = (-R_2\theta^n, R_1\theta^n) \quad \text{in} \quad \mathbb{R}^2 \times \mathbb{R}_+, \\ \theta^{n+1}|_{t=0} = \theta_0 \quad \text{in} \quad \mathbb{R}^2, \end{cases} \tag{4.10}$$

for $n \geq 0$.

We will prove the uniform estimate on $\theta^n$. Let $X_T^n \equiv \|\theta^n\|_{L_T^2\dot{B}_{2,1}^{2-\alpha}}$ and $Y_T^n \equiv \|\theta^n\|_{L_T^1\dot{B}_{2,1}^2}$. By the argument in Step 1, we can show that there exists a bounded function $I(T)$ with $\lim_{T\to 0} I(T) = 0$ such that

$$\begin{aligned} X_T^0 &\leq I(T), \\ X_T^{n+1} &\leq I(T) + C_1 X_T^n X_T^{n+1} \qquad \text{for } n \geq 0. \end{aligned}$$

Taking $T_0 > 0$ sufficiently small satisfying $I(T_0) \leq 1/(4C_1)$, we have

$$X_T^n \leq 2I(T) \qquad \text{for} \quad n \geq 0. \tag{4.11}$$

On the other hand we can also prove that there exists a bounded function $I'(T)$ with $\lim_{T\to 0} I'(T) = 0$ such that

$$\begin{aligned} Y_T^0 &\leq I'(T), \\ Y_T^{n+1} &\leq I'(T) + C_2 X_T^n X_T^{n+1}. \end{aligned}$$

Combining with the above estimate and (4.11), we have

$$Y_T^{n+1} \leq I'(T) + C_4(I(T))^2 \qquad \text{for } n \geq 0. \tag{4.12}$$

Using the uniform estimate, we will prove the convergence of the sequence in $L_T^\infty \dot{B}_{2,1}^1$.

Let $\delta\theta^{n+1} = \theta^{n+1} - \theta^n$ and $\delta u^{n+1} = u^{n+1} - u^n$. Then we have following equations of the differences:

$$\begin{cases} \partial_t\delta\theta^{n+1} + (-\Delta)^\alpha\delta\theta^{n+1} + u^n \cdot \nabla\delta\theta^{n+1} + \delta u^n \cdot \nabla\theta^n = 0, \\ \delta u^n = (-R_2\delta\theta^n, R_1\delta\theta^n) \quad \text{in } \mathbb{R}^2 \times \mathbb{R}_+, \\ \delta\theta^{n+1}|_{t=0} = 0 \quad \text{in } \mathbb{R}^2, \end{cases}$$

for $n \geq 0$.

Similarly to a priori estimates, we have

$$\frac{1}{2}\frac{d}{dt}\|\delta\theta_j^{n+1}\|_{L^2}^2+2^{2\alpha j}\|\delta\theta_j^{n+1}\|_{L^2}^2 \leq -\langle \Delta_j(u^n\cdot\nabla\delta\theta^{n+1})+\Delta_j(\delta u^n\cdot\nabla\theta^n), \delta\theta_j^{n+1}\rangle,$$

where $\delta\theta_j^n \equiv \Delta_j\theta^{n+1} - \Delta_j\theta^n$. Thanks to the divergence free condition, we have

$$\langle u^n \cdot \nabla\delta\theta_j^{n+1}, \delta\theta_j^{n+1}\rangle = 0.$$

By Hölder's inequality, we have

$$\frac{d}{dt}\|\delta\theta_j^{n+1}\|_{L^2}+2^{2\alpha j}\|\delta\theta_j^{n+1}\|_{L^2} \leq \|[u^n, \Delta_j]\nabla\delta\theta^{n+1}\|_{L^2} + \|\Delta_j(\delta u^n \cdot \nabla\theta^n)\|_{L^2}.$$

This implies

$$\begin{aligned}&\|\delta\theta_j^{n+1}(t)\|_{L^2}\\ &\quad\leq C\int_0^t e^{-2^{2\alpha j}c(t-s)}(\|[u^n, \Delta_j]\nabla\delta\theta^{n+1}\|_{L^2} + \|\Delta_j(\delta u^n \cdot \nabla\theta^n)\|_{L^2})ds.\end{aligned} \tag{4.13}$$

Let $s = 2$ and $t = 0$ in Proposition 3.3. Then we have

$$\begin{aligned}\|[u^n, \Delta_j]\nabla\delta\theta^{n+1}\|_{L^2} &\leq c_j 2^{-j}\|u^n\|_{\dot{B}^2_{2,1}}\|\nabla\delta\theta^{n+1}\|_{\dot{B}^0_{2,1}}\\ &\leq c_j 2^{-j}\|\theta^n\|_{\dot{B}^2_{2,1}}\|\delta\theta^{n+1}\|_{\dot{B}^1_{2,1}}.\end{aligned}$$

Multiplying $2^j$ on (4.13) and summing over $j \in \mathbb{Z}$, we have

$$\begin{aligned}&\|\delta\theta^{n+1}(t)\|_{\dot{B}^1_{2,1}}\\ &\leq C\int_0^t e^{-2^{2\alpha j}c(t-s)}(\|\theta^n\|_{\dot{B}^2_{2,1}}\|\delta\theta^{n+1}\|_{\dot{B}^1_{2,1}}+\|(\delta u^n \cdot \nabla\theta^n)\|_{\dot{B}^1_{2,1}})ds\\ &\leq C\int_0^t e^{-2^{2\alpha j}c(t-s)}(\|\theta^n\|_{\dot{B}^2_{2,1}}\|\delta\theta^{n+1}\|_{\dot{B}^1_{2,1}}+\|\delta\theta^n\|_{\dot{B}^1_{2,1}}\|\theta^n\|_{\dot{B}^2_{2,1}})ds,\end{aligned}$$

where we use Proposition 3.2 in the last line. Hence we have

$$\begin{aligned}\|\delta\theta^{n+1}\|_{L^\infty_T\dot{B}^1_{2,1}} &\leq C(\|\theta^n\|_{L^1_T\dot{B}^2_{2,1}}\|\delta\theta^{n+1}\|_{L^\infty_T\dot{B}^1_{2,1}} + \|\delta\theta^n\|_{L^\infty_T\dot{B}^1_{2,1}}\|\theta^n\|_{L^1_T\dot{B}^2_{2,1}}\\ &\leq C_5 Y_T^n(\|\delta\theta^{n+1}\|_{L^\infty_T\dot{B}^1_{2,1}} + \|\delta\theta^n\|_{L^\infty_T\dot{B}^1_{2,1}}).\end{aligned}$$

By (4.12), there exists $T_1 > 0$ such that $Y^n_{T_1} < 1/(3C_5)$ for $n \geq 0$. Then we have

$$\begin{aligned}
\|\delta\theta^{n+1}\|_{L^\infty_{T_1}\dot{B}^1_{2,1}} &\leq \frac{1}{2}\|\delta\theta^n\|_{L^\infty_{T_1}\dot{B}^1_{2,1}} \\
&\leq \frac{1}{2^{n+1}}\|\theta^0\|_{L^\infty_{T_1}\dot{B}^1_{2,1}} \\
&\leq \frac{C}{2^{n+1}}\|\theta_0\|_{\dot{B}^1_{2,1}}.
\end{aligned}$$

This shows the existence of the limit function $\theta \in L^\infty_{T_1}\dot{B}^1_{2,1}$ satisfying $\theta^n \to \theta$ in $L^\infty_{T_1}\dot{B}^1_{2,1}$ as $n \to \infty$. On the other hand, uniform estimates show that $\theta$ also belongs to $L^\infty_{T_1}\dot{B}^{2-2\alpha}_{2,1} \cap L^1_{T_1}\dot{B}^2_{2,1}$ by the uniqueness of the limit $\theta(t)$ in $\mathcal{Z}'$ for $t \in (0, T_1)$. Here we can easily observe that the limit function $\theta$ satisfies (DQG$_\alpha$).

Finally we prove the continuity (in time) of the solution in $\dot{B}^1_{2,1}$. The proof is the same as the argument in Chae-Lee [2]. Indeed $\theta^n$ satisfies

$$\partial_t\theta^{n+1} = -u^n \cdot \nabla\theta^{n+1} - (-\Delta)^\alpha\theta^{n+1},$$

where the right hand side belongs to $L^1(0, T_1; \dot{B}^1_{2,1})$ Since

$$\theta^{n+1}(t') - \theta^{n+1}(t) = -\int_t^{t'} \left(u^n \cdot \nabla\theta^{n+1}(s) + (-\Delta)^\alpha\theta^{n+1}(s)\right) ds,$$

we have

$$\begin{aligned}
&\|\theta^{n+1}(t') - \theta^{n+1}(t)\|_{\dot{B}^1_{2,1}} \\
\leq &\int_t^{t'} \left(\|u^n \cdot \nabla\theta^{n+1}(s)\|_{\dot{B}^1_{2,1}} + \|(-\Delta)^\alpha\theta^{n+1}(s)\|_{\dot{B}^1_{2,1}}\right) ds \\
\leq &C\int_t^{t'} \left(\|\theta^n(s)\|_{\dot{B}^1_{2,1}}\|\theta^{n+1}(s)\|_{\dot{B}^2_{2,1}} + \|\theta^{n+1}(s)\|_{\dot{B}^{1+2\alpha}_{2,1}}\right) ds \\
\leq &C\int_t^{t'} \left(\|\theta^n(s)\|_{\dot{B}^1_{2,1}}\|\theta^{n+1}(s)\|_{\dot{B}^2_{2,1}} + \|\theta^{n+1}(s)\|_{\dot{B}^1_{2,1}} + \|\theta^{n+1}(s)\|_{\dot{B}^2_{2,1}}\right) ds \\
\leq &\|\theta^n\|_{L^\infty(t',t;\dot{B}^1_{2,1})}\|\theta^{n+1}\|_{L^1(t',t;\dot{B}^2_{2,1})} + \|\theta^{n+1}\|_{L^1(t',t;\dot{B}^1_{2,1})} + \|\theta^{n+1}\|_{L^1(t',t;\dot{B}^2_{2,1})}.
\end{aligned}$$

By the absolutely continuity of the integral, the right hand side converges to 0 as $t'$ goes to $t$. Since $\theta^{n+1}$ converges to $\theta$ in $\dot{B}^1_{2,1}$ uniformly in time, we obtain the continuity of $\theta$ in $\dot{B}^1_{2,1}$.

$\square$

## 4.3 Proof of Theorem 2.4

We will establish the following uniform estimates of the solution for (4.10). Indeed we will prove that there exists a positive constant $T_2$ such that

$$\lim_{T\to 0} \sup_{n\geq 0} \sup_{0<t<T} t^{\frac{\beta}{2\alpha}} \|\theta^n(t)\|_{\dot{B}^{2-2\alpha+\beta}_{2,1}} = 0, \tag{4.14}$$

for $T < T_2$ and $0 < \beta < 2\alpha$. Since we proved the existence and the uniqueness of the solution in $L^\infty(0,T;\dot{B}^1_{2,1}) \cap L^1(0,T;\dot{B}^2_{2,1})$ in Theorem 2.2, the uniform estimate (4.14) guarantees the desired decay estimate.

We divide the proof into two cases: $0 < \beta < \alpha$ and $\alpha \leq \beta < \alpha$.

**Step 1:** Firstly we prove (4.14) for $0 < \beta < \alpha$. For $n = 0$ it follows from Proposition 4.1 that there exists a bounded function $J = J(T)$ with $\lim_{T\to 0} J(T) = 0$ such that

$$\sup_{0<t<T} t^{\frac{\beta}{2\alpha}} \|\theta^0(t)\|_{\dot{B}^{2-2\alpha+\beta}_{2,1}} \leq J(T), \tag{4.15}$$

where $J(T) \leq C\|\theta_0\|_{\dot{B}^{2-2\alpha}_{2,1}}$.

For $n \geq 0$, $\theta^n_j$ satisfies

$$\frac{d}{dt}\|\theta^{n+1}_j\|_{L^2} + c2^{2\alpha j}\|\theta^{n+1}_j\|_{L^2} \leq \|[u^n, \Delta_j]\nabla\theta^{n+1}\|_{L^2}. \tag{4.16}$$

Applying Proposition 3.3 for $s = 2 - 2\alpha + \beta$ and $t = 1 - 2\alpha + \beta$, then we have

$$\|[u^n, \Delta_j]\nabla\theta^{n+1}\|_{L^2} \leq Cc_j 2^{-(2-4\alpha+2\beta)j}\|\theta^n\|_{\dot{B}^{2-2\alpha+\beta}_{2,1}}\|\theta^{n+1}\|_{\dot{B}^{2-2\alpha+\beta}_{2,1}}.$$

Hence we obtain

$$\begin{aligned}\|\theta^{n+1}_j(t)\|_{L^2} &\leq e^{-2^{2\alpha j}ct}\|\theta_j(0)\|_{L^2} \\ &+ Cc_j 2^{-(2-4\alpha+2\beta)j}\int_0^t e^{-2^{2\alpha j}c(t-s)}\|\theta^n(s)\|_{\dot{B}^{2-4\alpha+2\beta}_{2,1}}\|\theta^{n+1}(s)\|_{\dot{B}^{2-4\alpha+2\beta}_{2,1}}ds.\end{aligned}$$

Multiplying $2^{(2-2\alpha+\beta)j}$ and summing over $j \in \mathbb{Z}$, we have

$$\begin{aligned}\|\theta^{n+1}(t)\|_{\dot{B}^{2-2\alpha+\beta}_{2,1}} &\leq \sum_{j\in\mathbb{Z}} 2^{(2-2\alpha+\beta)j}e^{-2^{2\alpha j}ct}\|\theta_j(0)\|_{L^2} \\ &+ C\sum_{j\in\mathbb{Z}} c_j 2^{(2\alpha-\beta)j}\int_0^t e^{-2^{2\alpha j}c(t-s)}\|\theta^n(s)\|_{\dot{B}^{2-2\alpha+\beta}_{2,1}}\|\theta^{n+1}(s)\|_{\dot{B}^{2-2\alpha+\beta}_{2,1}}ds.\end{aligned}$$

This is equivalent to

$$t^{\frac{\beta}{2\alpha}}\|\theta^{n+1}(t)\|_{\dot{B}^{2-2\alpha+\beta}_{2,1}} \le t^{\frac{\beta}{2\alpha}} \sum_{j\in\mathbb{Z}} 2^{(2-2\alpha+\beta)j} e^{-2^{2\alpha j}ct}\|\theta_j(0)\|_{L^2}$$
$$+ Ct^{\frac{\beta}{2\alpha}} \sum_{j\in\mathbb{Z}} c_j 2^{(2\alpha-\beta)j} \int_0^t e^{-2^{2\alpha j}c(t-s)}\|\theta^n(s)\|_{\dot{B}^{2-4\alpha+2\beta}_{2,1}}\|\theta^{n+1}(s)\|_{\dot{B}^{2-4\alpha+2\beta}_{2,1}} ds$$
$$\equiv I + II. \tag{4.17}$$

For the first term, we have

$$\sup_{0<t<T} t^{\frac{\beta}{2\alpha}} \sum_{j\in\mathbb{Z}} 2^{(2-2\alpha+\beta)j} e^{-2^{2\alpha j}ct}\|\theta_j(0)\|_{L^2} \le CJ(T)$$

by Proposition 4.1.

On the other hand, we observe that

$$2^{(2\alpha-\beta)j} e^{-2^{2\alpha j}c(t-s)} < C(t-s)^{-(2\alpha-\beta)/2\alpha} \qquad \text{for all } j \in \mathbb{Z}.$$

So the second term of (4.17) is estimated as follows:

$$\begin{aligned} II &\le Ct^{\frac{\beta}{2\alpha}} \int_0^t (t-s)^{-(2\alpha-\beta)/2\alpha}\|\theta^n(s)\|_{\dot{B}^{2-4\alpha+2\beta}_{2,1}}\|\theta^{n+1}(s)\|_{\dot{B}^{2-2\alpha+\beta}_{2,1}} ds \\ &\le C\Big(\sup_{0<t<T} t^{\frac{\beta}{2\alpha}}\|\theta^n(t)\|_{\dot{B}^{2-2\alpha+\beta}_{2,1}}\Big)\Big(\sup_{0<t<T} t^{\frac{\beta}{2\alpha}}\|\theta^{n+1}(t)\|_{\dot{B}^{2-2\alpha+\beta}_{2,1}}\Big) \\ &\qquad \times t^{\frac{\beta}{2\alpha}} \int_0^t (t-s)^{-(2\alpha-\beta)/2\alpha} s^{-\beta/\alpha} ds \\ &\le C\Big(\sup_{0<t<T} t^{\frac{\beta}{2\alpha}}\|\theta^n(t)\|_{\dot{B}^{2-2\alpha+\beta}_{2,1}}\Big)\Big(\sup_{0<t<T} t^{\frac{\beta}{2\alpha}}\|\theta^{n+1}(t)\|_{\dot{B}^{2-2\alpha+\beta}_{2,1}}\Big) \end{aligned}$$

for $0 < t < T$, where we use the assumption $0 < \beta < \alpha$ in the last line. Thus we have

$$\begin{aligned} &\sup_{0<t<T} t^{\frac{\beta}{2\alpha}}\|\theta^{n+1}(t)\|_{\dot{B}^{2-2\alpha+\beta}_{2,1}} \\ &\le C_6 J(T) + C_7\Big(\sup_{0<t<T} t^{\frac{\beta}{2\alpha}}\|\theta^n(t)\|_{\dot{B}^{2-2\alpha+\beta}_{2,1}}\Big)\Big(\sup_{0<t<T} t^{\frac{\beta}{2\alpha}}\|\theta^{n+1}(t)\|_{\dot{B}^{2-2\alpha+\beta}_{2,1}}\Big). \end{aligned}$$

Taking $T_2 > 0$ sufficiently small, we can estimate $J(T) < 1/(4C_6C_7)$ in the above inequality for $T < T_2$. Then we conclude that

$$\sup_{0<t<T} t^{\frac{\beta}{2\alpha}}\|\theta^n(t)\|_{\dot{B}^{2-2\alpha+\beta}_{2,1}} \le 2J(T).$$

for all $T < T_2$ and $n \ge 0$, which yields (4.14).

**Step 2:** We next prove (4.14) for $\alpha \le \beta < 2\alpha$. For $n = 0$, Proposition 4.1 shows that there exists a bounded monotone decreasing function $J'(T)$ with $\lim_{T\to 0} J'(T) = 0$ such that

$$\sup_{0<t<T} t^{\frac{\beta}{2\alpha}} \|\theta^0(t)\|_{\dot{B}^{2-2\alpha+\beta}_{2,1}} \le J'(T). \tag{4.18}$$

For $n \ge 0$, we apply Proposition 3.3 for $s = 2 - 3\alpha/2 + \beta/4$ and $s = 1 - 3\alpha/2 + \beta/4$ to (4.16), then we have

$$\begin{aligned}\frac{d}{dt}&\|\theta_j^{n+1}\|_{L^2} + c2^{2\alpha j}\|\theta_j^{n+1}\|_{L^2}\\ &\le Cc_j 2^{-(2-3\alpha+\beta/2)j}\|\theta^n\|_{\dot{B}^{2-3\alpha/2+\beta/4}_{2,1}}\|\theta^{n+1}\|_{\dot{B}^{2-3\alpha/2+\beta/4}_{2,1}}.\end{aligned}$$

Transforming to the integral inequality and summing over $j \in \mathbb{Z}$, we have

$$\begin{aligned} t^{\frac{\beta}{2\alpha}}\|\theta^{n+1}(t)\|_{\dot{B}^{2-2\alpha+\beta}_{2,1}} \le & t^{\frac{\beta}{2\alpha}} \sum_{j\in\mathbb{Z}} 2^{(2-2\alpha+\beta)j} e^{-2^{2\alpha j}ct}\|\theta_j(0)\|_{L^2}\\ &+Ct^{\frac{\beta}{2\alpha}}\sum_{j\in\mathbb{Z}} c_j 2^{(\alpha-\beta/2)j}\int_0^t e^{-2^{2\alpha j}c(t-s)}\\ &\times \|\theta^n(s)\|_{\dot{B}^{2-3\alpha/2+\beta/4}_{2,1}}\|\theta^{n+1}(s)\|_{\dot{B}^{2-3\alpha/2+\beta/4}_{2,1}} ds\\ &\equiv I + II.\end{aligned}$$

The first term is estimated as (4.18). So we estimate the second term. Since

$$2^{(\alpha-\beta/2)j}e^{-2^{2\alpha j}c(t-s)} < C(t-s)^{-(2\alpha+\beta)/4\alpha} \qquad \text{for all } j \in \mathbb{Z},$$

we have

$$\begin{aligned} II &\le Ct^{\frac{\beta}{2\alpha}}\int_0^t (t-s)^{-(2\alpha+\beta)/4\alpha}\|\theta^n(s)\|_{\dot{B}^{2-3\alpha/2+\beta/4}_{2,1}}\|\theta^{n+1}(s)\|_{\dot{B}^{2-3\alpha/2+\beta/4}_{2,1}} ds\\ &\le C\Big(\sup_{0<t<T} t^{\frac14+\frac{\beta}{8\alpha}}\|\theta^n(t)\|_{\dot{B}^{2-3\alpha/2+\beta/4}_{2,1}}\Big)\Big(\sup_{0<t<T} t^{\frac14+\frac{\beta}{8\alpha}}\|\theta^{n+1}(t)\|_{\dot{B}^{2-3\alpha/2+\beta/4}_{2,1}}\Big)\end{aligned}$$

for $0 < t < T$. Since $0 < 1/4 + \beta/(8\alpha) < \alpha$, it follows from Step 1 that

$$\sup_{0<t<T} t^{\frac14+\frac{\beta}{8\alpha}}\|\theta^n(t)\|_{\dot{B}^{2-3\alpha/2+\beta/4}_{2,1}} \le 2J(T) \qquad \text{for} \quad T < T_2.$$

Hence the second term is bounded by $4C(J(T))^2$ for $T < T_2$.

Combining the above estimates, we obtain the desired estimate (4.14) for $\alpha < \beta < 2\alpha$.

$\square$

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