Zeta function of learning theory and generalization error of three layered neural perceptron

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Abstract

Recently, the purpose of obtaining the maximum poles of certain zeta functions arises in the learning theory when one is looking for the generalization errors of hierarchical learning models asymptotically [3, 4]. The zeta function of a learning model is defined by the integral of its Kullback function and its a priori probability density function. Today, for several learning models, upper bounds of the main terms in their asymptotic forms were calculated, but not the exact values, so far. In this paper, we obtain the explicit value of the main term for a three layered neural network, which is one of hierarchical learning models.

Keywords zeta function, resolution of singularities, Generalization error, layered neural networks.

1 Introduction

The purpose of the learning system is such as image or speech recognition, artificial intelligence, control of a robot, genetic analysis, data mining or time series prediction. Their data are very complicated, not generated by simple normal distributions, since they are influenced by many factors. Learning models for analyzing such data have to have complicated structures, too. Hierarchical learning models such as a layered neural network, reduced rank regression, a normal mixture model and a Boltzmann machine are known as effective learning models. These models have been applied practically to such data. However, they can not be analyzed by using classic theories of regular statistical models. A few mathematical theories for such learning models have been known in the past. So it is necessary and crucial to construct fundamental mathematical theories.

Recently, it was proved that the maximum poles of certain zeta functions asymptotically give the generalization errors of hierarchical learning models [3, 4]. Furthermore, it was shown that the poles of the zeta function can be calculated by using desingularization. In spite of those mathematical foundations, for most examples, only upper bounds of the main terms were calculated but not the exact values, by two main reasons as follows.

(1) By Hironaka's Theorem [2], it is known that desingularization of an arbitrary polynomial can be obtained by using a blowing-up process. However desingularization of any polynomial in general, although it is known as a finite process, is very difficult. Furthermore, (a) most of Kullback functions are degenerate (over $\mathbb{R}$) with respect to their Newton polyhedrons, (b) singularities of Kullback functions are not isolated, (c) Kullback functions are not simple polynomials, i.e., they have parameters, for example, $p$ of $\sum_{m=1}^{r}(\sum_{m=1}^{r}c_{m}b_{m}^{2n-1})^{2}$, which is one of Kullback functions for the three layered neural networks.

We note that there are many classical results for calculating the maximum poles of the zeta functions whose dimension is two, using desingularization of plane curves. Also there have been many investigations for the case of the prehomogeneous spaces. Kullback functions do not occur in the prehomogeneous spaces. Therefore, to obtain desingularization of Kullback functions is a new problem even in mathematics, since most of these singularities have not been investigated.

(2) Since the main purpose is to obtain the maximum pole, desingularization is not enough. We need some techniques for comparing poles as real numbers. However, as far as we know, there had been no theorem for comparing poles yet.

In the paper [1], we have clarified the maximum pole and its order of the reduced rank regression which is the three layered neural network with linear hidden units. In this paper, we use a recursive blowing-up, an inductive comparing method and a toric resolution for obtaining those values of the three layered neural network.
2 Main Theorems and poles of the zeta function for the three layered neural network

In this section, Main Theorem 1 and Main Theorem 2 are stated below. Results in Main Theorem 2 are related to the three layered neural network. They are obtained from Main Theorem 1 which contains more general cases.

Let $w = (a_1^{(w)}, \ldots, a_p^{(w)}, b_1^{(w)}, \ldots, b_p^{(w)}) \in \mathbb{R}^{2p}$ be a parameter, $w^* = (a_1^{*}, \ldots, a_p^{*}, b_1^{*}, \ldots, b_p^{*}) \in \mathbb{R}^{2p}$ a constant value vector and $U^* = U_{a_1^{*}, \ldots, a_p^{*}, b_1^{*}, \ldots, b_p^{*}}$ a sufficiently small neighborhood of $w^*$. Let $Q$ be an arbitrary natural number.

Set
$$J_Q(z) = \int_{U^*} \left( \sum_{n=1}^{P} a_m^{(w)} b_m^{(w)} (Q(n-1)+1) - \sum_{m=1}^{P} a_m^{*} b_m^{*} (Q(n-1)+1) \right)^z \prod_{m=1}^{P} da_m^{(w)} db_m^{(w)},$$
(1)

where $z$ is an one-dimensional complex value and $P \geq 2p$.

In the case of the three layered neural network, $Q$ is two, which is shown later.

Let $b_1^{*Q}, \ldots, b_r^{*Q}$ be different real numbers in $\{b_i^{*Q} | b_i^{*Q} \neq 0\}$ from each other:

$\{b_1^{*Q}, \ldots, b_r^{*Q} | b_i^{*Q} \neq b_j^{*Q}, i \neq j\} = \{b_i^{*Q} | b_i^{*Q} \neq 0\}$, and $a_i^{*} = -(\sum_{m \in B_{	au}^{(w)}} a_m^{*} b_m^{*})/b_i^{*}$.

Also set $a_i^{*} = 0, b_i^{*} = 0$. Put
$$B_i^{(w)} = \{i | b_i^{*Q} = b_i^{*Q}\}, \quad s_\tau = \#B_i^{(w)}, \quad 1 \leq \tau \leq r,$

where $\#$ implies the number of elements. So $\sum_{\tau=0}^{s_\tau} = p$. Let $\tilde{r} (\tilde{r} \leq r)$ be the total number of $a_i^{*} \neq 0$.

We can assume that $a_i^{*} \neq 0, \ldots, a_{\tilde{r}}^{*} \neq 0, a_{\tilde{r}+1}^{*} = 0, \ldots, a_r^{*} = 0$. Then we have
$$\sum_{m=1}^{P} a_m^{*} b_m^{*} (Q(n-1)+1) = -\sum_{m=1}^{P} a_m^{*} b_m^{*} (Q(n-1)+1)$$
for any $n \in \mathbb{N}$. Set $H^{(1)} = \sum_{m \in B_i^{(w)}} a_m^{(w)} b_m^{(w)} (Q(n-1)+1) + a_i^{*} b_i^{*} (Q(n-1)+1)$. By the definitions, we have

$$J_{\tau}(z) = \int_{U^{* \cap R^{\gamma}}} \left( \sum_{n=1}^{P} H_m^{(n)} \right)^z \prod_{m \in B_{\tau}^{(w)}} da_m^{(w)} db_m^{(w)}.$$

Let $-\lambda_{Q}, -\lambda_{r}$ be the maximum poles of $J_Q(z)$ and $J_{\tau}(z)$, respectively, and the order of $-\lambda_{Q}$.

Remark 1 If $p+\tilde{r} = 2, R[w]/\{\sum_{n=1}^{P} (\sum_{\tau=1}^{r} H_{\tau})^2\}$ is not a normal ring. If $p+\tilde{r} \geq 3, R[w]/\{\sum_{n=1}^{P} (\sum_{\tau=1}^{r} H_{\tau})^2\}$ is a normal ring.

Main Theorem 1

(1) We have $\lambda_{Q} = \sum_{r=0}^{P} \lambda_{r}$.

(2) $\lambda_{0} = \frac{Q(\tilde{n}_{0}^{2} + \tilde{n}_{0}) + 2s_{0}}{4Q\tilde{n}_{0} + 4}$, $\tilde{n}_{0} = \max\{i \in Z | Q(i^2) - 2i \leq 2s_{0}\}$,

$\lambda_{r_{1}} = \frac{n_{r_{1}} + n_{r_{1}} + 2s_{r_{1}}}{4n_{r_{1}}}$, $n_{r_{1}} = \max\{i \in Z | i^2 + i \leq 2s_{r_{1}}\}$, \text{if} $1 \leq \tau_{1} \leq \tilde{r}$,

$\lambda_{r_{2}} = \frac{n_{r_{2}} + n_{r_{2}} + 2(s_{r_{1}} - 1)}{4n_{r_{2}}}$, $n_{r_{2}} = \max\{i \in Z | i^2 + i \leq 2(s_{r_{1}} - 1)\}$, \text{if} $\tilde{r} + 1 \leq \tau_{2} \leq r$.

(3) Set $\Theta = \{r_{0}, r_{1}, r_{2} | Q(\tilde{n}_{0}^{2} - \tilde{n}_{0}) + 2s_{0} = 2s_{n_{0}}, r_{0} = 0, s_{0} \geq 1$,

$\tilde{n}_{0} = \max\{i \in Z | i^2 + i \leq 2s_{r_{1}}\}$, \text{if} $1 \leq \tau_{1} \leq \tilde{r}$,

$\tilde{n}_{1} = \max\{i \in Z | i^2 + i \leq 2(s_{r_{1}} - 1)\}$, \text{if} $\tilde{r} + 1 \leq \tau_{2} \leq r$.

Then $\theta = \#\Theta + 1$.

Consider the three layered neural network with one input unit, $p$ hidden units, and one output unit which is trained to estimate the true distribution represented by the model with $\tilde{r} (\tilde{r} < p)$ hidden units. Denote an input value by $x \in \mathbb{R}$ with a probability density function $q(x)$ with compact support $W \subset [-1, 1]$. Then an output value $y$ of the three layered neural network is given by $y = f(x, w) + (\text{noise})$, where

$f(x, w) = \sum_{m=1}^{P} a_m^{(w)} \tanh(b_m^{(w)} x)$.

Consider a statistical model $p(y|x, w) = \frac{1}{2\pi} \exp(-\frac{1}{2}(y - f(x, w))^2)$. Assume that a true distribution is $p(y|x, w)$ which is included in the learning model, where $w = (a_1^{(w)}, \ldots, a_p^{(w)}, b_1^{(w)}, \ldots, b_p^{(w)}), |b_i^{(w)}| < \pi/2, i = 1, \ldots, p$. Let $W^*$ be the true parameter set: $W^* = \{w \in \mathbb{R}^p | f(x, w) \text{ for any } x\}$. Suppose that an a priori probability density function $\psi(w)$ is a Cauchy function with compact support $W$ where $\psi(w^*) > 0$. Then the maximum pole of $f_{w} K(w^*) \psi dw$ is equal to that of $J_Q(z) = \int_{W} \left( \sum_{m=1}^{P} a_m^{(w)} b_m^{(w)} 2^{-n-1} - \sum_{m=1}^{P} a_m^{*} b_m^{*} 2^{-n-1} \right)^2 dw$, where $P$ is a sufficient large integer. It is proved by a Taylor expansion at $0$ together with Lemma 5 in [3].
Remark 2 Let $\sigma(x) = \sum_{i=1}^{\infty} \alpha_i x^i$ with $\alpha_i \neq 0$. Then, the maximum pole of

$$f_W(f_{\tilde{p}}(\sum_{m=1}^{p} a_m^{(0)} \sigma(b_m^{(0)} x) - \sum_{m=1}^{p} a_m^{*} \sigma(b_m^{*} x))^2 q(x) dx) \psi(w) dw,$$

and its order are the same ones in Main Theorem 1. The function $\tanh x$ satisfies this condition with $Q = 2$.

Main Theorem 2

1. The maximum pole $-\lambda$ and its order $\theta$ are obtained by setting $Q = 2$ in Main Theorem 1. More precisely, we have $\lambda = \max_{\tilde{w} \in W^*} \lambda_{\tilde{w}}$ with its order $\theta$.

2. In particular, assume that $W^*$ includes all $\tilde{w}$ satisfying $f(x, \tilde{w}) = f(x, w^*)$:

$$W^* = \{ \tilde{w} \in \mathbb{R}^d \mid f(x, \tilde{w}) = f(x, w^*) \text{ for any } x \}.$$

Then for $p - \tilde{r} + 1 \leq 10$, we have $\lambda = \tilde{r} + 1 + \frac{j^2 + j + 2(p - \tilde{r} + 1)}{4j}$ and $\theta = \{ 1, \text{ if } j - 1 < 2(p - \tilde{r} + 1), 2, \text{ otherwise } \}$.

Remark 3 (1) We have the condition $\tilde{w} = (\tilde{a}_1, \ldots, \tilde{a}_p, \tilde{b}_1, \ldots, \tilde{b}_p) \in W^*$ if and only if for any $1 \leq \tau \leq \tilde{r}$, there exists $1 \leq \xi \leq p$ such that $b_{\xi}^{*} = b_{\tau}^{* Q}$. Hence, we can see a kind of phase transitions at $p - \tilde{r} + 1 = 10$.

Main Theorem 2 gives the generalization error of the three layered neural network asymptotically.

Define

$$\psi = \left( \prod_{n=1}^{P} \prod_{m=1}^{P} a_m^{(w)} b_m^{(w)^Q} - \sum_{m=1}^{P} a_m^{*} b_m^{* Q} \right)^{2} \prod_{m=1}^{P} da_m^{(w)} db_m^{(w)}$$

Put the auxiliary function $f_n,i(x_1, \ldots, x_l) = \{ \sum_{j : i+\cdots+i=j} x_1^{Qj_1} \cdots x_l^{Qj_l} \}$.

Lemma 1 Let $n, \ell \in \mathbb{N}$. Set $C_{\ell} = \sum_{n=1}^{P} \sum_{m=1}^{P} a_m^{(w)} b_m^{(w)^Q} - b_{i}^{* Q} \cdots (b_{i}^{* Q} - b_{i-1}^{* Q})$ for $i = 1, \ldots, \ell$.

Then we have

$$C_{\ell} = \sum_{n=1}^{P} \sum_{m=1}^{P} a_m^{(w)} b_m^{(w)^Q} - b_{i}^{* Q} \cdots (b_{i}^{* Q} - b_{i-1}^{* Q})$$

for $i \leq p$, and

$$C_{\ell} = \sum_{m=1}^{P} a_m^{(w)} b_m^{(w)^Q} - b_{i}^{* Q} \cdots (b_{i}^{* Q} - b_{i-1}^{* Q})$$

for $p < i \leq p + \tilde{r}$.

By Lemma 1, we have $\psi = \{ f_n,i, C_{\ell} + f_{n,1}(b_1^{* Q}) C_2 + \cdots + f_{n,\ell}(b_1^{* Q}, \ldots, b_\ell^{* Q}) C_\ell \}$$.

Q.E.D.

We can assume that $b_{1}^{* Q} = b_{2}^{* Q}, \ldots, b_{r}^{* Q} = b_{r}^{* Q}$ and that if $b_{\tau}^{* Q} \neq b_{\tau}^{* Q}$ then $b_{\tau}^{* Q} \neq b_{\tau}^{* Q}$ on $U^*$. Then, since $b_{\tau}^{* Q} \neq b_{\tau}^{* Q}$, we can change the variables from $a_m^{(w)^Q}$ to $d_i$ by $d_i = C_i$. Next consider the case $i > r$. There exist functions $g(i, m) \neq 0$ on $U^*$ such that

$$C_i = \sum_{m=1}^{P} a_m^{(w)} b_m^{(w)^Q} \{ (b_{\tau}^{* Q} - b_{\tau}^{* Q}) \cdots (b_{i}^{* Q} - b_{i-1}^{* Q}) \}$$

for $i < p$.

Q.E.D.
Let \( a_i = \begin{cases} a_{i}^{(w)}, & i = r + 1, \ldots, p, \\ a_{i}^{*}, & i = p + 1, \ldots, p + \tilde{r}, \end{cases} \) and \( b_i = \begin{cases} b_{i}^{(w)}, & \text{if } b_{i}^{*} = 0, \\ b_{i}^{(w)} - b_{i}^{*}, & \text{if } b_{i}^{*} = \beta^{**Q} \neq 0, a_{i}^{**} \neq 0, \\ b_{i}^{(w)} - b_{i}^{*}, & \text{if } b_{i}^{*} = \beta^{**Q} \neq 0, a_{i}^{**} = 0, \\ 0, & \text{if } 1 \leq i \leq p, \\ b_{i}^{*}, & \text{if } i = p + 1, \ldots, p + \tilde{r}. \end{cases} \)

When we distinguish \( a^{*}, b^{*} \) from \( a^{(w)}, b^{(w)} \), we call \( a = a^{*}, b = b^{*} \) constants. Let

\[
J_{i}^{(i)} = b_{i}^{*}, \quad i = 1, \ldots, \tilde{r}, r + 1, \ldots, p + \tilde{r}.
\]  

(2)

Put \( p' = p + \tilde{r} \). Then we have

\[
C_i = \sum_{1 \leq m \leq p'} g(i, m)a_m b_m \prod_{1 \leq i \leq \tau}(b_m - b_i) + \sum_{1 \leq m \leq p'} g(i, m)a_m b_m \prod_{1 \leq i \leq \tau}(b_m - b_i),
\]  

(3)

for \( r < i < p' \). By Lemmas 2 and 3 in [1], the maximum pole of \( f_w \Psi \) and its order are equal to those of \( f_w \Psi \), where

\[
\Psi' = \{d_1^2 + \cdots + d_2^2 + C_2^2 + \cdots + C_{p+1}^2\}^{\tau} \prod_{m=1}^{p'} \prod_{m=m+1}^{p} \prod_{m=m+1}^{r} \prod_{m=m+1}^{r} \prod_{m=m+1}^{p} db_i^{(w)}.
\]  

(4)

Since we often change variables during a blowing-up process, it is more convenient for us to use the same symbols \( \epsilon_m \) rather than \( \epsilon_m^{(w)}, \epsilon_m^{(w')}, \ldots, \) etc, for the sake of simplicity. For instance,

"Let \( \begin{cases} \epsilon_1 = v_{11} \\ \epsilon_m = v_{11} \epsilon_m^{(w)}, \end{cases} \) instead of "Let \( \begin{cases} \epsilon_1 = v_{11} \\ \epsilon_m = v_{11} \epsilon_m^{(w)} \end{cases} \)"

### 3 Proof of Main Theorem 1: Part 1

Take \( J^{(a)} \in \mathbb{R}^a \). Denote \( J^{(a)} = (J^{(a)}, *, \alpha) \) and \( \alpha > \alpha' \) by \( J^{(a)} > J^{(a')} \). Also denote \( J^{(a)} = (0, \ldots, 0) \) by \( J^{(a)} = 0 \). We need the following inductive statements of \( k, K, \alpha \) for calculating poles by using the blowing-up process.

**Inductive statements**

Set \( E = \{m \mid \epsilon_m \text{ is non-constant}\}, E_\tau = \{m \mid J_m^{(a)} = (b_{i}^{**Q}, \ldots, p^**Q, \ldots, K_{f})\}, s(J^{(a)}) = \#(m \mid m \geq k + 1, J_m^{(a)} = J^{(a)}, m \in E), s(I^{(a)}) = \#(m \mid k + 1 \leq m \leq k + 1 - 1, J_m^{(a)} = J^{(a)}, m \in E) \) for \( J^{(a)} \in \mathbb{R}^a \).

(a) \( k = k_0 + \ldots + k_r, K = K_0 + \ldots + K_r, K_0 \geq k_0, \) and \( K_r = k + 1 \) for \( 1 \leq \tau \leq r, \) where \( k_0, \ldots, k_r \in \mathbb{Z}_+ \) and \( K_0, \ldots, K_r \in \mathbb{Z}_+ \). Set \( k_0^{(a)} = k_0 \).

(b) \( \Psi' = \{t_1^{(1)}, t_2^{(1)}, \ldots, t_{r+1}^{(1)}\}(d_1^2 + d_2^2 + \cdots + d_K^2) + \sum_{k=1}^{p'} C_k^2 \prod_{r=0}^{r} \prod_{m=1}^{K+r} db_i^{(w)} \prod_{m=m+1}^{K+r} db_i^{(w)} \prod_{m=m+1}^{p} db_i^{(w)} \).

(5)

Here, \( t_i, r \in \mathbb{Z}_+ \). Also, there exist \( J^{(a)} \in \mathbb{R}^a \), \( t(i, J^{(a)}, (l, \tau)) \in \mathbb{Z}_+ \) and functions \( g(i, m) \neq 0 \) such that

\[
C_i = \sum_{1 \leq m \leq p'} g(i, m)a_m b_m \prod_{1 \leq i \leq \tau}(b_m - b_i) + \sum_{1 \leq m \leq p'} g(i, m)a_m b_m \prod_{1 \leq i \leq \tau}(b_m - b_i),
\]  

(3)

(c) \( J^{(a)} \neq J^{(a')} \) for \( k < \tau < i \leq K, J^{(a)} \in R(J^{(a)}) \cup \{0\} \) for \( k < i \leq K \).

(d) \( t(i, J_m^{(a)}, (l, \tau)) \geq t_{r+2} \) for all \( J_m^{(a)} \), \( i \leq m < p' \) and there exist \( D^{(a)}_0, D^{(a)} \in \mathbb{Z}_+ \) such that

\[
\sum_{J_m^{(a)} = 0} D^{(a)}_0(t(i, J_m^{(a)})) + \sum_{J_m^{(a)} = 0} D^{(a)}_0(t(i, J_m^{(a)})) + \sum_{J_m^{(a)} = 0} D^{(a)}_0(t(i, J_m^{(a)})) + \sum_{J_m^{(a)} = 0} D^{(a)}_0(t(i, J_m^{(a)})).
\]  

(6)
(e) There exist \(g(l, r, \tau') \in \mathbb{Z}_+\), \(\eta_1(l, r, \tau') \in \mathbb{Z}_+\) such that \(t(l, r, \tau') = \frac{1}{2} \sum_{\xi=1}^{g(l, r, \tau')} (1 + \eta_1(l, r, \tau') - \cdots - \eta_{K-1}(l, r, \tau'))\) for all \(\tau'\), and \(g(l, r, \tau') \leq \sum_{j(l, r) \geq j(l, r)} D_{j(l, r)}\) for \(m \in \mathbb{E}_r\).

\[0 \leq \eta_1(l, r, \tau') \leq Q_0 \leq \eta_1(l, r, \tau') + \eta_2(l, r, \tau') \leq 2Q_0, \cdots \leq 2Q_0, 0 \leq \eta_1(l, r, \tau') + \eta_2(l, r, \tau') + \cdots + \eta_{K_0}(l, r, \tau') \leq Q_0, \text{ and} \]

\[0 \leq \eta_1(l, r, \tau') \leq 1, 0 \leq \eta_2(l, r, \tau') + \cdots + \eta_{K-1}(l, r, \tau') \leq 2, \cdots \leq 0 \leq \eta_1(l, r, \tau') + \eta_2(l, r, \tau') + \cdots + \eta_{K-1}(l, r, \tau') \leq K_1 - 1, \eta_1(l, r, \tau') = 0, \text{ for } \tau' \geq 1.\]

(f) Let \(\varphi(l, r, \tau') := \{ s_0 + \eta_1(l, r, \tau') + 2n_2(l, r, \tau') + \cdots + K_0 \eta_1(l, r, \tau'), s_1 + \eta_1(l, r, \tau') + 2n_2(l, r, \tau') + \cdots + K_0 \eta_1(l, r, \tau'), \ldots, \text{ if } \tau' = 0, \]

\[s_1 + \eta_1(l, r, \tau') + 2n_2(l, r, \tau') + \cdots + K_0 \eta_1(l, r, \tau'), \text{ if } 1 \leq \tau' \leq \tilde{\tau}, \]

\[s_1 + \eta_1(l, r, \tau') + 2n_2(l, r, \tau') + \cdots + K_0 \eta_1(l, r, \tau'), \text{ if } \tilde{\tau} < \tau' \leq r.\]

There exist \(\varphi(l, r, \tau') \in \mathbb{Z}_+\) such that

\[q(l, r, \tau') = \sum_{\tau=1}^{q(l, r, \tau')} \varphi(l, r, \tau') + \sum_{m=1}^{q(l, r, \tau')} (-s(l, r, \tau') + \cdots + D_{j(l, r)}), \text{ and} \]

\[q(l, r, \tau') + 1 = \sum_{\tau=1}^{q(l, r, \tau')} \varphi(l, r, \tau').\]

The end of inductive statements.

Statements (d), (e) and (f) are needed to compare poles. The definitions of all variables will be given later on in the proof.

Special Case Fix \(\tilde{\tau}\). We call the case satisfying the following two conditions Special Case.

- \(j_{m}^{(a)} = \text{the same for any } m \in \mathbb{E}_r, \)
- \(s(J_m^{(a)}) = \left\{ \begin{array}{ll}
s_0 - k_f^{(a)} & \text{if } 0 \leq \tilde{\tau} \leq \tilde{\tau}, \\
s_1 - k_f^{(a)} & \text{if } \tilde{\tau} < \tilde{\tau} \leq r, \end{array} \right. \)

\[s(K+1, j^{(a)}) = \left\{ \begin{array}{ll}K_{f} - k_f^{(a)} & \text{if } 0 \leq \tilde{\tau} \leq \tilde{\tau}, \text{ for } j^{(a)} \leq j_{m}^{(a)}. \end{array} \right.\]

Then we have the followings: \(a') \ K_{f} = \left\{ \begin{array}{ll}K_{f} & \text{if } \tilde{\tau} = 0, \\
K_{f} + 1 & \text{if } \tilde{\tau} \geq 1.\end{array} \right. \)

\(d') t(i, j_{m}^{(a)}, (l, r)) = \left\{ \begin{array}{ll}
\sum_{J_{m}^{(a)} \geq J_{m}^{(a)}} D_{j(l, r)} (Q s(i, 0)) + 1, & \text{if } \tilde{\tau} = 0, m \in \mathbb{E}_0, \\
\sum_{J_{m}^{(a)} \geq J_{m}^{(a)}} D_{j(l, r)} (s(i, j^{(a)}) + 1), & \text{if } \tilde{\tau} < \tilde{\tau} \leq r, m \in \mathbb{E}_r, \\
\sum_{J_{m}^{(a)} \geq J_{m}^{(a)}} D_{j(l, r)} (s_i, j^{(a)}), & \text{if } 1 \leq \tilde{\tau} \leq \tilde{\tau}, m \in \mathbb{E}_r. \end{array} \right.\)

\(e') g(l, r, \tau') = \sum_{J_{m}^{(a)} \geq j_{m}^{(a)}} D_{j(l, r)} \frac{t(l, r, \tau')}{2} = \sum_{\xi=1}^{g(l, r, \tau')} (1 + \eta_1(l, r, \tau') + \cdots + \eta_1(l, r, \tau')).\]

For \(j^{(a)} \leq j_{m}^{(a)}\), there are \(\xi\)'s as many as \(D_{j(l, r)} \leq g(l, r, \tau')\) such that \(\eta_{1}(l, r, \tau') \leq \{ Q \text{ if } \tilde{\tau} = 0, \text{ if } \tilde{\tau} \geq 1, \}

\[\eta_1(l, r, \tau') = \cdots = \eta_1(l, r, \tau'), \eta_1(l, r, \tau') = 0, \eta_1(l, r, \tau') = \cdots = \eta_{K-1}(l, r, \tau'), \text{ if } \tilde{\tau} = 0, \]

\[\eta_1(l, r, \tau') = \cdots = \eta_1(l, r, \tau'), \eta_1(l, r, \tau') = 0, \eta_1(l, r, \tau') = \cdots = \eta_{K-1}(l, r, \tau'), \text{ if } 1 \leq \tilde{\tau} \leq r.\]

(f') Set \(\varphi(l, r, \tau') := \{ s_0 + \eta_1(l, r, \tau') + 2n_2(l, r, \tau') + \cdots + K_0 \eta_1(l, r, \tau'), s_1 + \eta_1(l, r, \tau') + 2n_2(l, r, \tau') + \cdots + K_0 \eta_1(l, r, \tau'), \ldots, \text{ if } \tilde{\tau} = 0, \]

\[s_1 + \eta_1(l, r, \tau') + 2n_2(l, r, \tau') + \cdots + K_0 \eta_1(l, r, \tau'), \text{ if } \tilde{\tau} < \tilde{\tau} \leq r, \]

Then \(q(l, r, \tau') = \sum_{\tau=1}^{q(l, r, \tau')} \varphi(l, r, \tau').\)

**Remark 4** Special Case \(E_f\) satisfies \(j^{(a)} = ((b_0^*)^2, 0, 0, 0), j^{(a)} = R \in R^{(a)}\) if \(0 \leq \tilde{\tau} \leq \tilde{\tau}, \text{ for all } m \in \mathbb{E}_f.\)

**3.1 Step 1**

Set \(K_0 = 0, K_1 = K_2 = \cdots = K_r = 1, k_0 = k_1 = \cdots = k_r = 0, \alpha = 1.\)

\(RJ^{(1)} = \{ J_{m}^{(1)} \mid b_i^* \neq 0, a_i^* = 0, r < i \leq p\}, t(l, r) = 0, q_{l, r} = 0, \text{ and } t(i, j(l, r)) = 0, \text{ where } J_{m}^{(1)} \text{ is defined by Eq.}(2)\). Also set all parameters in \((d')~(f)\) by 0. Then, by putting \(e_i = b_i \in \text{Eq.}(3)\), we have the inductive statements when \(K = r, k = 0\) and \(\alpha = 1.\)

**3.2 Step 2**

We assume the case \(k, K, \alpha\). Let us concentrate on \(C_{K+1}\).

Set \(J^{(a)} = \{ J_{m}^{(a)} \in \mathbb{R}^{(a)} \mid (l, r), (K + 1, j_{m}^{(a)}, (l, r)) > t_{l, r}/2, \}

\(JC^{(a)} = \{ J \in \mathbb{R}^{(a)} \mid (K + 1, j_{m}^{(a)}, (l, r)) = t_{l, r}/2 \text{ for all } (l, r)\},\)
$C^{(a)} = \{ m \geq k+1; t(K+1, J_m^{(a)}, (l, \tau)) = t_{r_\tau}/2 \text{ for all } (l, \tau) \}, C^{(a)}_v = \{ m \in C^{(a)}; a_m, e_m \text{ are not constant } \}$,
$\omega = \{ (l, \tau) | 1 \leq l \leq k, 0 \leq \tau \leq \tau \}$ and $\Omega = \{ A \subset \omega | \forall J \in J^{(a)}, \exists (l, \tau) \in A \text{ s.t. } t(K+1, J, (l, \tau)) > \frac{t_{r_\tau}}{2} \}$.

Fix $A^{(a)} \in \Omega$ whose number of elements is in minimum: $\# A^{(a)} = \min_{A \subset \Omega} \# A$.

Also let
$$T_{i, j} = \sum_{(l, \tau) \in A^{(a)}} t_{i, j, (l, \tau)} + \begin{cases} Qs(i, J) + 1, & \text{if } J = 0, J \in J^{(a)}; \\ s(i, J) + 1, & \text{if } J \in J^{(a)} \setminus J^{(a)}, \\ 0, & \text{otherwise}, \end{cases}$$

$$T = \sum_{(l, \tau) \in A^{(a)}} t_{r_\tau} + 2, \quad Q = \sum_{(l, \tau) \in A^{(a)}} q_{r_\tau} + K + \# A^{(a)} + \# C^{(a)} - 1. \quad (7)$$

Let $C^{(a)}_v = \{ m \in C^{(a)} | J_m^{(a)} \notin R^{(a)}, J_m^{(a)} \neq 0, J_m^{(a)} \neq J_m^{(a)} \text{ for all } k < t' \leq K \}$.

**Case 1** $C^*_v \neq \phi$.

### 3.2.1 Transformation in Case 1

If $a_m, e_m$ with $m \in C^{(a)}_v$ are all constants, we reorder those as $a_{k+1}, \ldots, a_{k+l}$, i.e.,
$$C^{(a)}_v = \{ K + 1, \ldots, K + l \}.$$

Then we have
$$C^{K+1}_{K+1} \frac{t_{l, \tau}^{1/2}}{t_{l, \tau}^{1/2}} \cdots \frac{t_{l, \tau}^{1/2}}{t_{l, \tau}^{1/2}}, \frac{t_{l, \tau}^{1/2}}{t_{l, \tau}^{1/2}} \cdots \frac{t_{l, \tau}^{1/2}}{t_{l, \tau}^{1/2}},$$
in the neighborhood of $\{ v_{l, \tau} = \cdots = v_{l, \tau} = 0 \}$. Therefore, we obtain the poles $-\frac{q_{r_\tau} + 1}{t_{r_\tau}}, 0 \leq \tau \leq r, 1 \leq k'' \leq k_r$.

If there exist non-constants $a_m, e_m$ with $m \in C^{(a)}_v$, we can assume that $K + 1 \in C^{(a)}_v$ by reordering and that $K + 1 \in E$. Then we have $C^{K+1}_{K+1} \frac{t_{l, \tau}^{1/2}}{t_{l, \tau}^{1/2}} \cdots \frac{t_{l, \tau}^{1/2}}{t_{l, \tau}^{1/2}}, \frac{t_{l, \tau}^{1/2}}{t_{l, \tau}^{1/2}} \cdots \frac{t_{l, \tau}^{1/2}}{t_{l, \tau}^{1/2}}$. Put $K_r \rightarrow K + 1 + 1$ and we have the inductive statements of $K \rightarrow K + 1$.

**Case 2** $C^*_v = \phi$.

Construct the blowing-up of $\Psi'$ in (5) along the submanifold $\{ d_1 = \cdots = d_K = v_\tau = e_m = 0, (l, \tau) \in A^{(a)}, m \in C^{(a)} \}$.

### 3.2.2 Transformation (i)

Let $d_1 = u_1, d_2 = u_1d_2, 2 \leq \ell \leq K, v_\tau = u_1 v_\tau, (l, \tau) \in A^{(a)}$ and $e_m = u_1e_m, m \in C^{(a)}$. Substituting them into Eq.(5) gives the poles
$$-\frac{\sum_{(l, \tau) \in A^{(a)}} q_{r_\tau} + K + \# A^{(a)} + \# C^{(a)}}{\sum_{(l, \tau) \in A^{(a)}} t_{r_\tau} + 2} \frac{q_{r_\tau} + 1}{t_{r_\tau}}, 0 \leq \tau \leq r, 1 \leq l \leq k_r.$$

### 3.2.3 Transformation (ii)

Fix $\{ k', \tau' \} \in A^{(a)}$. Let $d_{k'} = v_{k'} u_1, 1 \leq \ell \leq K, v_\tau = u_1 v_\tau, (l, \tau) \in A^{(a)} - \{ (k', \tau') \}$ and $e_m = v_{k'} e_m, m \in C^{(a)}$. Set $t_{k', \tau'} \rightarrow t, t(i, J, (k', \tau')) \rightarrow t_{i, j},$ and $q_{k', \tau'} \rightarrow Q$ by using Eq.(6), (7).

Then, we have $t_{k', \tau'}/2 = (\sum_{(l, \tau) \in A^{(a)}} t_{r_\tau} + 2)/2 = \sum_{(l, \tau) \in A^{(a)}} t_{r_\tau} + (1 + \eta_{1, (l, \tau), r^{(\ell)}} + \cdots + \eta_{K_r, (l, \tau), r^{(\ell)}}) + 1,$
and $q_{k', \tau'} + 1 = \sum_{(l, \tau) \in A^{(a)}} (q_{r_\tau} + 1) + K + \# C^{(a)} = \sum_{(l, \tau) \in A^{(a)}} \left\{ \sum_{r^{(\ell)}} \eta_{1, (l, \tau), r^{(\ell)}} + \cdots + \eta_{K_r, (l, \tau), r^{(\ell)}} \right\} + K + \sum_{(l, \tau) \in A^{(a)}} t_{r_\tau} + 2$.
\[ + \sum_{r=0}^{r} \sum_{(l,r) \in A^{(a)}} \left( g(l,r),r + \sum_{j=1}^{r} \sum_{(l,r) \in A^{(a)}} D_{j(n),(l,r)} + 1 \right) + \sum_{(l,r) \in A^{(a)}} [\phi(l,r),r + K]. \]

Put \( D_{j(n),(l,r)} \rightarrow \sum_{(l,r) \in A^{(a)}} D_{j(n),(l,r)} + 1 \) if \( j(n) \in JC^{(a)}. \) Also, put \( D_{j(n),(l,r)} \rightarrow \sum_{(l,r) \in A^{(a)}} D_{j(n),(l,r)} \) if \( j(n) \notin JC^{(a)}. \) Then we have (d) and that \( q_{k',r'} + 1 = \sum_{(l,r) \in A^{(a)}} \left( \sum_{\xi=1}^{g_{(k',r'),\overline{r}}} \varphi_{(k',r'),\overline{r}} + \sum_{K_{n},(k',r'),\overline{r}} + \sum_{(l,r) \in A^{(a)}} \phi(l,r),r \right) \]

For all \( 0 \leq r' \leq r, \) one of the following steps (I), (II), Special Case 1 and Special Case 2 is proceeded with respect to each condition of the case \( r'. \) Fix \( r' = \tilde{r}. \)

(I) Assume that there exist \( \{\xi(0), \theta(0), \tilde{\tau}\} \) such that \( 0 \leq \eta_{1,(l_{0},r_{0})}, \theta_{2,(l_{0},r_{0})}, + \cdots + \eta_{K_{t},(l_{0},r_{0})}, \tilde{\tau} < \sum_{(l,r) \in A^{(a)}} g(l,r),\tilde{r} + 1 \).

\[ \sum_{\xi=1}^{g_{(k',r'),\overline{r}}} \varphi_{(k',r'),\overline{r}} + \sum_{K_{n},(k',r'),\overline{r}} + \sum_{(l,r) \in A^{(a)}} \phi(l,r),r \]

By the assumption (I), there exists \( \xi_{0} \) such that \( 0 \leq \eta_{1,(l_{0},r_{0})},\theta_{2,(l_{0},r_{0})}, + \cdots + \eta_{K_{t},(l_{0},r_{0})}, \tilde{\tau} < \sum_{(l,r) \in A^{(a)}} g(l,r),\tilde{r} + 1 \).

(II) Assume \( 0 \leq \eta_{1,(l_{0},r_{0})},\theta_{2,(l_{0},r_{0})}, + \cdots + \eta_{K_{t},(l_{0},r_{0})}, \tilde{\tau} = \sum_{(l,r) \in A^{(a)}} g(l,r),\tilde{r} + 1 \).

The assumption (d) gives \( \sum_{J_{r}^{(a)}} D_{j(n),(l_{r})}(Q_{s}(1,0^{(a)}) + 1) + \sum_{j=1}^{r} D_{j(n),(l_{r})} s(t_{r}^{(a)} + 1) + \sum_{(l,r) \in A^{(a)}} \phi(l,r),n \)

for all \( J_{r}^{(a)} \in \mathbb{R}, i \leq m \leq p^{(b)}. \) If not Special Case, we have \( g(l,r),\tilde{r} \leq \sum_{J_{r}^{(a)}} D_{j(n),(l_{r})} \) for \( m \in E_{r}, \)

since \( s(K_{r} + 1, J_{r}^{(a)}) \leq \sum_{(l,r) \in A^{(a)}} D_{j(n),(l_{r})} \leq \sum_{J_{r}^{(a)}} D_{j(n),(l_{r})} \) for \( m \in E_{r}, \)

Let \( \psi_{(k',r'),\overline{r}} = \sum_{(l,r) \in A^{(a)}} \varphi_{(l,r),\overline{r}} + \sum_{k_{1} \leq \sum_{m \in E_{r}} g_{(k_{1},r'),\overline{r}} + 1} + \sum_{(l,r) \in A^{(a)}} \phi(l,r),\overline{r} - 1 \]

Put \( \psi_{(k',r'),\overline{r}} = \sum_{(l,r) \in A^{(a)}} g(l,r),\overline{r} + 1, \) which satisfies \( g_{(k',r'),\overline{r}} \leq \sum_{J_{r}^{(a)}} D_{j(n),(l_{r})} \) by (8). Set \( \psi_{(k',r'),\overline{r}} = \left\{ \begin{array}{ll} s_{r} & \text{if } 0 \leq \tilde{r} \leq r, \\
 s_{r} + 1 & \text{if } 1 \leq \tilde{r} \leq \tilde{r}. \end{array} \right. \) Let \( \psi_{(k',r'),\overline{r}} \) be \( \psi_{(l,r),\overline{r}} \) for \( (l,r) \in A^{(a)}. \)
\[ \xi = 1, \ldots, g(l, r), \), \] and \( \eta^{(1)}_{(k', \tau'), \overline{r}}, \ldots, \eta^{(g(k', \tau'), \overline{r}), \overline{r})} \) be \( \eta^{(\xi)}_{(l, r), \overline{r}}, (l, r) \in A^{(a)}, \xi = 1, \ldots, g(l, r), \), by numbering in the same order for every \( \ell \).

Put \( \eta^{(g(k', \tau'), \overline{r})} = 0 \) for \( \ell \geq 1 \), and \( \phi^{(l, r), \overline{r}} = \left\{ \begin{array}{ll} -k + k + \sum_{(l, r) \in A^{(a)}} \phi^{(l, r), \overline{r}}, & \text{if } \tilde{r} = 0, \\
- k + k - \sum_{(l, r) \in A^{(a)}} \phi^{(l, r), \overline{r}}, & \text{if } 1 \leq \tilde{r} \leq r. 
\end{array} \right. \)

Then we obtain
\[ \frac{t_{k', \tau'}}{2} = \sum_{\xi=1}^{g(k', \tau'), \overline{r}} (1 + \eta^{(\xi)}_{(l, r), \overline{r}}) + \sum_{k+1 \leq m \leq J_{m}^{(a)}} D_{J_{m}^{(a)}, (l, r)}(K_{0} - k_{0}^{(a)}), \quad \sum_{J_{n}^{(a)} \geq J^{(l)}} D_{J^{(l)}, (l, r)}(Q(K_{0} - k_{0}^{(a)}), 1), \quad \text{if } \tilde{r} = 0, \]
\[ \sum_{k+1 \leq m \leq J_{m}^{(a)}} D_{J_{m}^{(a)}, \overline{r}}(K_{0} - k_{0}^{(a)}), 1), \quad \text{if } 1 \leq \tilde{r} \leq r. \]

We have
\[ \frac{t_{k', \tau'}}{2} = \sum_{\xi=1}^{g(k', \tau'), \overline{r}} (1 + \eta^{(\xi)}_{(l, r), \overline{r}}) + \sum_{k+1 \leq m \leq J_{m}^{(a)}} D_{J_{m}^{(a)}, \overline{r}}(K_{0} - k_{0}^{(a)}), 1), \quad \sum_{J_{n}^{(a)} \geq J^{(l)}} D_{J^{(l)}, \overline{r}}(Q(K_{0} - k_{0}^{(a)}), 1), \quad \text{if } \tilde{r} = 0, \]
\[ \sum_{k+1 \leq m \leq J_{m}^{(a)}} D_{J_{m}^{(a)}, \overline{r}}(K_{0} - k_{0}^{(a)}), 1), \quad \text{if } 1 \leq \tilde{r} \leq r. \]

The end of (II)

**Special Case 1** Assume that \( J_{m}^{(a)} \notin JC^{(a)} \) for all \( m \in E_{\tau} \). By \( (e') \), we have
\[ \frac{t_{k', \tau'}}{2} = \sum_{(l, r) \in A^{(a)}} g^{(l, r), \overline{r}}. \]

By using \( t(K + 1, J_{m}^{(a)}, (l, r)) = \sum_{(l, r) \in A^{(a)}} t(K + 1, J_{m}^{(a)}, (l, r)) \geq \frac{t_{k', \tau'}}{2} \) and
\[ t(K + 1, J_{m}^{(a)}, (k', \tau')) = \left\{ \begin{array}{ll} \sum_{(l, r) \in A^{(a)}} \sum_{J_{n}^{(a)} \geq J^{(l)}} D_{J^{(l)}, (l, r)}(Q(K_{0} - k_{0}^{(a)}), 1), & \text{if } \tilde{r} = 0, \\
\frac{t_{k', \tau'}}{2} = \sum_{\xi=1}^{g(k', \tau'), \overline{r}} (1 + \eta^{(\xi)}_{(l, r), \overline{r}}) + \sum_{k+1 \leq m \leq J_{m}^{(a)}} D_{J_{m}^{(a)}, \overline{r}}(K_{0} - k_{0}^{(a)}), 1), & \text{if } 1 \leq \tilde{r} \leq r. 
\end{array} \right. \]

So, there exist \( (l_{0}, r_{0}) \in A^{(a)} \) and \( \xi_{0} \) such that \( \eta^{(\xi_{0})}_{(l_{0}, r_{0}), \overline{r}} < \left\{ \begin{array}{ll} Q & \text{if } \tilde{r} = 0, \\
1 & \text{if } 1 \leq \tilde{r} \leq r. \end{array} \right. \)

Put
\[ g^{(k', \tau'), \overline{r}} = \sum_{(l, r) \in A^{(a)}} g^{(l, r), \overline{r}}. \]

Let \( \varphi^{(1)}(k', \tau'), \ldots, \varphi^{(g(k', \tau'), \overline{r})}(k', \tau'), (l, r) \in A^{(a)}, \xi = 1, \ldots, g(l, r), \), be \( \varphi^{(\xi)}(l, r), (l, r) \in A^{(a)}, \xi = 1, \ldots, g(l, r), \), by numbering in the same order for all \( \ell \). Choose \( \xi_{1} \) with \( \varphi^{(\xi_{1})}_{(k', \tau'), \overline{r}} = \varphi^{(\xi_{1})}_{(l_{0}, r_{0}), \overline{r}} \) and \( \varphi^{(\xi_{1})}_{(k', \tau'), \overline{r}} \rightarrow \varphi^{(\xi_{1})}_{(k', \tau'), \overline{r}} \rightarrow \varphi^{(\xi_{1})}_{(k', \tau'), \overline{r}} \rightarrow K_{\overline{r}}. \)

Then we obtain
\[ \frac{t_{k', \tau'}}{2} = \sum_{\xi_{1}=1}^{g(k', \tau'), \overline{r}} (1 + \eta^{(\xi_{1})}_{(k', \overline{r}), \overline{r}}) + \sum_{\xi_{1}=1}^{g(k', \tau'), \overline{r}} \eta^{(\xi_{1})}_{(k', \overline{r}), \overline{r}} + q^{(g(k', \tau'), \overline{r})}, \]

The end of Special Case 1

**Special Case 2** Assume that \( J_{m}^{(a)} \in JC^{(a)} \) for all \( m \in E_{\tau} \). Since \( (e') \) yields
\[ t_{l_{n}} / 2 = \sum_{\xi=1}^{g_{(l_{n}), \overline{r}} \geq J_{m}^{(a)}} \eta^{(\xi)}_{(l_{n}), \overline{r}}, \quad \sum_{J_{n}^{(a)} \geq J^{(l)}} D_{J^{(l)}, (l, r)}(Q(K_{0} - k_{0}^{(a)}), 1), \quad \text{if } \tilde{r} = 0, \]
we have
\[ \sum_{\xi=1}^{g_{(l_{n}), \overline{r}} \geq J_{m}^{(a)}} \eta^{(\xi)}_{(l_{n}), \overline{r}} = \left\{ \begin{array}{ll} Q & \text{if } \tilde{r} = 0, \\
1 & \text{if } 1 \leq \tilde{r} \leq r. \end{array} \right. \]

for all \( \xi = 1, \ldots, g(l, r) \). Put
\[ g^{(k', \tau'), \overline{r}} = \sum_{(l, r) \in A^{(a)}} g^{(l, r), \overline{r}}. \]

Let \( \varphi^{(1)}(k', \tau'), \ldots, \varphi^{(g(k', \tau'), \overline{r})}(k', \tau'), (l, r) \in A^{(a)}, \xi = 1, \ldots, g(l, r), \), and \( \eta^{(1)}_{(k', \tau'), \overline{r}}, \ldots, \eta^{(g(k', \tau'), \overline{r})}_{(k', \tau'), \overline{r}} \) be \( \eta^{(\xi)}_{(l, r), \overline{r}}, (l, r) \in A^{(a)}, \xi = 1, \ldots, g(l, r), \), by numbering in the same order for any \( \ell \). Then we have
\[ \frac{t_{k', \tau'}}{2} = \sum_{\xi=1}^{g_{(k', \tau'), \overline{r}} \geq J_{m}^{(a)}} (1 + \eta^{(\xi)}_{(k', \tau'), \overline{r}}) + \sum_{J_{n}^{(a)} \geq J^{(l)}} D_{J^{(l)}, (l, r)}(Q(K_{0} - k_{0}^{(a)}), 1), \]

\begin{align*}
\eta_{\ell, (k', \tau'), \overline{\tau}}^{(\xi), \overline{\sigma}} + 1 \quad \text{and} \quad q_{(k', \tau'), \overline{\tau}} = \sum_{\xi=1}^{g_{(k', \tau'), \overline{\tau}}^{(\xi), \overline{\sigma}}} \psi_{(k', \tau'), \overline{\tau}}^{(\xi), \overline{\sigma}}(K_{\overline{\tau}} + s_{\overline{\tau}} - k_{\overline{\tau}}, \overline{\tau}) + \begin{cases} K_{\overline{\tau}} + s_{\overline{\tau}} - k_{\overline{\tau}}, & \text{if } \tilde{r} = 0 \\ K_{\overline{\tau}} + s_{\overline{\tau}} - k_{\overline{\tau}}, & \text{if } 1 \leq \tilde{r} \leq \tilde{r} \\ K_{\overline{\tau}} + s_{\overline{\tau}} - k_{\overline{\tau}}, & \text{if } \tilde{r} < \tilde{r} \leq \tilde{r} \\
\end{cases} \sum_{\xi=1}^{g_{(k', \tau'), \overline{\tau}}^{(\xi), \overline{\sigma}}} \psi_{(k', \tau'), \overline{\tau}}^{(\xi), \overline{\sigma}}
\end{align*}

The end of Special Case 2

3.2.4 Transformation (iii)

Fix $k_{c} \in C'_{(\alpha)}$. Assume that $k_{c} \in E_{\tau'}$. Let $d_{\ell} = u_{k_{c}+1, \tau'}d_{\ell}, 1 \leq \ell \leq K, e_{k_{c}} = u_{k_{c}+1, \tau'}, u_{\tau'} = v_{k_{c}+1, \tau'}u_{\tau'}(\tau, \tau') \in A'(\alpha)$ and $e_{m} = u_{k_{c}+1, \tau'}e_{m}, m \in C_{(\alpha)} - \{k_{c}\}$.

(A) If $k < k_{c} \leq K$, then we can assume $k_{c} = k + 1$ by the symmetry of the formulas $C_{i}$.

(III) Do the same procedure in Transformation (ii) by substituting $(k_{c} + 1, \tau')$ into $(k', \tau')$.

Adding it, set $\phi_{(\ell, \tau), \tau'} = -g(\ell, \tau', \overline{\tau}) + \sum_{j_{c}^{(\alpha)} \geq j_{(\ell, \tau), \tau'}} D_{j_{(\ell, \tau), \tau'}}(\tau') + \phi_{(\ell, \tau), \tau''}$ for all $\ell, \tau$ and $\tau''$.

The end of (III)

Then, putting $k_{c}' = k_{c} + 1$ and $k \rightarrow k + 1$ completes Statements (a)~(f).

Special Case We need (a') for Special Case. After setting in (A), we have $k + 1 \in C'_{(\alpha)}$. So, we can transform in Case 1 with non-constants $a_{k+1}, e_{k+1}$. Then we have $K_{\tau'} \rightarrow K_{\tau'} + 1$ and the inductive statements of $K \rightarrow K + 1$.

(B) Next consider the case $K_{\tau'} + 1 \leq k_{c} \leq p'$.

If $j_{k_{c}}^{(\alpha)} \notin R(J^{(\alpha)}$ and $J_{k_{c}}^{(\alpha)} \neq 0$, then there exists $i' \leq K$ such that $J_{i'}^{(\alpha)} = j_{k_{c}}^{(\alpha)}$ because of the Case 2 assumption. By the symmetry of the formulas $C_{i}$, we can assume $i' = k_{c}$. So the case results in (A). Consider the case $J = J_{k_{c}}^{(\alpha)} \in R(J$ or $J = j_{k_{c}}^{(\alpha)} = 0$. Again by the symmetry, we can assume that $k_{c} = K + 1$. By the transformation (iii), we have $\eta_{k_{c}+1, \tau'} = \frac{C_{k_{c}+1}}{K_{\tau'} + 1, K_{\tau'} + 1} = \frac{v_{k_{c}+1}a_{K_{\tau'} + 1, K_{\tau'} + 1} + \cdots}{\prod_{m}(u_{k_{c}+1, \tau'}v_{k_{c}+1, \tau'})^{2}}$. Now, there is no $a_{K_{\tau'} + 1}$ in the formulas $C_{i}$, $i \geq K + 2$. So, we can change the variable from $a_{K_{\tau'} + 1}$ to $d_{K_{\tau'} + 1}$ by $d_{K_{\tau'} + 1} = a_{K_{\tau'} + 1, \tau'}(K + 1, K + 1) + \cdots$.

By repeating Step 2, the conditions Case 2 (ii), (iii)(A) disappear. So, $K$ is increased with these finite steps. $K = p + 1$ completes the blowing-up process.

4 Proof of Main Theorem 1 : Part 2

Part 1 shows the blowing-up process. To obtain the maximum pole and its order, we prepare the following theorem.
Theorem 2  Assume that all \( B^{(w)} \) are Special Case. Then we have the pole \( \lambda_Q^* \) whose order \( \theta \) in Main Theorem 1.

4.1 Proof of Theorem 2 : Step 1
Let \( \zeta_1 \) be the total number of \( s_1 = 1 \) and \( \zeta_2 \) the total number of \( s_1 = 1 \) with \( a^{**}_{s} \neq 0 \) among \( 1 \leq r \leq \tilde{r} \).
We can assume that \( s_1 = \ldots = s_{\zeta_1} = 1 \), \( B^{(w)} = \{ \tau \} \) for \( 1 \leq \tau \leq \zeta_2 \) and \( s_{\tau+1} = \ldots = s_{\tau+\zeta_2} = 1 \), \( B^{(\tau)} = \{ \tau \} \) for \( \tilde{r} + 1 \leq \tau \leq \zeta_1 - \zeta_2 + \tilde{r} \). Also, we can assume that \( a_{r+1} = a^{**}_{s} \), \ldots , \( a_{r+\zeta_2} = a^{**}_{s} \)
in Eq.(3). Then \( C_{r+1} = \sum_{m=1}^{\zeta_2} g(r, m) a^{**}_{m} b_m + \sum_{m=1}^{\zeta_2} g(r+1, m) a^{**}_{m} b_m + \sum_{m=1}^{\zeta_2} g(r+2, m) a^{**}_{m} b_m + \sum_{m=1}^{\zeta_2} g(r+3, m) a^{**}_{m} b_m + \sum_{m=1}^{\zeta_2} g(r+4, m) a^{**}_{m} b_m + \sum_{m=1}^{\zeta_2} g(r+5, m) a^{**}_{m} b_m \)
by \( d_{r+1} = C_{r+1} \). Also we have \( C_{r+2} = \sum_{m=2}^{\zeta_2} g(r+1, m) a^{**}_{m} b_m + \sum_{m=2}^{\zeta_2} g(r+2, m) a^{**}_{m} b_m + \sum_{m=2}^{\zeta_2} g(r+3, m) a^{**}_{m} b_m + \sum_{m=2}^{\zeta_2} g(r+4, m) a^{**}_{m} b_m + \sum_{m=2}^{\zeta_2} g(r+5, m) a^{**}_{m} b_m \)
from \( b_2 \) to \( d_{r+2} = C_{r+2} \).

Note that \( b_r \) with \( s_r = 1 \) disappear in the expressions \( C_1 \) because \( g(i,m) \)’s include \( b_r \). Put \( Y' = \# \Theta \).
For simplicity, number the set
\[
\Theta = \{ \tau_{i_1}, \ldots, \tau_{i_Y} \} = \left\{ \{ \tau_{i_1}, \ldots, \tau_{i_Y} \} \mid \begin{array}{l}
Q(n_{1}^{2} - n_0) + 2n_0 = 2s_{m}, \\
n_{1} > 1, s_0 = 0,
\end{array} \right\}
\]
and \( \{ \tau_{i_1}, \ldots, \tau_{i_Y} \} = \{ \tau_{m}, \tau_{i_1}, \ldots, \tau_{i_Y}, \} \) if \( 1 \leq \tau \leq Y \). Then we have \( n_{i} \leq s_{i} \), if \( 1 \leq \tau \leq r \). Let \( I \) be the lowest common multiple of \( n_{1}, \ldots, n_{Y} \) and let \( \xi = \frac{1}{n_{i}} \) for \( 1 \leq i \leq Y \). The greatest common divisor of \( \xi \)’s is clearly 1. Using induction, we can construct a \( Y \times Y \) matrix \( L_1 = (l_{i,j}) \in M_Y(Z) \) such that \( (1) \ l_{i,j}^{(1)} = \xi \) for \( i = 1, \ldots, Y \), (2) \( \det L_1 = 1 \), (3) \( l_{i,j}^{(1)} : \) for any \( i \geq j \), (4) \( n_{j-1}^{(1)} / n_{j} \), (5) \( \frac{1}{r} \leq \xi \leq 1 \).

Assume that \( Y = 2 \). Since the greatest common divisor of \( \xi_1 \) and \( \xi_2 \) is 1, there exist \( l_{1,2}^{(1)} \) and \( l_{1,2}^{(2)} \) such that \( \xi_{1} - \xi_{2} \). Then \( \xi_{2} / \xi_1 = \xi_{2} / \xi_1 \).

Let \( \xi \) be the greatest common divisor of \( \xi_2, \ldots, \xi_Y \) and let \( \xi = \xi_{i} \). It is clear that the greatest common divisor of \( \xi_1 \)’s is 1. We can assume that we have a matrix \( L_2 = (l_{i,j}) \), \( 2 \leq i, j \leq Y \), satisfying \( (1)-(4) \) for \( \xi_1, \xi_2, \ldots, \xi_Y \).

Since the greatest common divisor of \( \xi_1 \) and \( \xi_2 \) is 1, there exist \( l_{1,2}^{(1)} \) and \( l_{2,2}^{(2)} \) such that \( l_{1,2}^{(1)} - l_{1,2}^{(2)} \). Then we have \( n_{j-1}^{(1)} / n_{j} \).

Let \( L_1 = \begin{pmatrix}
\xi_1 & 0 & \ldots & 0 \\
\xi_2 & \xi_1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\xi_Y & \xi_Y & \ldots & \xi_1 \\
\end{pmatrix}
\). We have det \( L_1 = l_{1,2}^{(1)} \xi / l_{1,2}^{(2)} \xi_2 \) det \( L_2 = l_{1,2} \xi / l_{1,2}^{(1)} \xi_2 \) det \( L_2 = -\det L_2 \).

So, \( L_1 \) satisfies the conditions (1)-(4). Next construct a \( Y \times Y \) matrix
\[
L = (l_{i,j}) \in M_Y(Z)
\]
such that \( (i) \ l_{i,j}^{(1)} = \xi_i \) for \( i = 1, \ldots, Y \), (ii) \( \det L = \pm 1 \), (iii) for every pair of \( (\beta_i, \beta_Y) \) with \( 1 \leq \beta_i \leq n_{Y} \), \( 1 \leq \beta_Y \leq n_{i} \), \( 1 \leq i \leq Y \), there exists \( 0 \leq j \leq Y \) such that \( \beta_i l_{i,j} = \beta_Y l_{Y,j}, \beta_i l_{i,j} > \beta_Y l_{Y,j} \).
\[ \beta_{i+1} > \cdots > \beta_{v} \text{ for } i \geq j, \] and
\[ \beta_{i-1} > \cdots > \beta_{v} \text{ for } i < j. \]
(iii) \( I := n_{ij} \) for \( i, j \), (iv) \( n'_{i,j} > I, i < j \).

For any positive integers \( M_{1,2} \), let \( L^{(2)} = (l_{1}^{(2)} \cdots l_{M_{Y}}^{(2)}) \) satisfy
\[ M_{1,2} \text{'s are large enough, then } \alpha_{i+1}^{(2)} > \beta_{i}^{(2)} \text{ for all } i. \] Also we have \( I_{i}^{(2)} := I_{i}^{(1)} + \sum_{j=1}^{M_{Y}} I_{ij} \geq 2 \). That is, \( I_{i}^{(2)} = n_{i}^{(2)} \), \( i \geq j \), and
\[ n'_{i,j} > I_{i}^{(2)}, (j \geq 2) \text{, where } I_{i}^{(2)} := I_{i}^{(3)} = I_{i}^{(1)} + \sum_{j=1}^{M_{Y}} I_{ij} \text{ for } j \geq 2. \]

Again by using large integers \( M_{2,2} \) and by setting \( L^{(3)} = (l_{1}^{(3)} \cdots l_{M_{Y}}^{(3)}) \), we have
\[ M_{1,2} \text{'s are large enough, then } \alpha_{i+1}^{(3)} > \beta_{i}^{(3)} \text{ for all } i. \] Also we have \( \alpha_{i+1}^{(3)} > \beta_{i}^{(3)} \text{ for } j \geq 2 \text{ when } \alpha_{i}^{(3)} = \beta_{i}^{(3)} \text{, } \alpha_{i+1}^{(3)} > \beta_{i}^{(3)} \text{ for } j \geq 2 \text{.}

From \( I_{i}^{(3)} := I_{i}^{(2)} + \sum_{j=1}^{M_{Y}} I_{ij} \), \( i \geq j \geq 3 \), we have
\[ n'_{i,j} > I_{i}^{(3)}, (j \geq 2) \text{, where } I_{i}^{(3)} := I_{i}^{(3)} = I_{i}^{(1)} + \sum_{j=1}^{M_{Y}} I_{ij} \text{ for } j \geq 2. \]

Again by using large integers \( M_{2,2} \) and by setting \( L^{(3)} = (l_{1}^{(3)} \cdots l_{M_{Y}}^{(3)}) \), we have
\[ M_{1,2} \text{'s are large enough, then } \alpha_{i+1}^{(3)} > \beta_{i}^{(3)} \text{ for all } i. \] Also we have \( \alpha_{i+1}^{(3)} > \beta_{i}^{(3)} \text{ for } j \geq 2 \text{ when } \alpha_{i}^{(3)} = \beta_{i}^{(3)} \text{, } \alpha_{i+1}^{(3)} > \beta_{i}^{(3)} \text{ for } j \geq 2 \text{.}

From \( I_{i}^{(3)} := I_{i}^{(2)} + \sum_{j=1}^{M_{Y}} I_{ij} \), \( i \geq j \geq 3 \), we have
\[ n'_{i,j} > I_{i}^{(3)}, (j \geq 2) \text{, where } I_{i}^{(3)} := I_{i}^{(3)} = I_{i}^{(1)} + \sum_{j=1}^{M_{Y}} I_{ij} \text{ for } j \geq 2. \]

Again by using large integers \( M_{2,2} \) and by setting \( L^{(3)} = (l_{1}^{(3)} \cdots l_{M_{Y}}^{(3)}) \), we have
\[ M_{1,2} \text{'s are large enough, then } \alpha_{i+1}^{(3)} > \beta_{i}^{(3)} \text{ for all } i. \] Also we have \( \alpha_{i+1}^{(3)} > \beta_{i}^{(3)} \text{ for } j \geq 2 \text{ when } \alpha_{i}^{(3)} = \beta_{i}^{(3)} \text{, } \alpha_{i+1}^{(3)} > \beta_{i}^{(3)} \text{ for } j \geq 2 \text{.}

From \( I_{i}^{(3)} := I_{i}^{(2)} + \sum_{j=1}^{M_{Y}} I_{ij} \), \( i \geq j \geq 3 \), we have
\[ n'_{i,j} > I_{i}^{(3)}, (j \geq 2) \text{, where } I_{i}^{(3)} := I_{i}^{(3)} = I_{i}^{(1)} + \sum_{j=1}^{M_{Y}} I_{ij} \text{ for } j \geq 2. \]
\( \sigma_{B_{L}} \in \Delta' \). \( \Delta' \) is clearly a refinement of \( \tilde{\Delta} \). Since the toric variety \( X(\Delta') \) defined by \( \Delta' \) is non-singular, we have the proper map \( \pi: X(\Delta') \to X(\Delta) \cong \mathbb{R}^{r'} \) such that \( \pi_{\sigma}: \mathbb{R}^{r'} = (v_{1}^{\sigma}, \ldots, v_{r}^{\sigma}) \to U_{\sigma} \to \mathbb{R}^{r'} \), where \( A_{\sigma} = (a_{1}, \ldots, a_{p'}) \), \( \sigma = \sum_{i=1}^{r} a_{i} \in \Delta' \) and \( U_{\sigma} \) is the related subset of \( X(\Delta') \) to \( \sigma \). In particular, \( u = \pi_{\sigma_{B_{L}}} \in \mathbb{R}^{r'} \). Using \( B_{L} = \pm 1 \), we have:

\[
\mathrm{d}\pi_{\sigma_{B_{L}}} = \pm 1 (u_{1L}^{s_{1}'+1} v_{12}^{s_{1}'-1} + \cdots + u_{Y2}^{s_{Y}-1} v_{Y1}^{s_{Y}'+1}) \prod_{j=1}^{Y} (v_{11}^{l_{j1} s_{j}'-1} v_{12}^{l_{j2} s_{j}'-1} \cdots v_{1Y}^{l_{jY} s_{j}'-1} v_{22}^{l_{j1} s_{j}'-1} v_{32}^{l_{j2} s_{j}'-1} \cdots v_{s_{j}Y}^{1}).
\]

The differential form \( \prod_{j=1}^{Y} u_{ij}^{s_{j}'-1} \cdots d v_{ij} \) is:

\[
\prod_{j=1}^{Y} u_{ij}^{s_{j}'-1} \cdots d v_{ij} = \prod_{j=1}^{Y} (v_{1i}^{s_{j}'-1} v_{1j}^{s_{j}'-1} \cdots v_{ij}^{s_{j}'-1} v_{2j}^{s_{j}'-1} \cdots v_{s_{j}j}^{0} \prod_{j=1}^{Y} (v_{1j}^{l_{j1} s_{j}'-1} v_{12}^{l_{j2} s_{j}'-1} \cdots v_{1Y}^{l_{jY} s_{j}'-1} v_{2j}^{l_{j1} s_{j}'-1} v_{3j}^{l_{j2} s_{j}'-1} \cdots v_{s_{j}j}^{0}).
\]

Therefore, the differential form \( \Psi' = (d_{1}^{2} + \cdots + d_{2}^{2} + C_{r}^{2} + \cdots + C_{p'}^{2}) \) is:

\[
\prod_{j=1}^{Y} u_{ij}^{s_{j}'-1} \cdots d v_{ij} = \prod_{j=1}^{Y} (v_{1j}^{l_{j1} s_{j}'-1} v_{12}^{l_{j2} s_{j}'-1} \cdots v_{s_{j}j}^{0} \prod_{j=1}^{Y} (v_{1j}^{l_{j1} s_{j}'-1} v_{12}^{l_{j2} s_{j}'-1} \cdots v_{s_{j}j}^{0}).
\]

By constructing a blowing-up of \( \Psi' \) on \( U_{\sigma_{B_{L}}} \), let us show inductively that we obtain:

\[
P_{\Psi'}(v_{1}^{\sigma}, \cdots, v_{p}^{\sigma}, v_{p+1}^{\sigma}) = C_{1} \cdots C_{p} \prod_{j=1}^{Y} (v_{1j}^{l_{j1} s_{j}'-1} v_{12}^{l_{j2} s_{j}'-1} \cdots v_{s_{j}j}^{0}).
\]

where \( C_{i} = \sum_{j=1}^{Y} (v_{1j}^{l_{j1} s_{j}'-1} v_{12}^{l_{j2} s_{j}'-1} \cdots v_{s_{j}j}^{0}) \prod_{m=1}^{r} d_{m}. \]

Put \( V_{j} = v_{1j}^{l_{j1} s_{j}'-1} v_{12}^{l_{j2} s_{j}'-1} \cdots v_{s_{j}j}^{1} \), then we have:

\[
C_{i} = \sum_{j=1}^{Y} (v_{1j}^{l_{j1} s_{j}'-1} v_{12}^{l_{j2} s_{j}'-1} \cdots v_{s_{j}j}^{1}) \prod_{m=1}^{r} d_{m}. \]

where \( k_{i} = s(K+1,i) \), \( K = K_{1} + \cdots + K_{Y} + \zeta + \zeta_{2} \), \( t_{i} = \min \{1, \ldots, Y\} (l_{j1}(Q_{j}k_{j} + 1) - t_{i}) \) for \( 1 \leq i \leq Y \). We have Eq. (13) with \( k_{i} = 0 \) and \( q_{i} = \sum_{j=1}^{Y} (l_{j1}s_{j}^{1} + (K + \zeta_{1} + \zeta_{2})t_{i} - 1 - (\zeta_{1} + \zeta_{2})t_{i} + \sum_{j=1}^{Y} (l_{j1}s_{j}^{1} + K_{j}t_{i}) - 1} \).
The page contains a complex mathematical equation involving variables such as $t$, $q$, $K$, and $i$, along with some indeterminate constants $\beta_i$. The equation appears to be a part of a larger proof or derivation, possibly in the field of mathematics or physics. The specific terms and variables are not clearly defined in the image, suggesting a high level of mathematical notation and context-specific vocabulary.

4.3 Proof of Theorem 2: Step 3

Set $Z_i(t) = \begin{cases} l_j(i_j + (n_j' + 1)n_j'/2) - n_j'(i_jn_j' - t) & t_j' \neq 0, \\ l_j(i_j + (n_j' + 1)n_j'/2) - n_j'(i_jn_j' - t) & t_j' = 0, \end{cases}$

for $1 \leq i \leq Y$. The expression involves terms like $l_j(i_j + (n_j' + 1)n_j'/2)$, which seem to be related to some form of summation or accumulation over indices $j$ and $i$. The context of this equation is not entirely clear from the image alone, but it is likely part of a larger proof or derivation, possibly in the field of mathematics or physics.
If $Q(n_0^2 - \bar{n}_0) + 2\bar{n}_0 = 2s_0$ then $\frac{n_0}{2} = \frac{Q(n_0 + \bar{n}_0) + 2s_0}{4Q\bar{n}_0 + 4} = \cdots = \frac{Q(n_0 - 1 + (\bar{n}_0 - 1)^2) + 2s_0}{4Q(\bar{n}_0 - 1) + 4}$.

So, if $Q(n_0^2 - \bar{n}_0) + 2\bar{n}_0 = 2s_0$ and $\tau_j' = 0$, then we have $Z_{ji}(t) = \frac{Q(n_0 + \bar{n}_0) + 2s_0}{4Q\bar{n}_0 + 4}$. (15)

If $(n_0 - 1)^2 + n_0 - 1 = 2s_0'$ then $n_0' = \frac{n_0' + n_0'^2 + 2s_0'}{4n_0'} = \frac{(n_0' - 1) + (n_0' - 1)^2 + 2s_0'}{4(n_0' - 1)}$.

Therefore for $\tau_j' \neq 0$, we have $n_0' = n_0$, and $Z_{ji}(t) = \frac{n_0' + n_0'^2 + 2s_0'}{4n_0'}$. (16)

Set $d_1 = u_1, d_2 = u_1d_2, \ldots, d_{K+1} = u_1d_{K+1}, v_{11} = u_1v_{11}$ in Eq.(14). Then we have

$$
\Psi = \{u_1^n_1v_1^2_1-2v_1^2_1\cdots v_{1Y}^2v_1^{(K+1)}v_{11}^{(1)+\cdots v_{1Y}(1+D_{J^{(\mu)}\dagger})}G(v') \prod_{m=1}^{r+1} \prod_{m=1}^{p} \prod_{m=K+2}^{p} \prod_{m=K+2}^{p} \prod_{m=0}^{r} \prod_{m=0}^{r} db_{m}^{(w)} dv.
$$

So the poles $\frac{q_{i-K}(2t_{i-1})}, \frac{q_{i+1}(I_i)}{2t_{i}}$ \(1 \leq i \leq Y\) are obtained. By Eq.(15) and (16), we have

$$
\begin{align*}
\frac{q_{i-K}}{2(I_i-1)} &= \sum_{j=1}^{\xi_{i-1}} 1 + \sum_{j=1}^{\xi_{i-1}} 1 + \sum_{j=1}^{\xi_{i-1}} 1 + \sum_{j=1}^{\xi_{i-1}} 1 + \sum_{j=1}^{\xi_{i-1}} 1 + \sum_{j=1}^{\xi_{i-1}} 1, \\
\frac{q_{i+1}}{2I_i} &= \sum_{j=1}^{\xi_{i-1}} 1 + \sum_{j=1}^{\xi_{i-1}} 1 + \sum_{j=1}^{\xi_{i-1}} 1 + \sum_{j=1}^{\xi_{i-1}} 1.
\end{align*}
$$

By using $I_i = n_j'I_i, i \leq j$, we have $Z_{ji}(I_i) = \{u_{1}^{2I_{1}}v_{11}^{2t_{1}-2}v_{12}^{2I_{2}}\cdots v_{1Y}^{2I_{Y}}(1+d_{2}^{2}+\cdots+d_{r+\zeta_{2}}^{2}+G_{r+\zeta_{2}+1}^{(w)})d_{\tau+\zeta_{2}+1}^{2}+\cdots+G_{K+1}^{(w)})d_{K+1}^{2})$.

Thus, we have $\frac{q_{i+1}}{2I_i} = \cdots = \frac{q_{Y+1}+1}{2Y+1} = \lambda_{Q}^{*}$. Also if $Y = Y'$ then $\frac{q_{i-K}}{2(I_i-1)} = \lambda_{Q}^{*}$.  

5 Proof of Main Theorem 1: Part 3

Finally, we prove that the pole in Theorem 2 and its order are max.

Lemma 3 If $f_1, g_1, g_2, \ldots, f_m, g_m > 0$, then \(\sum_{i=1}^{m} f_i \geq \min_{i} \{f_i\}\).

Lemma 4 Let $K \in Z_+$. Assume that $\eta_k \in Z_{+}, k = 1, \ldots, K$ satisfy $0 \leq \eta_1 \leq Q, 0 \leq \eta_1 + \eta_2 \leq 2Q, \ldots, 0 \leq \eta_1 + \eta_2 + \cdots + \eta_K \leq QS$. Let $t = \eta_1 + \cdots + \eta_K = Q(i-1) + m$, $i \in \mathbb{N}$, $0 \leq m < Q-1$, and $\varphi = \eta_1 + 2\eta_2 + \cdots + K\eta_K$. Then $\lambda_{Q}^{*} = Q(i-1)/2 + m$.

Next assume that $\eta_k \in Z_{+}, k = 1, \ldots, K$ satisfy $0 \leq \eta_1 \leq 0, 0 \leq \eta_1 + \eta_2 \leq 2, \ldots, 0 \leq \eta_1 + \eta_2 + \cdots + \eta_K \leq K-1$. Then by setting $t = 1 + \eta_1 + \cdots + \eta_K$, and $\varphi = \eta_1 + 2\eta_2 + \cdots + \eta_K$, we have $Q(i-1)/2 + m$. 

Lemma 5 For any $s_0, s_1, m$ with $0 \leq m < Q - 1$, we have $Q(i-1)/2 + m + s_0 \geq Q(n^2 + n) + 2s_0$.

where $n = \max\{i \in Z | Q(i^2 - i) + 2i \leq 2s_0\}$, or we have $\frac{i^2 + 2i + 2s_1}{4s_0} \geq \frac{(n^2 + n) + 2s_1}{4n}$, where $n - 1 = \max\{i \in Z | i^2 + i \leq 2s_0\}$

Lemma 6 If some $J_{\lambda}'$ is not Special Case, then the poles in Section 3 are smaller than $\lambda_{Q}^{*}$ in Main Theorem 1.

Proof Assume that $B_\theta$ is not Special Case. Then by $g_{t,(i,r),\theta} < \sum_{j \geq j_{\lambda}'} D_{J^{(\mu)},(i,r)}$ for $m \in B_\theta$, together with Statement (e) and (f), we have $2^{\frac{q_{i+1}}{2t_{i}}} > 2^{\sum_{r=0}^{\infty} q_{t_{i},r}^{(i,r),\theta} + \sum_{r=0}^{\infty} q_{t_{i},r}^{(i,r),\theta} + \sum_{r=0}^{\infty} q_{t_{i},r}^{(i,r),\theta}} \geq \lambda_{Q}^{*}$.

By Lemmas 3, 4, and 5, we obtain $2^{\frac{q_{i+1}}{2t_{i}}} \geq \lambda_{Q}^{*}$ Q.E.D.
Theorem 7  The order of the maximum pole $\lambda_0^*$ is $\theta = \# \Theta + 1$.

Proof  By Lemma 6, it is enough to consider Special Case for all $B_i^{(s)}$. Especially, consider the case $J_{m}^{(a)} = \{b_{ij}^{0}, 0, \ldots, 0\}$, which satisfies Statements (a)~(f). Suppose that $Y, Y', Y_{i}, Y'_{i}, B_{ij} = \{b_{ij}, b_{ij}', b_{ij}''\}$, $u_{ij}, u = (u_{11}, u_{12}, \ldots, u_{1Y}) \times (u_{21}, u_{22}, \ldots, u_{2Y}, u_{31}, \ldots, u_{s_{1}'Y}), u_{ij}''$, and $s_i''$ are as before. Let $N(i, j)$ be the number such that $N(i, j)$th element of $u$ is $u_{ij}$.

Let $B_L$ be an $(s_i'' + s_i' + \cdots + 1) \times (s_i'' + s_i' + \cdots + 1)$ matrix. For simplicity, denote the $(k, l)$th element of $B_L$ by $B_{ij}(i', j')$ if $k = N(i, j), l = N(i', j')$.

We proceed Section 3 from $\Psi'$ in Eq.(9).

Transformation (ii) or (iii) for $\psi$ corresponds to $B_i^{(s)} \psi$, where $B_i^{(s)}$ is a $(s_i'' + s_i' + \cdots + 1) \times (s_i'' + s_i' + \cdots + 1)$ matrix: $B_{ij}(i', j') = \begin{cases} 1, & (i, j) = (i', j'), \\ 1, & (i, j) \in A^{(a)}, (i', j') = (k', \tau'), \\ 1, & (i, j') = (k_0 + 1, \tau), \\ 1, & (i, j') = (k + 1, \tau), \\ 0, & \text{otherwise,} \end{cases}$ where

Therefore, we have $D_{ij}(i', j') = B_{ij}((k_0 + 1, j), (k, l))$. If $-2\eta_{\xi\kappa}$ is the maximum pole, we need that

(1) $B_{ij}((k_0 + 1, j), (k, l)) = 0$ for $k_0^{(a)} \geq 1$ and $1 \leq j \leq Y$, (2) $g_{(k_0, l), \tau} = B_{ij}((1, j), (k, l))$ for $1 \leq j \leq Y$, (3) $t_{\eta\tau} = \sum_{i=1}^{\#A^{(a)}} (1 + \eta_{i, (k, l), \tau}) + \cdots + \eta_{n_{ij}, (k_0, \xi), \tau_{j}''} = B_{ij}((1, j), (k, l))n_j''$ and

are linear, since $t_{\eta\tau} = B_{ij}((1, j), (k, l))n_j''$ for $Y' + 1 \leq j \leq Y$. Then $\det B_L = 0$. This is the contradiction. So, the total number of $(k, l)$ giving the maximum pole is less than $Y' + 1$. Next assume that $-\sum_{(k, l) \in A^{(a)}} \frac{2\eta_{\xi\kappa}}{\sum_{(k_0, l_0) \in A^{(a)}} \#A^{(a)} + \#C^{(a)}}$ is the maximum pole. We have (1) $B_{ij}((k_0 + 1, j), (k, l)) = 0$ for $k_0^{(a)} \geq 1, 1 \leq j \leq Y$ and $(k, l) \in A^{(a)}$, (2) $g_{(k_0, l), \tau} = B_{ij}((1, j), (k, l))$ for $1 \leq j \leq Y$ and $(k, l) \in A^{(a)}$, (3) There exist $(k_0, l_0) \in A^{(a)}$ and $\xi_0$ such that $t_{\eta\tau} = B_{ij}((1, j), (k_0, l_0))n_j''$ and

have $t_{\eta\tau} = \sum_{i=1}^{\#A^{(a)}} (1 + \eta_{i, (k, l), \tau}) + \cdots + \eta_{n_{ij}, (k_0, \xi), \tau_{j}''} = B_{ij}((1, j), (k, l))n_j''$ and

Therefore, we similarly have the contradiction to the order $Y' + 2$.

The end of the proof of Main Theorem 1

References