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UNIQUENESS OF ENTIRE SOLUTIONS FOR A REACTION-DIFFUSION EQUATION

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1. Introduction

We consider the reaction-diffusion equation of 1-space dimension:

$$u_t = u_{xx} - f(u),\ x \in \mathbf{R},\ t \in \mathbf{R}, \tag{1.1}$$

where $f(u) \in C^1([0,1])$ and

$$f(0) = f(1) = 0, \quad f'(0) > 0, \quad f'(1) \neq 0. \tag{1.2}$$

It is called **bistable case**, if $f'(1) > 0$. This is the case for Allen-Cahn equation [1]. It is called **monostable case**, if $f'(1) < 0$ and $f > 0$ in $(0,1)$, e.g., Fisher (KPP) equation [9, 15].

We are interested in the interaction of these two homogeneous steady states $u \equiv 0$ and $u \equiv 1$. One such example is the so-called **traveling wave solutions** (TWS) connecting equilibria $u = 0$ and $u = 1$ with speed c, i.e., a solution of the form $u(x,t) = Q(z)$, $z = x - ct$:

$$\ddot{Q} + c\dot{Q} = f(Q),\ 0 \le Q \le 1,\ z \in \mathbf{R}, \tag{1.3}$$

$$Q(-\infty) = 0,\ Q(+\infty) = 1. \tag{1.4}$$

Concerning the existence, uniqueness, and stability of TWS of (1.1), we refer the reader to the papers by, e.g., Kolmogorov-Petrovsky-Piskunov [15], Fisher [9], Kanel [14], Aronson-Weinberger [2, 3], Fife-McLeod [7, 8], Uchiyama [16], Bramson [4], etc.

In particular, for the bistable case, it is shown by Fife-McLeod [7] that there is a TWS of (1.1) which is unique up to translation, if f has exactly one interior zero in $(0,1)$. For the bistable case with multiple interior zeros, say, $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_{2m} = 1$, $m \ge 2$, there exists a TWS connecting 0 and 1, if the corresponding wave speed c_k connecting α_{2k-2} and α_{2k}, $k = 1, \cdots, m$, satisfying $c_1 > \cdots > c_m$ (cf. also [7]).

For the Fisher-KPP equation, it is well-known that there is a $c_{min} > 0$ such that a TWS exists if and only if $c \ge c_{min}$.

We define **entire solution** as a solution of (1.1) which is defined for all $(x,t) \in \mathbf{R}^2$. Note that a TWS is a 1-front entire solution.

Our main question here is to construct so-called 2-front entire solutions which behave as two (opposite) traveling wave fronts approaching each other from both sides of the x-axis and then annihilating in a finite time.

We shall first recall some existence results for 2-front entire solutions in §2. Then we shall concentrate on the uniqueness of 2-front entire solutions for the bistable case in §3. For convenience, from now on we shall simply call 2-front entire solution as entire solution.

2. Existence and Partial Uniqueness

For the existence of entire solution, in the monostable case, Hamel-Nadirashvili [12, 13] constructed a rich class of entire solutions for $c > c_{\min}$ for general n spatial dimension under the assumption that $f'(0) = \max_{s\in[0,1]} f'(s)$. An *"almost"* uniqueness result is also given.

The first work for the existence of entire solutions in the bistable case is done by Yagisita [17]. In 2003, he constructed entire solutions for bistable nonlinearity f with a single interior zero in $(0, 1)$ including the case $c = 0$, but no detailed motion of fronts.

The above two methods are quite complicated and involved. In [10], Fukao-Morita-Ninomiya considered the case when $f(u) = u(1-u)(a-u)$, $a \neq 1/2$ (and so $c \neq 0$). The main idea of their method is to construct suitable super/sub-solutions using the exact expression of TWS.

In [11], we extend the idea of [10] to derive the existence and *partial uniqueness* (i.e., uniqueness in the class of solutions that are sandwiched between a pair of sub-super-solution) of entire solutions for both monostable and bistable cases, if a TWS with speed $c \neq 0$ exists. Here in the monostable case we also assume that $f'(0) = \max_{s\in[0,1]} f'(s)$, but for any $c \geq c_{\min}$ (including the case when $c = c_{\min}$).

This method is quite simple and is easy to be applied to various cases such as discrete diffusive KPP equation (see, e.g., [11]). It is important to remark that the asymptotic behaviors of TWS $Q(z)$ as $z \to \pm\infty$ play an important role in this construction.

Indeed, the existence of entire solutions follows by the following **monotone iteration scheme** from a sub-super-solution pair.

Suppose that $(\underline{u}, \overline{u})$ is a uniformly bounded sub-super-solution pair of (1.1) on $\mathbf{R}\times(-\infty, T]$. For each $\tau < T$, let $w(\tau; x, t)$ be the solution of the initial value problem

$$\begin{aligned} w_t &= w_{xx} - f(w) \quad \text{in } \mathbf{R} \times (\tau, T], \\ w(\tau; \cdot, \tau) &= \underline{u}(\cdot, \tau) \text{ on } \mathbf{R} \times \{\tau\}. \end{aligned}$$

Such a solution exists and satisfies $\underline{u} \leq w \leq \overline{u}$ on $\mathbf{R} \times [\tau, T]$.

Consider the family $\{w(\tau;\cdot,\cdot)\}_{\tau<T}$. This family is uniformly bounded from above by $\overline{u}$. It is also monotonic. Hence the limit

$$u(x,t) := \lim_{\tau\to-\infty} w(\tau;x,t) \quad \forall (x,t) \in \mathbf{R}\times(-\infty,T]$$

exists. By a parabolic regularity theory, such convergence is locally uniform and u is a classical solution of (1.1) that satisfies $\underline{u} \le u \le \overline{u}$.

By constructing a **quasi-invariant manifold** (so that a **deterministic** sub-super-solution pair can be constructed), in [5] we construct entire solutions for the monostable case for any $c \ge c_{\min}$, without the assumption that $f'(0) = \max_{s\in[0,1]} f'(s)$. A sub-super-solution pair is called **deterministic** via translation if there exist functions $\xi(\cdot), \rho(\cdot)$ such that

$$\overline{u}(x,t) \le \underline{u}(x+\xi(t), t+\rho(t)) \quad \forall x \in \mathbf{R}, t \le T,$$
$$\lim_{t\to-\infty}\{|\rho(t)| + |\xi(t)|\} = 0.$$

Here in [5] only the partial uniqueness is proved for the monostable case. Also, the existence and uniqueness of entire solutions for the bistable case with $c \neq 0$ is derived. See Theorems 2.1 and 3.1 below. The case for $c = 0$ is treated in [6].

Theorem 2.1. *Assume that $f \in C^2(\mathbf{R})$, $f(0) = f(1) = 0$, $f'(0) > 0$, $f'(1) > 0$, and a TWS (c, Q) with $c > 0$ exists. Then (1.1) admits an entire solution $u = U$ satisfying*

$$U(x,t) = U(-x,t), \quad U_t(x,t) < cU_x(x,t) < 0 \qquad \forall x > 0, t \in \mathbf{R},$$
$$U(x,t+h(t)) < Q(x-ct)Q(-ct-x) < U(x,t-h(t)) \qquad \forall x \in \mathbf{R}, t < 0, \tag{2.1}$$

where $h(t) = M[1 - Q(c\,|t|)\,]$ and M is some positive constant.

Here the set $\{Q(x+q)Q(p-x) \mid p, q > 0\}$ is called a quasi-invariant manifold, since $Q(x-ct)Q(-ct-x)$ is very "close" to a solution of (1.1) for $-t \gg 1$.

3. Uniqueness for Bistable Case

The main uniqueness theorem in [5] is as follows.

Theorem 3.1. *Let α_0, β_0 be constants such that $f \neq 0$ in $(0, \alpha_0] \cup [\beta_0, 1)$. If u is a non-constant entire solution of (1.1) with $0 \le u \le 1$ and the initial condition:*
$\exists\, d > 0$, $T \in \mathbf{R}$, and functions $l(\cdot)$ and $r(\cdot)$ such that

$$u(x,t) \le \alpha_0, \ \forall\, x \in (-\infty, l(t)] \cup [r(t), \infty),$$
$$u(x,t) \ge \beta_0, \ \forall\, x \in [l(t)+d, r(t)-d]$$

for all $t \le T$, then, under the assumption of Theorem 2.1,

$$u(x,t) = U(x+x_0, t+t_0), \ \forall\, (x,t) \in \mathbf{R}^2$$

for some $(x_0, t_0) \in \mathbf{R}^2$.

Proof. Here we shall give an outline of the proof as follows.

Step 1. Let γ_0 be a fixed constant such that $\gamma_0 > \alpha_0$ and $f > 0$ in $(0, \gamma_0]$. Then we can find $T_1 \in \mathbf{R}$ such that $M(t) := \sup_{x\in\mathbf{R}} u(x,t) > \gamma_0$, $\forall t \le T_1$.

Step 2. Define for $t \le T_2 := \min\{T, T_1\}$:

$$\tilde{l}(t) = \min\{x \mid u(x,t) = \gamma_0\},$$
$$\tilde{r}(t) = \max\{x \mid u(x,t) = \gamma_0\}.$$

Then $l(t) \le \tilde{l}(t) < \tilde{r}(t) \le r(t)$. Moreover,

$$\lim_{t\to-\infty} [\tilde{r}(t) - \tilde{l}(t)] = \infty.$$

Set $p(t) = [\tilde{r}(t) - \tilde{l}(t)]/2$ and $m(t) = [\tilde{r}(t) + \tilde{l}(t)]/2$. Then, for all $t \ll -1$,

$$u(x + m(t), t) \le \gamma_0, \quad \text{if } |x| \ge p(t),$$
$$u(x + m(t), t) \ge \beta_0, \quad \text{if } |x| \le p(t) - d.$$

Step 3. We prove the asymptotic wave resemblance:

$$\lim_{t\to-\infty} \inf_{z\in\mathbf{R},\tau\in\mathbf{R}} \|u(z+\cdot,t) - U(\cdot,\tau)\|_{L^\infty(\mathbf{R})} = 0. \tag{3.1}$$

Note that, as $t \to -\infty$,

$$U(x,t) \sim Q(x - ct)Q(-x - ct).$$

Step 4. There exist positive constants ε_0, B, ν such that

$$U^{\pm}(x,t) = U(x, \tau + t \mp B\varepsilon[1 - e^{-\nu t}]) \pm \varepsilon e^{-\nu t}$$

is a sub-super-solution pair for every $\tau \in \mathbf{R}$ and $\varepsilon \in (0, \varepsilon_0]$.

Step 5. Fix an arbitrary $\bar{t} \in \mathbf{R}$. Define

$$\eta(\bar{t}) := \inf_{z\in\mathbf{R},\tau\in\mathbf{R}} \|u(z+\cdot,\bar{t}) - U(\cdot,\tau)\|_{L^\infty(\mathbf{R})}.$$

Fix any small positive $\varepsilon \in (0, \varepsilon_0]$. By (3.1), there exist $t_1 < \bar{t}$, $z_1 \in \mathbf{R}$, and $\tau_1 \in \mathbf{R}$ such that

$$U(x,\tau_1) - \varepsilon \le u(z_1 + x, t_1) \le U(x,\tau_1) + \varepsilon \quad \forall x \in \mathbf{R}.$$

By comparison, for all $t \ge 0$,

$$\begin{aligned} & U(x, \tau_1 + t + B\varepsilon[1 - e^{-\nu t}]) - \varepsilon e^{-\nu t} \\ \le\ & u(z_1 + x, t_1 + t) \\ \le\ & U(x, \tau_1 + t - B\varepsilon[1 - e^{-\nu t}]) + \varepsilon e^{-\nu t} \quad \forall x \in \mathbf{R}. \end{aligned}$$

Set $t = \bar{t} - t_1$, $\hat{\tau} = \tau_1 + t - B\varepsilon[1 - e^{-\nu t}]$, we conclude that

$$\eta(\bar{t}) \le (2 + 2B\|U_t\|_\infty)\, \varepsilon.$$

Since ε is arbitrary, $\eta(\bar{t}) = 0$.

Step 6. Take a sequence $\{t_j\}$ such that $t_j \to -\infty$ as $j \to \infty$. Since $\eta(t_j) = 0$ for all j, there are sequences $\{z_j\}$ and $\{\tau_j\}$ such that

$$U(x, \tau_j) - 1/j \le u(z_j + x, t_j) \le U(x, \tau_j) + 1/j$$

for all $x \in \mathbf{R}$. Consequently, for $j \gg 1$, for all $t \ge 0$,

$$\begin{aligned} & U(x, \tau_j + t + B[1 - e^{-\nu t}]/j) - e^{-\nu t}/j \\ \le\ & u(z_j + x, t_j + t) \\ \le\ & U(x, \tau_j + t - B[1 - e^{-\nu t}]/j) + e^{-\nu t}/j \quad \forall x \in \mathbf{R}. \end{aligned}$$

Then for all $x \in \mathbf{R}$, for all $t \ge t_j$, for all $j \gg 1$,

$$\begin{aligned} & U(x - z_j, \tau_j - t_j + t + B[1 - e^{-\nu(t-t_j)}]/j) - e^{-\nu(t-t_j)}/j \\ \le\ & u(x, t) \\ \le\ & U(x - z_j, \tau_j - t_j + t - B[1 - e^{-\nu(t-t_j)}]/j) + e^{-\nu(t-t_j)}/j. \end{aligned}$$

Since u is a non-constant entire solution such that $0 \le u \le 1$ in $\mathbf{R}^2$, we have $0 < u < 1$ in $\mathbf{R}^2$. Then, from the properties of U, both $\{-z_j\}$ and $\{\tau_j - t_j\}$ are bounded and have finite limits x_0 and t_0 as $j \to \infty$. We conclude that $u(x, t) = U(x + x_0, t + t_0)$ in $\mathbf{R}^2$. □

Now we turn to the **balanced bistable case** (i.e., $c = 0$). In this case, we have $f = F'$, where F satisfies

(3.2) $\quad F \in C^4(\mathbf{R}),\ F''(0) > 0,\ F''(1) > 0, \quad F(0) = F(1) = 0 < F(s)\ \forall\, s \ne 0, 1.$

Note that (1.1) admits a monotonic standing wave $u(x, t) = Q(x)$:

$$\ddot{Q}(z) = f(Q(z)) \quad \forall\, z \in \mathbf{R}, \quad Q(-\infty) = 0,\ Q(\infty) = 1.$$

In the sequel, Q always refers to the particular solution defined by

$$\mu z = \int_1^{Q(z)} \Big(\frac{\mu}{\sqrt{2F(s)}} - \frac{1}{(1-s)}\Big) ds - \ln[1 - Q(z)]$$

for $z \in \mathbf{R}$, where $\mu := \sqrt{f'(1)}$. It has the expansion

$$Q(z) = 1 - e^{-\mu z} + \tfrac{f''(1)}{6f'(1)}\, e^{-2\mu z} + O(1) e^{-3\mu z}$$

as $z \to \infty$.

In [6], we prove the following existence and uniqueness theorem.

Theorem 3.2. *Assume (3.2). Then (1.1) admits a solution u with the initial condition at $t = -\infty$:*

(3.3) $$\liminf_{t \to -\infty}\ \inf_{p > q} \Big\{ \sup_{x > (p+q)/2} \Big| u(x, t) - Q(p - x) \Big| + \sup_{x < (p+q)/2} \Big| u(x, t) - Q(x - q) \Big| \Big\} = 0.$$

In addition, the solution is unique up to space and time translations, i.e., if u_1 *and* u_2 *are solutions of (1.1), (3.3), then there exist constants* ξ, η *such that*

$$u_1(x,t) = u_2(x+\xi, t+\eta) \qquad \forall (x,t) \in \mathbf{R}^2. \tag{3.4}$$

Furthermore, the solution satisfies

$$\begin{aligned}
&\lim_{t\to\infty} \|u(\cdot,t)\|_{L^\infty(\mathbf{R})} = 0,\\
&\lim_{|x|\to\infty} u(x,t) = 0 \quad \forall t \in \mathbf{R},\\
&\lim_{t\to-\infty} \sup_{y\in\mathbf{R}} |u(y+z,t) - Q(y+p(t))Q(-y+p(t))| = 0
\end{aligned}$$

for some translation $z \in \mathbf{R}$, *where*

$$p(t) := \tfrac{1}{2\mu} \ln|2\alpha\mu t|, \tag{3.5}$$

$$\mu := \sqrt{f'(1)}, \quad \alpha = \frac{2f'(1)}{\int_0^1 \sqrt{2F(s)}ds}. \tag{3.6}$$

Note that the "initial" condition (3.3) can also be replaced by the following condition: *There exist constants* $L > 0$ *and* $T \in \mathbf{R}$, *and functions* $p(\cdot)$ *and* $q(\cdot)$ *such that*

$$\begin{cases} u(x,t) \le \alpha_0 & \forall\, x \in (-\infty, q(t)] \cup [p(t), \infty),\\ u(x,t) \ge \beta_0 & \forall\, x \in [q(t)+L, p(t)-L], \end{cases}$$

for all $t \le T$, *where* α_0, β_0 *are constants satisfying*

$$f > 0 \quad in \quad (0,\alpha_0], \qquad f < 0 \quad in\ [\beta_0, 1).$$

For the existence, we construct $(c(p), \Phi(y,p))$ for $p \ge 0$ and $y \in \mathbf{R}$ (a quasi-invariant manifold) such that

$$\begin{aligned}
&\Phi(-y,p) = \Phi(y,p),\\
&\Phi_{yy} - f(\Phi) = c\Phi_p + O(1)e^{-6\mu p},\\
&c = -\alpha\, e^{-2\mu p} + O(1)pe^{-4\mu p},\\
&\Phi(y,p) = \hat\Phi(y,p)\Big\{1 + O(1)e^{-2\mu p}[1+|y-p|^2]\Big\},\\
&\Phi_p(y,p) = \hat\Phi_p(y,p)\Big\{1 + O(1)e^{-2\mu p}(1+|y-p|^2)\Big\},\\
&\Phi_y(y,p) = \hat\Phi_y(y,p) + O(1)e^{-2\mu p}[1+|y-p|^2]\hat\Phi_p,
\end{aligned}$$

where $\hat\Phi(y,p) := Q(p-y)Q(p+y)$.

The proof of uniqueness for the case $c = 0$ follows more or less the same line as the case for $c \ne 0$. We shall not repeat it here. Instead we point out some of the differences here only. For details, we refer the reader to [6].

We define

$$l(t) = \min\{x \mid u(x,t) = \alpha_0\},\ r(t) = \max\{x \mid u(x,t) = \alpha_0\},\ Q(m_0) = \alpha_0.$$

After deriving the estimate of exponential tails, i.e., there exist constants $T_1 \le 0$, $K > 0$ and $\varepsilon > 0$ such that for all $t \le T_1$,

$$0 < u(x,t) \le \alpha_0 e^{-\varepsilon \min\{|x-r(t)|,|x-l(t)|\}}$$

for all $x \in (-\infty, l(t)] \cup [r(t), \infty)$, and

$$0 < 1 - u(x,t) \le K e^{-\varepsilon \min\{|x-r(t)|,|x-l(t)|\}}$$

for all $x \in [l(t), r(t)]$, we obtain that, as $t \to -\infty$,

$$\|u(\cdot,t) - Q(m_0 + r(t) - \cdot)Q(m_0 + \cdot - l(t))\|_{L^2(\mathbf{R})} \to 0. \tag{3.7}$$

Then we define the quasi-invariant manifold by

$$\mathcal{M} := \{\Psi(\cdot, z, p) \mid z \in \mathbf{R}, p > p_0\} \subset L^2(\mathbf{R}),$$

where p_0 is a large positive constant and

$$\Psi(x,z,p) := \Phi(x-z,p) \qquad \forall x \in \mathbf{R}, z \in \mathbf{R}, p \ge 0.$$

For convenience, we use the notation

$$\langle \phi, \psi \rangle = \int_{\mathbf{R}} \phi(y)\psi(y)dy, \qquad \|\phi\| = \sqrt{\langle \phi, \phi \rangle}.$$

Also, we use the notation $\phi \perp \psi$ when $\langle \phi, \psi \rangle = 0$.

By applying the Implicit Function Theorem, the following lemma follows from (3.7).

Lemma 3.3. *There exists a constant $T_2 < 0$ with the property that for each $t \le T_2$ there exist unique $z = z(t) \in \mathbf{R}$ and $p = p(t) \ge p_0 + 1$ such that $u(x,t) = \Psi(x,z,p) + \phi(x,t)$ for all $x \in \mathbf{R}$, where*

$$\|\phi\| = \operatorname{dist}(u(\cdot,t), \mathcal{M}) := \min_{\psi \in \mathcal{M}} \left\| u(\cdot,t) - \psi \right\|.$$

In addition, (z,p) satisfies the orthogonality condition:

$$\langle \Psi - u, \Psi_z \rangle = 0, \qquad \langle \Psi - u, \Psi_p \rangle = 0. \tag{3.8}$$

Furthermore, $z(t), p(t), \phi$ are smooth functions.

To study the dynamics of $z(t), p(t)$, we need to study the spectrum of linearized operator of (1.1) around Φ. Therefore, we consider the linear operator

$$\mathcal{L}\phi := \phi_{yy} - f'(\Phi(y,p))\phi \tag{3.9}$$

where p is any large enough constant. The following lemma shows that the self-adjoint operator $\mathcal{L}$ has two eigenvalues of order $e^{-2\mu p}$, and all the remaining eigenvalues are strictly negative.

Lemma 3.4. *Let $\mathcal{L}$ be defined as in (3.9). Then for all $p \geq 0$,*

$$\mathcal{L}\Phi_y = O(1)e^{-2\mu p}, \quad \mathcal{L}\Phi_p = O(1)e^{-2\mu p}. \tag{3.10}$$

In addition, there exist positive constants ν, p_0 such that for all $p \geq p_0$,

$$\langle \mathcal{L}\phi, \phi \rangle \leq -3\nu\Big(\|\phi\|^2 + \|\phi_y\|^2\Big)$$

for all $\phi \in H^2(\mathbf{R}), \phi \perp \Phi_y, \phi \perp \Phi_p$.

Using this lemma, we can derive the following lemma of super-slow interfacial motion.

Lemma 3.5. *There exists a large negative constant T_3 and unique functions $z(t), p(t)$ defined on $(-\infty, T_3]$ such that for all $t \leq T_3$,*

$$\begin{aligned}
&u(x,t) = \Phi(x - z(t), p(t)) + \phi(x,t),\\
&\langle \phi, \Phi_x \rangle = \langle \phi, \Phi_p \rangle = 0, \quad \|\phi(\cdot, t)\| = O(1)e^{-6\mu p},\\
&\dot{z}(t) = O(1)e^{-8\mu p},\\
&0 > \dot{p}(t) = -\alpha e^{-2\mu p} + O(1)pe^{-4\mu p}.
\end{aligned}$$

Consequently,

$$\lim_{t \to -\infty} \Big\{ p(t) - \frac{1}{2\mu}\ln(2\alpha\mu|t|) \Big\} = 0, \tag{3.11}$$

$$z(t) = z(-\infty) + \frac{O(1)}{|t|^3}, \tag{3.12}$$

where $z(-\infty)$ is a finite number.

Finally, the uniqueness is proved by a change of coordinates. We refer to [6] for more details.

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