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Zeta function of learning theory and generalization error of three layered neural perceptron

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Abstract

Recently, the purpose of obtaining the maximum poles of certain zeta functions arises in the learning theory when one is looking for the generalization errors of hierarchical learning models asymptotically [3, 4]. The zeta function of a learning model is defined by the integral of its Kullback function and its *a priori* probability density function. Today, for several learning models, upper bounds of the main terms in their asymptotic forms were calculated, but not the exact values, so far. In this paper, we obtain the explicit value of the main term for a three layered neural network, which is one of hierarchical learning models.

Keywords zeta function, resolution of singularities, Generalization error, layered neural networks.

1 Introduction

The purpose of the learning system is such as image or speech recognition, artificial intelligence, control of a robot, genetic analysis, data mining or time series prediction. Their data are very complicated, not generated by simple normal distributions, since they are influenced by many factors. Learning models for analyzing such data have to have complicated structures, too. Hierarchical learning models such as a layered neural network, reduced rank regression, a normal mixture model and a Boltzmann machine are known as effective learning models. These models have been applied practically to such data. However, they can not be analyzed by using classic theories of regular statistical models. A few mathematical theories for such learning models have been known in the past. So it is necessary and crucial to construct fundamental mathematical theories.

Recently, it was proved that the maximum poles of certain zeta functions asymptotically give the generalization errors of hierarchical learning models [3, 4]. Furthermore, it was shown that the poles of the zeta function can be calculated by using desingularization. In spite of those mathematical foundations, for most examples, only upper bounds of the main terms were calculated but not the exact values, by two main reasons as follows.

(1) By Hironaka's Theorem [2], it is known that desingularization of an arbitrary polynomial can be obtained by using a blowing-up process. However desingularization of any polynomial in general, although it is known as a finite process, is very difficult. Furthermore, (a) most of Kullback functions are degenerate (over $\mathbb{R}$) with respect to their Newton polyhedrons, (b) singularities of Kullback functions are not isolated, (c) Kullback functions are not simple polynomials, i.e., they have parameters, for example, p of $\sum_{n=1}^{p}(\sum_{m=1}^{p} a_m b_m^{2n-1})^2$, which is one of Kullback functions for the three layered neural networks.

We note that there are many classical results for calculating the maximum poles of the zeta functions whose dimension is two, using desingularization of plane curves. Also there have been many investigations for the case of the prehomogeneous spaces. Kullback functions do not occur in the prehomogeneous spaces. Therefore, to obtain desingularization of Kullback functions is a new problem even in mathematics, since most of these singularities have not been investigated.

(2) Since the main purpose is to obtain the maximum pole, desingularization is not enough. We need some techniques for comparing poles as real numbers. However, as far as we know, there had been no theorem for comparing poles yet.

In the paper [1], we have clarified the maximum pole and its order of the reduced rank regression which is the three layered neural network with linear hidden units. In this paper, we use a recursive blowing-up, an inductive comparing method and a toric resolution for obtaining those values of the three layered neural network.

2 Main Theorems and poles of the zeta function for the three layered neural network

In this section, Main Theorem 1 and Main Theorem 2 are stated below. Results in Main Theorem 2 are related to the three layered neural network. They are obtained from Main Theorem 1 which contains more general cases.

Let $w = (a_1^{(w)}, \cdots, a_p^{(w)}, b_1^{(w)}, \cdots, b_p^{(w)}) \in \mathbb{R}^{2p}$ be a parameter, $w^* = (a_1^*, \cdots, a_p^*, b_1^*, \cdots, b_p^*) \in \mathbb{R}^{2p}$ a constant value vector and $U^* = U_{a_1^*,\cdots,a_p^*,b_1^*,\cdots,b_p^*}$ a sufficiently small neighborhood of w^*. Let Q be an arbitrary natural number.

Set
$$J_Q^*(z) = \int_{U^*} \{\sum_{n=1}^{P}(\sum_{m=1}^{p} a_m^{(w)} b_m^{(w)Q(n-1)+1} - \sum_{m=1}^{p} a_m^* b_m^{*\,Q(n-1)+1})^2\}^z \prod_{m=1}^{p} da_m^{(w)} db_m^{(w)}, \tag{1}$$
where z is an one-dimensional complex value and $P \geq 2p$.

In the case of the three layered neural network, Q is two, which is shown later.

Let $b_1^{**Q}, \ldots, b_r^{**Q}$ be different real numbers in $\{b_i^{*Q} \mid b_i^{*Q} \neq 0\}$ from each other: $\{b_1^{**Q}, \ldots, b_r^{**Q} \mid b_i^{**Q} \neq b_j^{**Q}, i \neq j\} = \{b_i^{*Q} \mid b_i^{*Q} \neq 0\}$, and $a_i^{**} = -(\sum_{\{m \mid b_m^{*Q} = b_i^{**Q}\}} a_m^* b_m^*)/b_i^{**}$.

Also set $a_0^{**} = 0$, $b_0^{**} = 0$. Put $\begin{cases} B_\tau^{(w)} = \{i \mid b_i^{*Q} = b_\tau^{**Q}\}, & s_\tau = \#B_\tau^{(w)}, \quad 1 \leq \tau \leq r, \\ B_0^{(w)} = \{i \mid b_i^* = 0\}, & s_0 = \#B_0^{(w)}, \end{cases}$ where $\#$ implies the number of elements. So $\sum_{\tau=0}^{r} s_\tau = p$. Let $\tilde{r}$ ($\tilde{r} \leq r$) be the total number of $a_\tau^{**} \neq 0$. We can assume that $a_1^{**} \neq 0, \cdots, a_{\tilde{r}}^{**} \neq 0$, $a_{\tilde{r}+1}^{**} = 0, \cdots, a_r^{**} = 0$. Then we have $\sum_{m=1}^{p} a_m^* b_m^{*\,Q(n-1)+1} = -\sum_{m=1}^{\tilde{r}} a_m^{**} b_m^{**\,Q(n-1)+1}$ for any $n \in \mathbb{N}$. Set $H_\tau^{(n)} = \sum_{m \in B_\tau^{(w)}} a_m^{(w)} b_m^{(w)Q(n-1)+1} + a_\tau^{**} b_\tau^{**\,Q(n-1)+1}$. By the definitions, we have $J_Q^*(z) = \int_{U^*} \{\sum_{n=1}^{P}(\sum_{\tau=1}^{r} H_\tau^{(n)})^2\}^z \prod_{m=1}^{p} da_m^{(w)} db_m^{(w)}$. Let

$$J_\tau(z) = \int_{U^* \cap \mathbb{R}^{s_\tau}} \{\sum_{n=1}^{P} H_\tau^{(n)2}\}^z \prod_{m \in B_\tau^{(w)}} da_m^{(w)} db_m^{(w)}.$$

Let $-\lambda_Q^*$, $-\lambda_\tau$ be the maximum poles of $J_Q^*(z)$ and $J_\tau(z)$, respectively, and θ the order of $-\lambda_Q^*$.

Remark 1 If $p+\tilde{r} = 2$, $\mathbf{R}[w]/(\sum_{n=1}^{p+\tilde{r}}(\sum_{\tau=1}^{r} H_\tau)^2)$ is not a normal ring. If $p+\tilde{r} \geq 3$, $\mathbf{R}[w]/(\sum_{n=1}^{p+\tilde{r}}(\sum_{\tau=1}^{r} H_\tau)^2)$ is a normal ring.

Main Theorem 1

(1) *We have* $\lambda_Q^* = \sum_{\tau=0}^{r} \lambda_\tau$.

(2)
$$\begin{aligned}
\lambda_0 &= \frac{Q(\tilde{n}_0^2 + \tilde{n}_0) + 2s_0}{4Q\tilde{n}_0 + 4}, && \tilde{n}_0 = \max\{i \in \mathbb{Z} \mid Q(i^2 - i) + 2i \leq 2s_0\}, \\
\lambda_{\tau_1} &= \frac{n_{\tau_1} + n_{\tau_1}^2 + 2s_{\tau_1}}{4n_{\tau_1}}, && n_{\tau_1} - 1 = \max\{i \in \mathbb{Z} \mid i^2 + i \leq 2s_{\tau_1}\}, \quad \text{if } 1 \leq \tau_1 \leq \tilde{r}, \\
\lambda_{\tau_2} &= \frac{n_{\tau_2} + n_{\tau_2}^2 + 2(s_{\tau_2} - 1)}{4n_{\tau_2}}, && n_{\tau_2} - 1 = \max\{i \in \mathbb{Z} \mid i^2 + i \leq 2(s_{\tau_2} - 1)\}, \quad \text{if } \tilde{r} + 1 \leq \tau_2 \leq r.
\end{aligned}$$

(3) *Set*
$$\begin{aligned}
\Theta = \{\tau_0, \tau_1, \tau_2 \mid\ & Q(\tilde{n}_0^2 - \tilde{n}_0) + 2\tilde{n}_0 = 2s_{\tau_0}, \tau_0 = 0, s_{\tau_0} \geq 1, \\
& (n_{\tau_1} - 1)^2 + n_{\tau_1} - 1 = 2s_{\tau_1}, s_{\tau_1} > 1, 1 \leq \tau_1 \leq \tilde{r}, \\
& (n_{\tau_2} - 1)^2 + n_{\tau_2} - 1 = 2(s_{\tau_2} - 1), s_{\tau_2} > 1, \tilde{r} < \tau_2 \leq r\}.
\end{aligned}$$

Then $\theta = \#\Theta + 1$.

Consider the three layered neural network with one input unit, p hidden units, and one output unit which is trained to estimate the true distribution represented by the model with $\tilde{r}$ ($\tilde{r} < p$) hidden units. Denote an input value by $x \in \mathbb{R}$ with a probability density function $q(x)$ with compact support $\tilde{W} \subset [-1, 1]$. Then an output value y of the three layered neural network is given by $y = f(x, w) +$(noise), where $f(x, w) = \sum_{m=1}^{p} a_m^{(w)} \tanh(b_m^{(w)} x)$. Consider a statistical model $p(y|x, w) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}(y - f(x, w))^2)$. Assume that a true distribution is $p(y|x, w^*)$ which is included in the learning model, where $w^* = (a_1^*, \ldots, a_p^*, b_1^*, \ldots, b_p^*)$, $|b_i^*| < \pi/2$, $i = 1, \ldots, p$. Let W^* be the true parameter set: $W^* = \{\tilde{w} \in W \mid f(x, \tilde{w}) = f(x, w^*)$ for any $x\}$. Suppose that an *a priori* probability density function $\psi(w)$ is a $C^\infty-$ function with compact support W where $\psi(w^*) > 0$. Then the maximum pole of $\int_W K(w)^z \psi dw$ is equal to that of $J_2^*(z) = \int_W \{\sum_{n=1}^{P}(\sum_{m=1}^{p} a_m^{(w)} b_m^{(w)2n-1} - \sum_{m=1}^{p} a_m^* b_m^{*\,2n-1})^2\}^z dw$, where P is a sufficient large integer. It is proved by a Taylor expansion at 0 together with Lemma 5 in [3].

Remark 2 Let $\sigma(x)=\sum_{i=1}^{\infty}\alpha_i x^{Q(i-1)+1}$ with $\alpha_i\neq 0$. Then, the maximum pole of $\int_W(\int_{\tilde W}(\sum_{m=1}^{p}a_m^{(w)}\sigma(b_m^{(w)}x)-\sum_{m=1}^{p}a_m^*\sigma(b_m^*x))^2 q(x)\mathrm{d}x)^z\psi(w)\mathrm{d}w$, and its order are the same ones in Main Theorem 1. The function $\tanh x$ satisfies this condition with $Q=2$.

Main Theorem 2

(1) *The maximum pole $-\lambda$ and its order θ are obtained by setting $Q=2$ in Main Theorem 1. More precisely, we have $\lambda=\max_{\tilde w\in W^*}\lambda_Q^*$ with its order θ.*

(2) *In particular, assume that W^* includes all $\tilde w$ satisfying $f(x,\tilde w)=f(x,w^*)$:*

$$W^*=\{\tilde w\in\mathbb{R}^d \mid f(x,\tilde w)=f(x,w^*) \text{ for any } x\}.$$

Then for $p-\tilde r+1\geq 10$, we have $\lambda=\tilde r+\dfrac{i^2+i+p-\tilde r}{4i+2}$ *and* $\theta=\begin{cases}1, & \text{if } i^2<p-\tilde r,\\ 2, & \text{if } i^2=p-\tilde r,\end{cases}$ where $i=\max\{\ell\in\mathbb{Z}\mid \ell^2\leq(p-\tilde r)\}$. *For $p-\tilde r+1<10$, we have* $\lambda=\tilde r-1+\dfrac{j^2+j+2(p-\tilde r+1)}{4j}$ *and*

$$\theta=\begin{cases}1, & \text{if } (j-1)^2+j-1<2(p-\tilde r+1),\\ 2, & \text{if } (j-1)^2+j-1=2(p-\tilde r+1),\end{cases}\quad \textit{where } j-1=\max\{\ell\in\mathbb{Z}\mid \ell^2+\ell\leq 2(p-\tilde r+1)\}.$$

Remark 3 (1) We have the condition $\tilde w=(\tilde a_1,\ldots,\tilde a_p,\tilde b_1,\ldots,\tilde b_p)\in W^*$ if and only if for any $1\leq\tau\leq\tilde r$, there exists $1\leq\xi\leq p$ such that $\tilde b_\xi^Q=b_\tau^{**Q}$. (2) You can see a kind of phase transitions at $p-\tilde r+1=10$.

Main Theorem 2 gives the generalization error of the three layered neural network asymptotically.

$$\begin{aligned}\text{Define}\quad \Psi &= \{\sum_{n=1}^{P}(\sum_{m=1}^{p}a_m^{(w)}b_m^{(w)Q(n-1)+1}-\sum_{m=1}^{p}a_m^*b_m^{*\,Q(n-1)+1})^2\}^z\prod_{m=1}^{p}\mathrm{d}a_m^{(w)}\mathrm{d}b_m^{(w)}\\ &= \{\sum_{n=1}^{P}(\sum_{m=1}^{p}a_m^{(w)}b_m^{(w)Q(n-1)+1}+\sum_{m=1}^{\tilde r}a_m^{**}b_m^{**Q(n-1)+1})^2\}^z\prod_{m=1}^{p}\mathrm{d}a_m^{(w)}\mathrm{d}b_m^{(w)}.\end{aligned}$$

Put the auxiliary function $f_{n,l}$ by $f_{n,l}(x_1,\cdots,x_l)=\begin{cases}\sum_{j_1+\cdots+j_l=n-l}x_1^{Qj_1}\cdots x_l^{Qj_l}, & \text{if } n-l\geq 0,\\ 0, & \text{if } n-l<0.\end{cases}$

Lemma 1 *Let $n,\ell\in\mathbb{N}$. Set $C_i'=\sum_{m=i}^{\ell}a_m'b_m'(b_m'^Q-b_1'^Q)\cdots(b_m'^Q-b_{i-1}'^Q)$ for $i=1,\cdots,\ell$.*

Then we have $\displaystyle\sum_{m=1}^{\ell}a_m'b_m'^{\,Q(n-1)+1}=f_{n,1}(b_1')C_1'+f_{n,2}(b_1',b_2')C_2'+\cdots+f_{n,\ell}(b_1',\cdots,b_\ell')C_\ell'$.

Proof It is proved by $f_{n,l}(x_1,\cdots,x_{l-1},y_l)-f_{n,l}(x_1,\cdots,x_{l-1},z_l)=(y_l^Q-z_l^Q)f_{n,l+1}(x_1,\cdots,x_{l-1},z_l,y_l)$. Q.E.D.

Let $C_i=\displaystyle\sum_{m=i}^{p}a_m^{(w)}b_m^{(w)}(b_m^{(w)Q}-b_1^{(w)Q})\cdots(b_m^{(w)Q}-b_{i-1}^{(w)Q})+\sum_{m=1}^{\tilde r}a_m^{**}b^{**}{}_m(b_m^{**Q}-b_1^{(w)Q})\cdots(b_m^{**Q}-b_{i-1}^{(w)Q})$ for $i\leq p$, and

$$C_i=\sum_{m=i-p}^{\tilde r}a_m^{**}b_m^{**}(b_m^{**Q}-b_1^{(w)Q})\cdots(b_m^{**Q}-b_p^{(w)Q})(b_m^{**Q}-b_1^{**Q})\cdots(b_m^{**Q}-b_{i-p-1}^{**Q})\text{ for } p<i\leq p+\tilde r.$$

By Lemma 1, we have $\Psi=\{\sum_{n=1}^{P}(f_{n,1}(b_1^{(w)})C_1+f_{n,2}(b_1^{(w)},b_2^{(w)})C_2+\cdots+f_{n,p}(b_1^{(w)},\cdots,b_p^{(w)})C_p$ $+f_{n,p+1}(b_1^{(w)},\cdots,b_p^{(w)},b_1^{**})C_{p+1}+\cdots+f_{n,p+\tilde r}(b_1^{(w)},\cdots,b_p^{(w)},b_1^{**},\cdots,b_{\tilde r}^{**})C_{p+\tilde r})\}^z\prod_{m=1}^{p}\mathrm{d}a_m^{(w)}\mathrm{d}b_m^{(w)}$.

We can assume that $b_1^{*Q}=b_1^{**Q},\cdots,b_r^{*Q}=b_r^{**Q}$ and that if $b_m^{*\,Q}\neq b_{m'}^{*\,Q}$ then $b_m^{(w)Q}\neq b_{m'}^{(w)Q}$ on U^*. Then since $b_i^*\neq 0$, $b_i^{*Q}-b_j^{*Q}\neq 0$ for $1\leq i<j\leq r$, we can change the variables from $a_i^{(w)}$ to d_i by $d_i=C_i$. Next consider the case $i>r$. There exist functions $g(i,m)\neq 0$ on U^* such that

$$\begin{aligned}C_i=&\sum_{\substack{i\leq m\leq p\\ b_m^*=0}}g(i,m)a_m^{(w)}b_m^{(w)}\prod_{\substack{1\leq i'\leq i-1,\\ b_{i'}^*=0}}(b_m^{(w)Q}-b_{i'}^{(w)Q})+\sum_{\substack{i\leq m\leq p\\ b_m^*\neq 0}}g(i,m)a_m^{(w)}\prod_{\substack{1\leq i'\leq i-1,\\ b_{i'}^{*Q}=b_m^{*Q}}}(b_m^{(w)Q}-b_{i'}^{(w)Q})\\ &+\sum_{m=1}^{\tilde r}g(i,m+p)a_m^{**}\prod_{\substack{1\leq i'\leq i-1,\\ b_{i'}^{*Q}=b_m^{**Q}}}(b_m^{**Q}-b_{i'}^{(w)Q})\text{ for } r<i\leq p,\text{ and}\end{aligned}$$

$$C_i=\sum_{m=i-p}^{\tilde r}g(i,m+p)a_m^{**}\prod_{\substack{1\leq i'\leq p,\\ b_{i'}^{*Q}=b_m^{**Q}}}(b_m^{**Q}-b_{i'}^{(w)Q})\text{ for } p<i\leq p+\tilde r.$$

Let $a_i = \begin{cases} a_i^{(w)} & i = r+1,\ldots,p, \\ a_{i-p}^{**} & i = p+1,\ldots,p+\tilde{r}, \end{cases}$ and $b_i = \begin{cases} b_i^{(w)}, & \text{if } b_i^* = 0, \\ b_i^{(w)Q} - b_i^{*Q}, & \text{if } b_i^{*Q} = b_\tau^{**Q} \neq 0, a_\tau^{**} \neq 0, \\ b_i^{(w)Q} - b_\tau^{(w)Q}, & \text{if } b_i^{*Q} = b_\tau^{**Q} \neq 0, a_\tau^{**} = 0, \\ & 1 \leq i \leq p, \\ 0, & i = p+1,\cdots,p+\tilde{r}. \end{cases}$

When we distinguish a^*, b^* from $a^{(w)}$, $b^{(w)}$, we call $a = a^*$, $b = b^*$ constants. Let

$$J_i^{(1)} = b_i^{*Q}, \quad i = 1,\ldots,\tilde{r}, r+1,\ldots,p+\tilde{r}. \tag{2}$$

Put $p' = p + \tilde{r}$. Then we have

$$\begin{aligned} C_i \;=\; & \sum_{\substack{i \leq m \leq p', \\ J_m^{(1)} = 0}} g(i,m) a_m b_m \prod_{\substack{1 \leq i' \leq i-1, \\ J_{i'}^{(1)} = J_m^{(1)}}} (b_m{}^Q - b_{i'}{}^Q) + \sum_{\substack{i \leq m \leq p', \\ J_m^{(1)} \neq 0, a_m^* = 0}} g(i,m) a_m b_m \prod_{\substack{1 \leq i' \leq i-1, \\ J_{i'}^{(1)} = J_m^{(1)}}} (b_m - b_{i'}) \\ & + \sum_{\substack{i \leq m \leq p', \\ J_m^{(1)} \neq 0, a_m^* \neq 0}} g(i,m) a_m \prod_{\substack{1 \leq i' \leq i-1, \\ J_{i'}^{(1)} = J_m^{(1)}}} (b_m - b_{i'}), \end{aligned} \tag{3}$$

for $r < i \leq p'$. By Lemmas 2 and 3 in [1], the maximum pole of $\int_W \Psi$ and its order are equal to those of $\int_W \Psi'$, where

$$\Psi' = \{d_1^2 + \cdots + d_r^2 + C_{r+1}^2 + \cdots + C_{\tilde{r}+p}^2\}^z \prod_{m=1}^{r} \mathrm{d}d_m \prod_{m=r+1}^{p} \mathrm{d}a_m \prod_{m=\tilde{r}+1}^{r} \mathrm{d}b_m^{(w)} \prod_{m=1}^{\tilde{r}} \mathrm{d}b_m \prod_{m=r+1}^{p} \mathrm{d}b_m. \tag{4}$$

Since we often change variables during a blowing-up process, it is more convenient for us to use the same symbols e_m rather than e'_m, e''_m, $\cdots$, etc, for the sake of simplicity. For instance,

"Let $\begin{cases} e_1 = v_{11} \\ e_m = v_{11} e_m, \end{cases}$" instead of "Let $\begin{cases} e_1 = v_{11} \\ e_m = v_{11} e'_m. \end{cases}$"

3 Proof of Main Theorem 1: Part 1

Take $J^{(\alpha)} \in \mathbb{R}^\alpha$. Denote $J^{(\alpha)} = (J^{(\alpha')}, *)$ and $\alpha > \alpha'$ by $J^{(\alpha)} > J^{(\alpha')}$. Also denote $J^{(\alpha)} = (0,\cdots,0)$ by $J^{(\alpha)} = 0^{(\alpha)}$ or $J^{(\alpha)} = 0$. Set $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. We need the following inductive statements of k, K, α for calculating poles by using the blowing-up process.

Inductive statements

Set $E = \{m \mid e_m \text{ is non-constant }\}$, $E_\tau = \{m \mid J_m^{(\alpha)} = (b_\tau^{**Q}, *)\}$, $s(J^{(\alpha)}) = \#\{m \mid m \geq k+1, J_m^{(\alpha)} = J^{(\alpha)}, m \in E\}$, and $s(i, J^{(\alpha)}) = \#\{m \mid k+1 \leq m \leq i-1, J_m^{(\alpha)} = J^{(\alpha)}, m \in E\}$ for $J^{(\alpha)} \in \mathbb{R}^\alpha$.

(a) $k = k_0 + \cdots + k_r$, $K = K_0 + \cdots + K_r$, $K_0 \geq k_0$, and $K_\tau \geq k_\tau + 1$ for $1 \leq \tau \leq r$, where $k_0,\ldots,k_r \in \mathbb{Z}_+$ and $K_0,\ldots,K_r \in \mathbb{Z}_+$. Set $k_i^{(\alpha)} = k_i$.

(b) $$\begin{aligned} \Psi' = \{ & \prod_{\tau=0}^{r} (v_{1\tau}^{t_{1\tau}} v_{2\tau}^{t_{2\tau}} \cdots v_{k_\tau \tau}^{t_{k_\tau \tau}}) \left(d_1^2 + d_2^2 + \cdots + d_K^2\right) + \sum_{i=K+1}^{p'} C_i^2\}^z \prod_{\tau=0}^{r} \prod_{l=1}^{k_\tau} v_{l\tau}^{q_{l\tau}} \\ & \prod_{m=1}^{K} \mathrm{d}d_m \prod_{\tau=0}^{r} \prod_{l=1}^{k_\tau} \mathrm{d}v_{l\tau} \prod_{m=K+1}^{p} \mathrm{d}a_m \prod_{\substack{k+1 \leq m \leq p \\ m \in E}} \mathrm{d}e_m \prod_{m=\tilde{r}+1}^{r} \mathrm{d}b_m^{(w)}. \end{aligned} \tag{5}$$

Here, $t_{l\tau}, q_{l\tau} \in \mathbb{Z}_+$. Also, there exist $RJ^{(\alpha)} \subset \mathbb{R}^\alpha$, $t(i, J, (l,\tau)) \in \mathbb{Z}_+$ and functions $g(i,m) \neq 0$ such that

$$\begin{aligned} C_i = & \prod_{\tau=0}^{r} (v_{1\tau}^{t(i,0,(1,\tau))} v_{2\tau}^{t(i,0,(2,\tau))} \cdots v_{k_\tau \tau}^{t(i,0,(k_\tau,\tau))}) \sum_{\substack{i \leq m \leq p' \\ J_m^{(\alpha)} = 0}} g(i,m) a_m e_m \prod_{\substack{k+1 \leq i' < i \\ J_{i'}^{(\alpha)} = 0}} (e_m^Q - e_{i'}^Q) \\ & + \sum_{J \in RJ^{(\alpha)}} \prod_{\tau=0}^{r} (v_{1\tau}^{t(i,J,(1,\tau))} v_{2\tau}^{t(i,J,(2,\tau))} \cdots v_{k_\tau \tau}^{t(i,J,(k_\tau,\tau))}) \sum_{\substack{i \leq m \leq p' \\ J_m^{(\alpha)} = J}} g(i,m) a_m e_m \prod_{\substack{k+1 \leq i' < i \\ J_{i'}^{(\alpha)} = J}} (e_m - e_{i'}) \\ & + \sum_{J \notin RJ^{(\alpha)} \cup \{0\}} \prod_{\tau=0}^{r} (v_{1\tau}^{t(i,J,(1,\tau))} v_{2\tau}^{t(i,J,(2,\tau))} \cdots v_{k_\tau \tau}^{t(i,J,(k_\tau,\tau))}) \sum_{\substack{i \leq m \leq p' \\ J_m^{(\alpha)} = J}} g(i,m) a_m \prod_{\substack{k+1 \leq i' < i \\ J_{i'}^{(\alpha)} = J}} (e_m - e_{i'}). \end{aligned}$$

(c) $J_{i'}^{(\alpha)} \neq J_i^{(\alpha)}$ for $k < i' < i \leq K$. $J_i^{(\alpha)} \notin RJ^{(\alpha)} \cup \{0\}$ for $k < i \leq K$.

(d) $t(i, J_m^{(\alpha)}, (l,\tau)) \geq t_{l\tau}/2$ for all $J_m^{(\alpha)}$, $i \leq m \leq p'$ and there exist $D_{J^{(\mu)},(l,\tau)} \in \mathbb{Z}_+$ such that $t(i, J_m^{(\alpha)}, (l,\tau)) =$
$$\sum_{J_m^{(\alpha)} \geq 0^{(\mu)}} D_{0^{(\mu)},(l,\tau)} (Qs(i, 0^{(\mu)}) + 1) + \sum_{\substack{J_m^{(\alpha)} \geq J^{(\mu)} \\ J^{(\mu)} \in RJ^{(\mu)}}} D_{J^{(\mu)},(l,\tau)} (s(i, J^{(\mu)}) + 1) + \sum_{\substack{J_m^{(\alpha)} \geq J^{(\mu)} \\ J^{(\mu)} \notin RJ^{(\mu)} \cup \{0\}}} D_{J^{(\mu)},(l,\tau)} s(i, J^{(\mu)}).$$

(e) There exist $g_{(l,\tau),\tau'} \in \mathbb{Z}_+$, $\eta^{(\xi)}_{\ell,(l,\tau),\tau'} \in \mathbb{Z}_+$ such that $\frac{t_{l\tau}}{2} = \sum_{\xi=1}^{g_{(l,\tau),\tau'}} (1+\eta^{(\xi)}_{1,(l,\tau),\tau'}+\cdots+\eta^{(\xi)}_{K_{\tau'},(l,\tau),\tau'})$ for all τ', and $g_{(l,\tau),\tau'} \le \sum_{J^{(\alpha)}_m \ge J^{(\mu)}} D_{J^{(\mu)},(l,\tau)}$ for $m \in E_{\tau'}$.

$0 \le \eta^{(\xi)}_{1,(l,\tau),0} \le Q, 0 \le \eta^{(\xi)}_{1,(l,\tau),0}+\eta^{(\xi)}_{2,(l,\tau),0} \le 2Q, \cdots, 0 \le \eta^{(\xi)}_{1,(l,\tau),0}+\eta^{(\xi)}_{2,(l,\tau),0}+\cdots+\eta^{(\xi)}_{K_0,(l,\tau),0} \le QK_0$, and $0 \le \eta^{(\xi)}_{2,(l,\tau),\tau'} \le 1, 0 \le \eta^{(\xi)}_{2,(l,\tau),\tau'}+\eta^{(\xi)}_{3,(l,\tau),\tau'} \le 2, \cdots, 0 \le \eta^{(\xi)}_{2,(l,\tau),\tau'}+\eta^{(\xi)}_{3,(l,\tau),\tau'}+\cdots+\eta^{(\xi)}_{K_{\tau'},(l,\tau),\tau'} \le K_{\tau'}-1$, $\eta^{(\xi)}_{1,(l,\tau),\tau'} = 0$, for $\tau' \ge 1$.

(f) Let $\varphi^{(\xi)}_{(l,\tau),\tau'} := \begin{cases} s_0+\eta^{(\xi)}_{1,(l,\tau),0}+2\eta^{(\xi)}_{2,(l,\tau),0}+\cdots+K_0\eta^{(\xi)}_{K_0,(l,\tau),0}, & \text{if } \tau'=0, \\ s_{\tau'}+1+\eta^{(\xi)}_{1,(l,\tau),\tau'}+2\eta^{(\xi)}_{2,(l,\tau),\tau'}+\cdots+K_{\tau'}\eta^{(\xi)}_{K_{\tau'},(l,\tau),\tau'}, & \text{if } 1 \le \tau' \le \tilde{r}, \\ s_{\tau'}+\eta^{(\xi)}_{1,(l,\tau),\tau'}+2\eta^{(\xi)}_{2,(l,\tau),\tau'}+\cdots+K_{\tau'}\eta^{(\xi)}_{K_{\tau'},(l,\tau),\tau'}, & \text{if } \tilde{r} < \tau' \le r. \end{cases}$

There exist $\phi_{(l,\tau),\tau'} \in \mathbb{Z}_+$ such that

$q_{(l,\tau),\tau'} = \sum_{\xi=1}^{g_{(l,\tau),\tau'}} \varphi^{(\xi)}_{(l,\tau),\tau'} + \phi_{(l,\tau),\tau'} + \sum_{m=k+1, m\in E_{\tau'}}^{p'} (-g_{(l,\tau),\tau'} + \sum_{J^{(\alpha)}_m \ge J^{(\mu)}} D_{J^{(\mu)},(l,\tau)})$, and

$q_{l\tau}+1 = \sum_{\tau'=0}^{r} q_{(l,\tau),\tau'}$. **The end of inductive statements**

Statements (d), (e) and (f) are needed to compare poles. The definitions of all variables will be given later on in the proof.

Special Case Fix $\tilde{\tau}$. We call the case satisfying the following two conditions *Special Case*.

- $J^{(\alpha)}_m$ is the same for any $m \in E_{\tilde{\tau}}$,
- $s(J^{(\alpha)}_m) = \begin{cases} s_{\tilde{\tau}} - k^{(\alpha)}_{\tilde{\tau}}, & \text{if } 0 \le \tilde{\tau} \le \tilde{r}, \\ s_{\tilde{\tau}} - 1 - k^{(\alpha)}_{\tilde{\tau}}, & \text{if } \tilde{r} < \tilde{\tau} \le r, \end{cases}$ $s(K+1, J^{(\mu)}) = \begin{cases} K_{\tilde{\tau}} - k^{(\mu)}_{\tilde{\tau}}, & \text{if } 0 \le \tilde{\tau} \le \tilde{r}, \\ K_{\tilde{\tau}} - k^{(\mu)}_{\tilde{\tau}} - 1, & \text{if } \tilde{r} < \tilde{\tau} \le r, \end{cases}$ for $J^{(\mu)} \le J^{(\alpha)}_m$.

Then we have the followings: (a') $K_{\tilde{\tau}} = \begin{cases} k_{\tilde{\tau}}, & \text{if } \tilde{\tau}=0, \\ k_{\tilde{\tau}}+1, & \text{if } \tilde{\tau} \ge 1. \end{cases}$

(d') $t(i, J^{(\alpha)}_m, (l,\tau)) = \begin{cases} \sum_{J^{(\alpha)}_m \ge J^{(\mu)}} D_{J^{(\mu)},(l,\tau)}(Qs(i,0^{(\mu)})+1), & \text{if } \tilde{\tau}=0, m \in E_0, \\ \sum_{J^{(\alpha)}_m \ge J^{(\mu)}} D_{J^{(\mu)},(l,\tau)}(s(i,J^{(\mu)})+1), & \text{if } \tilde{r} < \tilde{\tau} \le r, m \in E_{\tilde{\tau}}, \\ \sum_{J^{(\alpha)}_m \ge J^{(\mu)}} D_{J^{(\mu)},(l,\tau)} s(i,J^{(\mu)}), & \text{if } 1 \le \tilde{\tau} \le \tilde{r}, m \in E_{\tilde{\tau}}. \end{cases}$

(e') $g_{(l,\tau),\tilde{\tau}} = \sum_{J^{(\alpha)}_m \ge J^{(\mu)}} D_{J^{(\mu)},(l,\tau)}$. $\frac{t_{l\tau}}{2} = \sum_{\xi=1}^{g_{(l,\tau),\tilde{\tau}}} (1+\eta^{(\xi)}_{1,(l,\tau),\tilde{\tau}}+\cdots+\eta^{(\xi)}_{K_{\tilde{\tau}},(l,\tau),\tilde{\tau}})$.

For $J^{(\mu)} \le J^{(\alpha)}_m$, there are ξ's as many as $D_{J^{(\mu)},(l,\tau)} (\le g_{(l,\tau),\tilde{\tau}})$ such that $\eta^{(\xi)}_{K_{\tilde{\tau}},(l,\tau),\tilde{\tau}} \le \begin{cases} Q & \text{if } \tilde{\tau}=0, \\ 1 & \text{if } \tilde{\tau} \ge 1, \end{cases}$

$\eta^{(\xi)}_{1,(l,\tau),\tilde{\tau}} = \cdots = \eta^{(\xi)}_{k^{(\mu)}_{\tilde{\tau}},(l,\tau),\tilde{\tau}} = 0, \eta^{(\xi)}_{k^{(\mu)}_{\tilde{\tau}}+1,(l,\tau),\tilde{\tau}} = \cdots = \eta^{(\xi)}_{K_{\tilde{\tau}}-1,(l,\tau),\tilde{\tau}} = Q$, if $\tilde{\tau}=0$,

$\eta^{(\xi)}_{1,(l,\tau),\tilde{\tau}} = \cdots = \eta^{(\xi)}_{k^{(\mu)}_{\tilde{\tau}}+1,(l,\tau),\tilde{\tau}} = 0, \eta^{(\xi)}_{k^{(\mu)}_{\tilde{\tau}}+2,(l,\tau),\tilde{\tau}} = \cdots = \eta^{(\xi)}_{K_{\tilde{\tau}}-1,(l,\tau),\tilde{\tau}} = 1$, if $1 \le \tilde{\tau} \le r$.

(f') Set $\varphi^{(\xi)}_{(l,\tau),\tilde{\tau}} := \begin{cases} s_0+\eta^{(\xi)}_{1,(l,\tau),0}+2\eta^{(\xi)}_{2,(l,\tau),0}+\cdots+K_0\eta^{(\xi)}_{K_0,(l,\tau),0}, & \text{if } \tilde{\tau}=0, \\ s_{\tilde{\tau}}+1+\eta^{(\xi)}_{1,(l,\tau),\tilde{\tau}}+2\eta^{(\xi)}_{2,(l,\tau),\tilde{\tau}}+\cdots+K_{\tilde{\tau}}\eta^{(\xi)}_{K_{\tilde{\tau}},(l,\tau),\tilde{\tau}}, & \text{if } 1 \le \tilde{\tau} \le \tilde{r}, \\ s_{\tilde{\tau}}+\eta^{(\xi)}_{1,(l,\tau),\tilde{\tau}}+2\eta^{(\xi)}_{2,(l,\tau),\tilde{\tau}}+\cdots+K_{\tilde{\tau}}\eta^{(\xi)}_{K_{\tilde{\tau}},(l,\tau),\tilde{\tau}}, & \text{if } \tilde{r} < \tilde{\tau} \le r. \end{cases}$

Then $q_{(l,\tau),\tilde{\tau}} = \sum_{\xi=1}^{g_{(l,\tau),\tilde{\tau}}} \varphi^{(\xi)}_{(l,\tau),\tilde{\tau}}$.

Remark 4 Special Case $E_{\tilde{\tau}}$ satisfies $\begin{cases} J^{(\alpha)}_m = ((b^*_m)^Q, 0, \ldots, 0) & \text{if } 0 \le \tilde{\tau} \le \tilde{r}, \\ J^{(\alpha)}_m = ((b^*_m)^Q, 0 \text{ or } 1, \ldots, 0 \text{ or } 1) \text{ and } J^{(\alpha)}_m \in RJ^{(\alpha)} & \text{if } \tilde{r} < \tilde{\tau} \le r, \end{cases}$ for all $m \in E_{\tilde{\tau}}$.

3.1 Step 1

Set $K_0 = 0, K_1 = K_2 = \cdots = K_r = 1, k_0 = k_1 = \cdots = k_r = 0, \alpha = 1$, $RJ^{(1)} = \{J^{(1)}_i \mid b^*_i \ne 0, a^*_i = 0, r < i \le p\}, t_{l\tau} = 0, q_{l\tau} = 0$, and $t(i, J, (l,\tau)) = 0$, where $J^{(1)}_i$ is defined by Eq.(2). Also set all parameters in (d)∼ (f) by 0, Then, by putting $e_i = b_i$ in Eq.(3), we have the inductive statements when $K = r$, $k = 0$ and $\alpha = 1$.

3.2 Step 2

We assume the case k, K, α. Let us concentrate on C_{K+1}.

Set $JB^{(\alpha)} = \{J^{(\alpha)}_m \in \mathbb{R}^\alpha; \exists (l,\tau), t(K+1, J^{(\alpha)}_m, (l,\tau)) > t_{l\tau}/2\}$, $JC^{(\alpha)} = \{J \in \mathbb{R}^\alpha; t(K+1, J, (l,\tau)) = t_{l\tau}/2 \text{ for all } (l,\tau)\}$,

$C^{(\alpha)} = \{m \geq k+1; t(K+1, J_m^{(\alpha)}, (l,\tau)) = t_{l\tau}/2$ for all $(l,\tau)\}$, $C'^{(\alpha)} = \{m \in C^{(\alpha)}; a_m, e_m$ are not constant $\}$, $\omega = \{(l,\tau) | 1 \leq l \leq k_\tau, 0 \leq \tau \leq r\}$ and $\Omega = \{A \subset \omega | \forall J \in JB^{(\alpha)}, \exists (l,\tau) \in A$ s.t. $t(K+1, J, (l,\tau)) > \frac{t_{l\tau}}{2}\}$.

Fix $A^{(\alpha)} \in \Omega$ whose number of elements is minimum in Ω: $\#A^{(\alpha)} = \min_{A \in \Omega} \#A$.

Also let
$$T_{i,J} = \sum_{(l,\tau)\in A^{(\alpha)}} t(i,J,(l,\tau)) + \begin{cases} Qs(i,J)+1, & \text{if } J = 0, J \in JC^{(\alpha)}, \\ s(i,J)+1, & \text{if } J \in RJ^{(\alpha)} \cap JC^{(\alpha)}, \\ s(i,J), & \text{if } J \in JC^{(\alpha)} \setminus (RJ^{(\alpha)} \cup \{0\}), \\ 0, & \text{otherwise}, \end{cases} \tag{6}$$
$$T = \sum_{(l,\tau)\in A^{(\alpha)}} t_{l\tau} + 2, \qquad Q = \sum_{(l,\tau)\in A^{(\alpha)}} q_{l\tau} + K + \#A^{(\alpha)} + \#C'^{(\alpha)} - 1. \tag{7}$$

Let $C_*^{(\alpha)} = \{m \in C^{(\alpha)} \mid J_m^{(\alpha)} \notin RJ^{(\alpha)}, J_m^{(\alpha)} \neq 0, J_m^{(\alpha)} \neq J_{i'}^{(\alpha)}$ for all $k < i' \leq K\}$.

Case 1 $C_*^{(\alpha)} \neq \phi$.

3.2.1 Transformation in Case 1

If a_m, e_m with $m \in C_*^{(\alpha)}$ are all constants, we reorder those as $a_{K+1}, \ldots a_{K+l}$ and $e_{K+1}, \ldots e_{K+l}$, i.e., $C_*^{(\alpha)} = \{K+1, \ldots, K+l\}$. Then we have
$$\frac{C_{K+l}}{\prod_{\tau=0}^r (v_{1\tau}^{t_{1\tau}/2} v_{2\tau}^{t_{2\tau}/2} \cdots v_{k_\tau\tau}^{t_{k_\tau\tau}/2})} = a_{K+l} g(K+l, K+l) + \cdots \cong a_{K+l} g(K+l, K+l) \neq 0,$$
in the neighborhood of $\{v_{1\tau} = \cdots = v_{k_\tau\tau} = 0\}$. Therefore, we obtain the poles $-\frac{q_{k''\tau}+1}{t_{k''\tau}}, 0 \leq \tau \leq r, 1 \leq k'' \leq k_\tau$.

If there exist non-constants a_m, e_m with $m \in C_*^{(\alpha)}$, we can assume that $K+1 \in C_*^{(\alpha)}$ by reordering and that $K+1 \in E_\tau$. Then we have $\frac{C_{K+1}}{\prod_{\tau=0}^r (v_{1\tau}^{t_{1\tau}/2} v_{2\tau}^{t_{2\tau}/2} \cdots v_{k_\tau\tau}^{t_{k_\tau\tau}/2})} = a_{K+1} g(K+1, K+1) + \cdots$. Change the variable from a_{K+1} to d_{K+1} by $d_{K+1} = \frac{C_{K+1}}{\prod_{\tau=0}^r (v_{1\tau}^{t_{1\tau}/2} v_{2\tau}^{t_{2\tau}/2} \cdots v_{k_\tau\tau}^{t_{k_\tau\tau}/2})}$. Put $K_\tau \to K_\tau + 1$ and we have the inductive statements of $K \to K+1$.

Case 2 $C_*^{(\alpha)} = \phi$.

Construct the blowing-up of Ψ' in (5) along the submanifold $\{d_1 = \cdots = d_K = v_{l\tau} = e_m = 0, (l,\tau) \in A^{(\alpha)}, m \in C'^{(\alpha)}\}$.

3.2.2 Transformation (i)

Let $d_1 = u_1, d_\ell = u_1 d_\ell, 2 \leq \ell \leq K, v_{l\tau} = u_1 v_{l\tau}, (l,\tau) \in A^{(\alpha)}$ and $e_m = u_1 e_m, m \in C'^{(\alpha)}$. Substituting them into Eq.(5) gives the poles $-\frac{\sum_{(l,\tau)\in A^{(\alpha)}} q_{l\tau} + K + \#A^{(\alpha)} + \#C'^{(\alpha)}}{\sum_{(l,\tau)\in A^{(\alpha)}} t_{l\tau} + 2}$ and $-\frac{q_{l\tau}+1}{t_{l\tau}}, 0 \leq \tau \leq r, 1 \leq l \leq k_\tau$.

3.2.3 Transformation (ii)

Fix $(k',\tau') \in A^{(\alpha)}$. Let $d_\ell = v_{k'\tau'} d_\ell, 1 \leq \ell \leq K, v_{l\tau} = v_{k'\tau'} v_{l\tau}, (l,\tau) \in A^{(\alpha)} - \{(k',\tau')\}$ and $e_m = v_{k'\tau'} e_m, m \in C'^{(\alpha)}$. Set $t_{k'\tau'} \to T, t(i,J,(k',\tau')) \to T_{i,J}$ and $q_{k'\tau'} \to Q$ by using Eq.(6), (7).

Then, we have $t_{k'\tau'}/2 = (\sum_{(l,\tau)\in A^{(\alpha)}} t_{l\tau} + 2)/2 = \sum_{(l,\tau)\in A^{(\alpha)}} \sum_{\xi=1}^{g_{(l,\tau),\tau''}} (1 + \eta_{1,(l,\tau),\tau''}^{(\xi)} + \cdots + \eta_{K_{\tau''},(l,\tau),\tau''}^{(\xi)}) + 1$,

and
$$q_{k'\tau'} + 1 = \sum_{(l,\tau)\in A^{(\alpha)}} (q_{l\tau}+1) + K + \#C'^{(\alpha)} = \sum_{(l,\tau)\in A^{(\alpha)}} \left\{ \sum_{\tau''=0}^{r} \sum_{\xi=1}^{g_{(l,\tau),\tau''}} \varphi_{(l,\tau),\tau''}^{(\xi)} + \sum_{\tau''=0}^{r} \phi_{(l,\tau),\tau''} \right.$$
$$\left. + \sum_{\tau''=0}^{r} \sum_{\substack{k+1 \leq m \leq p', \\ m \in E_{\tau''}}} (-g_{(l,\tau),\tau''} + \sum_{J_m^{(\alpha)} \geq J^{(\mu)}} D_{J^{(\mu)},(l,\tau)}) \right\} + K + \sum_{J_m^{(\alpha)} \in JC^{(\alpha)}, m \in E} 1$$
$$= \sum_{(l,\tau)\in A^{(\alpha)}} \sum_{\tau''=0}^{r} \sum_{\xi=1}^{g_{(l,\tau),\tau''}} \varphi_{(l,\tau),\tau''}^{(\xi)} + \sum_{\tau''=0}^{r} \sum_{\substack{k+1 \leq m \leq p', \\ J_m^{(\alpha)} \notin JC^{(\alpha)}, m \in E_{\tau''}}} (- \sum_{(l,\tau)\in A^{(\alpha)}} g_{(l,\tau),\tau''} + \sum_{J_m^{(\alpha)} \geq J^{(\mu)}} \sum_{(l,\tau)\in A^{(\alpha)}} D_{J^{(\mu)},(l,\tau)})$$

$$+\sum_{\tau''=0}^{r}\sum_{\substack{k+1\leq m\leq p',\\ J_m^{(\alpha)}\in JC^{(\alpha)},\\ m\in E_{\tau''}}}(-\sum_{(l,\tau)\in A^{(\alpha)}} g_{(l,\tau),\tau''}+(\sum_{J_m^{(\alpha)}\geq J^{(\mu)}}\sum_{(l,\tau)\in A^{(\alpha)}} D_{J^{(\mu)},(l,\tau)})+1)+\sum_{(l,\tau)\in A^{(\alpha)}}\sum_{\tau''=0}^{r}\phi_{(l,\tau),\tau''}+K.$$

Put $D_{J^{(\alpha)},(k',\tau')}\to\sum_{(l,\tau)\in A^{(\alpha)}} D_{J^{(\alpha)},(l,\tau)}+1$ if $J^{(\alpha)}\in JC^{(\alpha)}$. Also put $D_{J^{(\mu)},(k',\tau')}\to\sum_{(l,\tau)\in A^{(\alpha)}} D_{J^{(\mu)},(l,\tau)}$ if $J^{(\mu)}\notin JC^{(\alpha)}$. Then we have (d) and $q_{k'\tau'}+1=\sum_{\tau''=0}^{r}\left\{\sum_{(l,\tau)\in A^{(\alpha)}}\sum_{\xi=1}^{g_{(l,\tau),\tau''}}\varphi^{(\xi)}_{(l,\tau),\tau''}\right.$

$$\left.+\sum_{\substack{k+1\leq m\leq p',\\ m\in E_{\tau''}}}(-\sum_{(l,\tau)\in A^{(\alpha)}} g_{(l,\tau),\tau''}+\sum_{J_m^{(\alpha)}\geq J^{(\mu)}} D_{J^{(\mu)},(k',\tau')})+K_{\tau''}+\sum_{(l,\tau)\in A^{(\alpha)}}\phi_{(l,\tau),\tau''}\right\}.$$

For all $0\leq\tau''\leq r$, one of the following steps (I), (II), Special Case 1 and Special Case 2 is proceeded with respect to each condition of the case τ''. Fix $\tau''=\tilde\tau$.

(I) Assume that there exist $(l_0,\tau_0)\in A^{(\alpha)}$ and ξ_0 such that $0\leq\eta^{(\xi_0)}_{1,(l_0,\tau_0),\tilde\tau}+\eta^{(\xi_0)}_{2,(l_0,\tau_0),\tilde\tau}+\cdots+\eta^{(\xi_0)}_{K_{\tilde\tau},(l_0,\tau_0),\tilde\tau}<\begin{cases}QK_{\tilde\tau} & \text{if }\tilde\tau=0,\\ K_{\tilde\tau}-1 & \text{if }\tilde\tau\geq1.\end{cases}$ Set $g_{(k',\tau'),\tilde\tau}\to\sum_{(l,\tau)\in A^{(\alpha)}} g_{(l,\tau),\tilde\tau}$. Let $\varphi^{(1)}_{(k',\tau'),\tilde\tau},\cdots,\varphi^{(g_{(k',\tau'),\tilde\tau})}_{(k',\tau'),\tilde\tau}$ be $\varphi^{(\xi)}_{(l,\tau),\tilde\tau}$, $(l,\tau)\in A^{(\alpha)}$, $\xi=1,\cdots,g_{(l,\tau),\tilde\tau}$, and $\eta^{(1)}_{\ell,(k',\tau'),\tilde\tau},\cdots,\eta^{(g_{(k',\tau'),\tilde\tau})}_{\ell,(k',\tau'),\tilde\tau}$ be $\eta^{(\xi)}_{\ell,(l,\tau),\tilde\tau}$, $(l,\tau)\in A^{(\alpha)}$, $\xi=1\cdots,g_{(l,\tau),\tilde\tau}$ by numbering in the same order for any ℓ. Then $t_{k'\tau'}/2=\sum_{\xi=1}^{g_{(k',\tau'),\tilde\tau}}(1+\eta^{(\xi)}_{1,(k',\tau'),\tilde\tau}+\cdots+\eta^{(\xi)}_{K_{\tilde\tau},(k',\tau'),\tilde\tau})+1$.

By the assumption (I), there exists ξ_1 such that $0\leq\eta^{(\xi_1)}_{1,(k',\tau'),\tilde\tau}+\eta^{(\xi_1)}_{2,(k',\tau'),\tilde\tau}+\cdots+\eta^{(\xi_1)}_{K_{\tilde\tau},(k',\tau'),\tilde\tau}<\begin{cases}QK_{\tilde\tau} & \text{if }\tilde\tau=0,\\ K_{\tilde\tau}-1 & \text{if }\tilde\tau\geq1.\end{cases}$ Let $\varphi^{(\xi_1)}_{(k',\tau'),\tilde\tau}\to\begin{cases}s_{\tilde\tau}+\eta^{(\xi_1)}_{1,(k',\tau'),\tilde\tau}+\cdots+K_{\tilde\tau}(\eta^{(\xi_1)}_{K_{\tilde\tau},(k',\tau'),\tilde\tau}+1), & \text{if }\tilde\tau=0,\tilde\tau<\tilde\tau\leq r,\\ s_{\tilde\tau}+1+\eta^{(\xi_1)}_{1,(k',\tau'),\tilde\tau}+\cdots+K_{\tilde\tau}(\eta^{(\xi_1)}_{K_{\tilde\tau},(k',\tau'),\tilde\tau}+1), & \text{if }1\leq\tilde\tau\leq\tilde\tau,\end{cases}$ $\eta^{(\xi_1)}_{K_{\tilde\tau},(k',\tau'),\tilde\tau}\to\eta^{(\xi_1)}_{K_{\tilde\tau},(k',\tau'),\tilde\tau}+1$, $\phi_{(k',\tau'),\tilde\tau}\to\sum_{(l,\tau)\in A^{(\alpha)}}\phi_{(l,\tau),\tilde\tau}$. Then we have $t_{k'\tau'}/2=\sum_{\xi=1}^{g_{(k',\tau'),\tilde\tau}}(1+\eta^{(\xi)}_{1,(k',\tau'),\tilde\tau}+\cdots+\eta^{(\xi)}_{K_{\tilde\tau},(k',\tau'),\tilde\tau})$ and $q_{(k',\tau'),\tilde\tau}=\sum_{\xi=1}^{g_{(k',\tau'),\tilde\tau}}\varphi^{(\xi)}_{(k',\tau'),\tilde\tau}+\sum_{k+1\leq m\leq p',m\in E_{\tilde\tau}}(-g_{(k',\tau'),\tilde\tau}$

$+\sum_{J_m^{(\alpha)}\geq J^{(\mu)}} D_{J^{(\mu)},(k',\tau')})+\phi_{(k',\tau'),\tilde\tau}$. **The end of (I)**

(II) Assume $0\leq\eta^{(\xi)}_{1,(l,\tau),\tilde\tau}+\eta^{(\xi)}_{2,(l,\tau),\tilde\tau}+\cdots+\eta^{(\xi)}_{K_{\tilde\tau},(l,\tau),\tilde\tau}=\begin{cases}QK_{\tilde\tau}, & \text{if }\tilde\tau=0,\\ K_{\tilde\tau}-1, & \text{if }\tilde\tau\geq1,\end{cases}$ for all $(l,\tau)\in A^{(\alpha)}$ and ξ.

The assumption (d) gives $\frac{t_{l\tau}}{2}=\sum_{\xi=1}^{g_{(l,\tau),\tilde\tau}}(1+\eta^{(\xi)}_{1,(l,\tau),\tilde\tau}+\cdots+\eta^{(\xi)}_{K_{\tilde\tau},(l,\tau),\tilde\tau})=g_{(l,\tau),\tilde\tau}\times\begin{cases}QK_{\tilde\tau}+1, & \text{if }\tilde\tau=0,\\ K_{\tilde\tau}, & \text{if }\tilde\tau\geq1,\end{cases}$ and

$$\frac{t_{l\tau}}{2}\leq\sum_{J\geq0^{(\mu)}} D_{0^{(\mu)},(l,\tau)}(Qs(i,0^{(\mu)})+1)+\sum_{\substack{J\geq J^{(\mu)}\\ J^{(\mu)}\in RJ^{(\mu)}}} D_{J^{(\mu)},(l,\tau)}(s(i,J^{(\mu)})+1)+\sum_{\substack{J\geq J^{(\mu)}\\ J^{(\mu)}\notin RJ^{(\mu)}\cup\{0\}}} D_{J^{(\mu)},(l,\tau)}s(i,J^{(\mu)})$$

for all $J=J_m^{(\alpha)}\in\mathbb{R}^\alpha$, $i\leq m\leq p'$. If not Special Case, we have $g_{(l,\tau),\tilde\tau}<\sum_{J_m^{(\alpha)}\geq J^{(\mu)}} D_{J^{(\mu)},(l,\tau)}$ for $m\in E_{\tilde\tau}$, since $s(K+1,J^{(\mu)})\leq\begin{cases}K_\tau, & \text{if }0\leq\tau\leq\tilde\tau,\\ K_\tau-1, & \text{if }\tilde\tau<\tau\leq r.\end{cases}$ Therefore we obtain

$$\sum_{(l,\tau)\in A^{(\alpha)}} g_{(l,\tau),\tilde\tau}<\sum_{(l,\tau)\in A^{(\alpha)}}\sum_{J_m^{(\alpha)}\geq J^{(\mu)}} D_{J^{(\mu)},(l,\tau)}\leq\sum_{J_m^{(\alpha)}\geq J^{(\mu)}} D_{J^{(\mu)},(k',\tau')}. \tag{8}$$

$$\text{Let } q_{(k',\tau'),\tilde\tau}=\sum_{(l,\tau)\in A^{(\alpha)}}\sum_{\xi=1}^{g_{(l,\tau),\tilde\tau}}\varphi^{(\xi)}_{(l,\tau),\tilde\tau}+\sum_{\substack{k+1\leq m\leq p'\\ m\in E_{\tilde\tau}}}(-\sum_{(l,\tau)\in A^{(\alpha)}} g_{(l,\tau),\tilde\tau}+\sum_{J_m^{(\alpha)}\geq J^{(\mu)}} D_{J^{(\mu)},(k',\tau')})+K_{\tilde\tau}$$

$$+\sum_{(l,\tau)\in A^{(\alpha)}}\phi_{(l,\tau),\tilde\tau}=\sum_{(l,\tau)\in A^{(\alpha)}}\sum_{\xi=1}^{g_{(l,\tau),\tilde\tau}}\varphi^{(\xi)}_{(l,\tau),\tilde\tau}+\sum_{k+1\leq m\leq p',m\in E_{\tilde\tau}}(-\sum_{(l,\tau)\in A^{(\alpha)}} g_{(l,\tau),\tilde\tau}-1$$

$$+\sum_{J_m^{(\alpha)}\geq J^{(\mu)}} D_{J^{(\mu)},(k',\tau')})+\left\{\begin{matrix}s_{\tilde\tau}-k_{\tilde\tau} & \text{if }0\leq\tilde\tau\leq\tilde\tau\\ s_{\tilde\tau}-1-k_{\tilde\tau} & \text{if }\tilde\tau<\tilde\tau\leq r\end{matrix}\right\}+K_{\tilde\tau}+\sum_{(l,\tau)\in A^{(\alpha)}}\phi_{(l,\tau),\tilde\tau}.$$

Put $g_{(k',\tau'),\tilde\tau}\to\sum_{(l,\tau)\in A^{(\alpha)}} g_{(l,\tau),\tilde\tau}+1$, which satisfies $g_{(k',\tau'),\tilde\tau}\leq\sum_{J_m^{(\alpha)}\geq J^{(\mu)}} D_{J^{(\mu)},(k',\tau')}$ by (8). Set $\varphi^{(g_{(k',\tau'),\tilde\tau})}_{(k',\tau'),\tilde\tau}=\begin{cases}s_{\tilde\tau}, & \text{if }\tilde\tau=0,\tilde\tau<\tilde\tau\leq r,\\ s_{\tilde\tau}+1, & \text{if }1\leq\tilde\tau\leq\tilde\tau.\end{cases}$ Let $\varphi^{(1)}_{(k',\tau'),\tilde\tau},\cdots,\varphi^{(g_{(k',\tau'),\tilde\tau}-1)}_{(k',\tau'),\tilde\tau}$ be $\varphi^{(\xi)}_{(l,\tau),\tilde\tau}$, $(l,\tau)\in A^{(\alpha)}$,

$\xi = 1, \ldots, g_{(l,\tau),\tilde{\tau}}$, and $\eta^{(1)}_{\ell,(k',\tau'),\tilde{\tau}}, \ldots, \eta^{(g_{(k',\tau'),\tilde{\tau}}-1)}_{\ell,(k',\tau'),\tilde{\tau}}$ be $\eta^{(\xi)}_{\ell,(l,\tau),\tilde{\tau}}$, $(l,\tau) \in A^{(\alpha)}$, $\xi = 1, \ldots, g_{(l,\tau),\tilde{\tau}}$, by numbering in the same order for any ℓ.

Put $\eta^{(g_{(k',\tau'),\tilde{\tau}})}_{\ell,(k',\tau'),\tilde{\tau}} = 0$ for $\ell \geq 1$, and $\phi_{(l,\tau),\tilde{\tau}} = \begin{cases} -k_{\tilde{\tau}} + K_{\tilde{\tau}} + \sum_{(l,\tau)\in A^{(\alpha)}} \phi_{(l,\tau),\tilde{\tau}}, & \text{if } \tilde{\tau} = 0, \\ -k_{\tilde{\tau}} + K_{\tilde{\tau}} - 1 + \sum_{(l,\tau)\in A^{(\alpha)}} \phi_{(l,\tau),\tilde{\tau}}, & \text{if } 1 \leq \tilde{\tau} \leq r. \end{cases}$

Then we obtain $\frac{t_{k'\tau'}}{2} = \sum_{\xi=1}^{g_{(k',\tau'),\tilde{\tau}}} (1 + \eta^{(\xi)}_{1,(k',\tau'),\tilde{\tau}} + \cdots + \eta^{(\xi)}_{K_{\tilde{\tau}},(k',\tau'),\tilde{\tau}})$ and $q_{(k',\tau'),\tilde{\tau}} = \sum_{\xi=1}^{g_{(k',\tau'),\tilde{\tau}}} \varphi^{(\xi)}_{(k',\tau'),\tilde{\tau}}$
$+ \sum_{k+1\leq m\leq p', m\in E_{\tilde{\tau}}} (-g_{(k',\tau'),\tilde{\tau}} + \sum_{J^{(\alpha)}_m \geq J^{(\mu)}} D_{J^{(\mu)},(k',\tau')}) + \phi_{(k',\tau'),\tilde{\tau}}$. **The end of (II)**

Special Case 1 Assume that $J^{(\alpha)}_m \notin JC^{(\alpha)}$ for all $m \in E_{\tilde{\tau}}$. By (e'), we have $t_{k'\tau'}/2 = (\sum_{(l,\tau)\in A^{(\alpha)}} t_{l\tau} +$
$2)/2 = 1 + \sum_{(l,\tau)\in A^{(\alpha)}} \sum_{\xi=1}^{g_{(l,\tau),\tilde{\tau}}} \eta^{(\xi)}_{K_{\tilde{\tau}},(l,\tau),\tilde{\tau}} + \sum_{(l,\tau)\in A^{(\alpha)}} \sum_{J^{(\alpha)}_m \geq J^{(\mu)}} \begin{cases} D_{J^{(\mu)},(l,\tau)}(Q(K_0 - 1 - k^{(\mu)}_0) + 1), & \text{if } \tilde{\tau} = 0, \\ D_{J^{(\mu)},(l,\tau)}((K_{\tilde{\tau}} - 2 - k^{(\mu)}_{\tilde{\tau}}) + 1), & \text{if } 1 \leq \tilde{\tau} \leq r. \end{cases}$

By using $t(K+1, J^{(\alpha)}_m, (k',\tau')) = \sum_{(l,\tau)\in A^{(\alpha)}} t(K+1, J^{(\alpha)}_m, (l,\tau)) \geq \frac{t_{k'\tau'}}{2}$ and

$$t(K+1, J^{(\alpha)}_m, (k',\tau')) = \begin{cases} \sum_{(l,\tau)\in A^{(\alpha)}} \sum_{J^{(\alpha)}_m \geq J^{(\mu)}} D_{J^{(\mu)},(l,\tau)}(Q(K_0 - k^{(\mu)}_0) + 1), & \text{if } m \in E_0, \tilde{\tau} = 0, \\ \sum_{(l,\tau)\in A^{(\alpha)}} \sum_{J^{(\alpha)}_m \geq J^{(\mu)}} D_{J^{(\mu)},(l,\tau)}(K_{\tilde{\tau}} - k^{(\mu)}_{\tilde{\tau}}), & \text{if } m \in E_{\tilde{\tau}}, 1 \leq \tilde{\tau} \leq r, \end{cases}$$

we have $1 + \sum_{(l,\tau)\in A^{(\alpha)}} \sum_{\xi=1}^{g_{(l,\tau),\tilde{\tau}}} \eta^{(\xi)}_{K_{\tilde{\tau}},(l,\tau),\tilde{\tau}} \leq \begin{cases} Q \sum_{(l,\tau)\in A^{(\alpha)}} \sum_{J^{(\alpha)}_m \geq J^{(\mu)}} D_{J^{(\mu)},(l,\tau)} = Q \sum_{(l,\tau)\in A^{(\alpha)}} g_{(l,\tau),\tilde{\tau}}, & \text{if } \tilde{\tau} = 0, \\ \sum_{(l,\tau)\in A^{(\alpha)}} \sum_{J^{(\alpha)}_m \geq J^{(\mu)}} D_{J^{(\mu)},(l,\tau)} = \sum_{(l,\tau)\in A^{(\alpha)}} g_{(l,\tau),\tilde{\tau}}, & \text{if } 1 \leq \tilde{\tau} \leq r. \end{cases}$

So, there exist $(l_0,\tau_0) \in A^{(\alpha)}$ and ξ_0 such that $\eta^{(\xi_0)}_{K_{\tilde{\tau}},(l_0,\tau_0),\tilde{\tau}} < \begin{cases} Q & \text{if } \tilde{\tau} = 0, \\ 1 & \text{if } 1 \leq \tilde{\tau} \leq r. \end{cases}$

Put $g_{(k',\tau'),\tilde{\tau}} \to \sum_{(l,\tau)\in A^{(\alpha)}} g_{(l,\tau),\tilde{\tau}}$. Let $\varphi^{(1)}_{(k',\tau'),\tilde{\tau}}, \cdots, \varphi^{(g_{(k',\tau'),\tilde{\tau}})}_{(k',\tau'),\tilde{\tau}}$ be $\varphi^{(\xi)}_{(l,\tau),\tilde{\tau}}$, $(l,\tau) \in A^{(\alpha)}$, $\xi = 1 \cdots, g_{(l,\tau),\tilde{\tau}}$, and $\eta^{(1)}_{\ell,(k',\tau'),\tilde{\tau}}, \cdots, \eta^{(g_{(k',\tau'),\tilde{\tau}})}_{\ell,(k',\tau'),\tilde{\tau}}$ be $\eta^{(\xi)}_{\ell,(l,\tau),\tilde{\tau}}$, $(l,\tau) \in A^{(\alpha)}$, $\xi = 1, \cdots, g_{(l,\tau),\tilde{\tau}}$, by numbering in the same order for all ℓ. Choose ξ_1 with $\varphi^{(\xi_1)}_{(k',\tau'),\tilde{\tau}} = \varphi^{(\xi_0)}_{(l_0,\tau_0),\tilde{\tau}}$. Set $\eta^{(\xi_1)}_{K_{\tilde{\tau}},(k',\tau'),\tilde{\tau}} \to \eta^{(\xi_1)}_{K_{\tilde{\tau}},(k',\tau'),\tilde{\tau}} + 1$ and $\varphi^{(\xi_1)}_{(k',\tau'),\tilde{\tau}} \to \varphi^{(\xi_1)}_{(k',\tau'),\tilde{\tau}} + K_{\tilde{\tau}}$. Then we obtain $\frac{t_{k'\tau'}}{2} = \sum_{\xi=1}^{g_{(k',\tau'),\tilde{\tau}}} (1 + \eta^{(\xi)}_{1,(k',\tau'),\tilde{\tau}} + \cdots + \eta^{(\xi)}_{K_{\tilde{\tau}},(k',\tau'),\tilde{\tau}})$, and $q_{(k',\tau'),\tilde{\tau}} = \sum_{\xi=1}^{g_{(k',\tau'),\tilde{\tau}}} \varphi^{(\xi)}_{(k',\tau'),\tilde{\tau}}$ **The end of Special Case 1**

Special Case 2 Assume that $J^{(\alpha)}_m \in JC^{(\alpha)}$ for all $m \in E_{\tilde{\tau}}$. Since (e') yields

$$t_{l\tau}/2 = \sum_{\xi=1}^{g_{(l,\tau),\tilde{\tau}}} \eta^{(\xi)}_{K_{\tilde{\tau}},(l,\tau),\tilde{\tau}} + \sum_{J^{(\alpha)}_m \geq J^{(\mu)}} \begin{cases} D_{J^{(\mu)},(l,\tau)}(Q(K_0 - 1 - k^{(\mu)}_0) + 1), & \text{if } \tilde{\tau} = 0, \\ D_{J^{(\mu)},(l,\tau)}((K_{\tilde{\tau}} - 2 - k^{(\mu)}_{\tilde{\tau}}) + 1), & \text{if } 1 \leq \tilde{\tau} \leq r, \end{cases} \quad \text{and}$$

$$t(K+1, J^{(\alpha)}_m, (l,\tau)) = \begin{cases} \sum_{J^{(\alpha)}_m \geq J^{(\mu)}} D_{J^{(\mu)},(l,\tau)}(Q(K_0 - k^{(\mu)}_0) + 1), & \text{if } m \in E_0, \tilde{\tau} = 0, \\ \sum_{J^{(\alpha)}_m \geq J^{(\mu)}} D_{J^{(\mu)},(l,\tau)}(K_{\tilde{\tau}} - k^{(\mu)}_{\tilde{\tau}}), & \text{if } m \in E_{\tilde{\tau}}, 1 \leq \tilde{\tau} \leq r, \end{cases} \quad \text{we have}$$

$$\sum_{\xi=1}^{g_{(l,\tau),\tilde{\tau}}} \eta^{(\xi)}_{K_{\tilde{\tau}},(l,\tau),\tilde{\tau}} = \begin{cases} Q \sum_{J^{(\alpha)}_m \geq J^{(\mu)}} D_{J^{(\mu)},(l,\tau)} = Q g_{(l,\tau),\tilde{\tau}}, & \text{if } \tilde{\tau} = 0, \\ \sum_{J^{(\alpha)}_m \geq J^{(\mu)}} D_{J^{(\mu)},(l,\tau)} = g_{(l,\tau),\tilde{\tau}}, & \text{if } 1 \leq \tilde{\tau} \leq r, \end{cases} \quad \text{that is, } \eta^{(\xi)}_{K_{\tilde{\tau}},(l,\tau),\tilde{\tau}} = \begin{cases} Q, & \text{if } \tilde{\tau} = 0, \\ 1, & \text{if } 1 \leq \tilde{\tau} \leq r, \end{cases}$$

for all $\xi = 1, \ldots, g_{(l,\tau),\tilde{\tau}}$. Put $g_{(k',\tau'),\tilde{\tau}} \to \sum_{(l,\tau)\in A^{(\alpha)}} g_{(l,\tau),\tilde{\tau}}$. Let $\varphi^{(1)}_{(k',\tau'),\tilde{\tau}}, \cdots, \varphi^{(g_{(k',\tau'),\tilde{\tau}})}_{(k',\tau'),\tilde{\tau}}$ be $\varphi^{(\xi)}_{(l,\tau),\tilde{\tau}}$, $(l,\tau) \in A^{(\alpha)}$, $\xi = 1 \cdots, g_{(l,\tau),\tilde{\tau}}$, and $\eta^{(1)}_{\ell,(k',\tau'),\tilde{\tau}}, \cdots, \eta^{(g_{(k',\tau'),\tilde{\tau}})}_{\ell,(k',\tau'),\tilde{\tau}}$ be $\eta^{(\xi)}_{\ell,(l,\tau),\tilde{\tau}}$, $(l,\tau) \in A^{(\alpha)}$, $\xi = 1 \cdots, g_{(l,\tau),\tilde{\tau}}$, by numbering in the same order for any ℓ. Then we have $\frac{t_{k'\tau'}}{2} = \sum_{\xi=1}^{g_{(k',\tau'),\tilde{\tau}}} (1 + \eta^{(\xi)}_{1,(k',\tau'),\tilde{\tau}} + \cdots +$

$\eta^{(\xi)}_{K_{\tilde\tau},(k',\tau'),\tilde\tau})+1$ and $q_{(k',\tau'),\tilde\tau} = \sum_{\xi=1}^{g_{(k',\tau'),\tilde\tau}} \varphi^{(\xi)}_{(k',\tau'),\tilde\tau} + \begin{cases} K_{\tilde\tau}+s_{\tilde\tau}-k_{\tilde\tau}, & \text{if } \tilde\tau=0 \\ K_{\tilde\tau}+s_{\tilde\tau}-k_{\tilde\tau}, & \text{if } 1\le\tilde\tau\le\tilde r \\ K_{\tilde\tau}+s_{\tilde\tau}-1-k_{\tilde\tau}, & \text{if } \tilde r<\tilde\tau\le r \end{cases} = \sum_{\xi=1}^{g_{(k',\tau'),\tilde\tau}} \varphi^{(\xi)}_{(k',\tau'),\tilde\tau} +$ $\begin{cases} s_{\tilde\tau} & \text{if } \tilde\tau=0, \\ s_{\tilde\tau}+1 & \text{if } 1\le\tilde\tau\le\tilde r, \\ s_{\tilde\tau} & \text{if } \tilde r<\tilde\tau\le r. \end{cases}$ So we set $\varphi^{(g_{(k',\tau'),\tilde\tau}+1)}_{(k',\tau'),\tilde\tau} \to \begin{cases} s_{\tilde\tau} & \text{if } \tilde\tau=0, \tilde r<\tilde\tau\le r, \\ s_{\tilde\tau}+1 & \text{if } 1\le\tilde\tau\le\tilde r, \end{cases}$ $\eta^{(g_{(k',\tau'),\tilde\tau}+1)}_{\ell,(k',\tau'),\tilde\tau}=0$ for $\ell\ge 1$, and $g_{(k',\tau'),\tilde\tau} \to g_{(k',\tau'),\tilde\tau}+1$. We have $\frac{t_{k'\tau'}}{2} = \sum_{\xi=1}^{g_{(k',\tau'),\tilde\tau}} (1+\eta^{(\xi)}_{1,(k',\tau'),\tilde\tau}+\cdots+\eta^{(\xi)}_{K_{\tilde\tau},(k',\tau'),\tilde\tau})$ and $q_{(k',\tau'),\tilde\tau} = \sum_{\xi=1}^{g_{(k',\tau'),\tilde\tau}} \varphi^{(\xi)}_{(k',\tau'),\tilde\tau}$.

The end of Special Case 2

If $JC^{(\alpha)} \neq \phi$, the following Step($*$) is needed.

Step($*$) Let $j^{(\alpha+1)}_m$ be any real number for each m with $J^{(\alpha)}_m \in JC^{(\alpha)}$. For m with $J^{(\alpha)}_m \notin JC^{(\alpha)}$, let $j^{(\alpha+1)}_m = j^{(\alpha)}_m$ where $J^{(\alpha)}_m = (J', j^{(\alpha)}_m)$, $J' \in \mathbb{R}^{\alpha-1}$ and $j^{(\alpha)}_m \in \mathbb{R}$. Consider a sufficiently small neighborhood of $e_m = j^{(\alpha+1)}_m$ and fix it. Put $J^{(\alpha+1)}_m = \begin{cases} (J^{(\alpha)}_m, (j^{(\alpha+1)}_m)^Q) & \text{if } J^{(\alpha)}_m = 0, \\ (J^{(\alpha)}_m, j^{(\alpha+1)}_m) & \text{if } J^{(\alpha)}_m \neq 0. \end{cases}$ Let $t(i,J,(l,\tau)) = t(i,J',(l,\tau))$ for $J=(J',*) \in \mathbb{R}^{\alpha+1}$ where $J' \in \mathbb{R}^{\alpha}$. Change $g(i,m)\neq 0$ and e_m properly, taking into account that the neighborhood of $e_m = j^{(\alpha+1)}_m$. Let $RJ^{(\alpha+1)}$ be the set of J satisfying $\sum_{\substack{i\le m\le p \\ J^{(\alpha+1)}_m=J}} g(i,m)a_m e_m \prod_{\substack{k+1\le i'<i \\ J^{(\alpha+1)}_{i'}=J}} (e_m - e_{i'})$ in C_i. Let $\alpha \to \alpha+1$.

The end of Step($*$)

Those new parameters defined in Step 2 satisfy Statements (a)∼(f) or (a')∼(f').

3.2.4 Transformation (iii)

Fix $k_c \in C'^{(\alpha)}$. Assume that $k_c \in E_{\tau'}$. Let $d_\ell = v_{k_{\tau'}+1,\tau'} d_\ell, 1 \le \ell \le K, e_{k_c} = v_{k_{\tau'}+1,\tau'}, v_{l\tau} = v_{k_{\tau'}+1,\tau'} v_{l\tau}, (l,\tau) \in A^{(\alpha)}$ and $e_m = v_{k_{\tau'}+1,\tau'} e_m, m \in C'^{(\alpha)} - \{k_c\}$.

(A) If $k < k_c \le K$, then we can assume $k_c = k+1$ by the symmetry of the formulas C_i.

(III) Do the same procedure in Transformation (ii) by substituting $(k_{\tau'}+1, \tau')$ into (k', τ'). Adding it, set $\phi_{(l,\tau),\tau''} \to (-g_{(l,\tau),\tau''} + \sum_{J^{(\alpha)}_{k+1} \ge J^{(\mu)}} D_{J^{(\mu)},(l,\tau),\tau''}) + \phi_{(l,\tau),\tau''}$ for all l, τ and τ''.

The end of (III)

Then, putting $k_{\tau'} \to k_{\tau'}+1$ and $k \to k+1$ completes Statements (a)∼(f).

Special Case We need (a') for Special Case. After setting in (A), we have $k+1 \in C^{(\alpha)}_*$. So, we can transform in Case 1 with non-constants a_{k+1}, e_{k+1}. Then we have $K_{\tau'} \to K_{\tau'}+1$ and the inductive statements of $K \to K+1$.

(B) Next consider the case $K+1 \le k_c \le p'$.

If $J^{(\alpha)}_{k_c} \notin RJ^{(\alpha)}$ and $J^{(\alpha)}_{k_c} \neq 0$, then there exists $i' \le K$ such that $J^{(\alpha)}_{i'} = J^{(\alpha)}_{k_c}$ because of the Case 2 assumption. By the symmetry of the formulas C_i, we can assume $i' = k_c$. So the case results in (A). Consider the case $\tilde J := J^{(\alpha)}_{k_c} \in RJ$ or $\tilde J := J^{(\alpha)}_{k_c} = 0$. Again by the symmetry, we can assume that $k_c = K+1$. By the transformation (iii), we have $\frac{C_{K+1}}{\prod_{\tau=0}^{r}(v_{1\tau}^{t_{1\tau}/2} v_{2\tau}^{t_{2\tau}/2} \cdots v_{k_\tau \tau}^{t_{k_\tau\tau}/2})} = v_{k+1}(a_{K+1}g(K+1,K+1)+\cdots)$. Now, there is no a_{K+1} in the formulas C_i, $i \ge K+2$. So, we can change the variable from a_{K+1} to d_{K+1} by $d_{K+1} = a_{K+1}g(K+1,K+1)+\cdots$: $v_{k+1}d_{K+1} = \frac{C_{K+1}}{\prod_{\tau=0}^{r}(v_{1\tau}^{t_{1\tau}/2} v_{2\tau}^{t_{2\tau}/2} \cdots v_{k_\tau \tau}^{t_{k_\tau\tau}/2})}$. Since a_{K+1}, e_{K+1} have vanished in all formulas C_i, we can set the variables $e_{k+1} \to e_{k+2}, \cdots, e_K \to e_{K+1}$, together with putting $J^{(\alpha)}_i$ and $RJ^{(\alpha)}$ properly. Proceed Step (III) for C_i, $K+2 \le i \le p'$. By putting $k \to k+1, K \to K+1$, $k_{\tau'} \to k_{\tau'}+1$ and $K_{\tau'} \to K_{\tau'}+1$, we have Statements (a)∼(f) or (a')∼(f').

By repeating Step 2, the conditions Case 2 (ii), (iii)(A) disappear. So, K is increased with these finite steps. $K = p+1$ completes the blowing-up process.

4 Proof of Main Theorem 1 : Part 2

Part 1 shows the blowing-up process. To obtain the maximum pole and its order, we prepare the following theorem.

Theorem 2 *Assume that all $B_\tau^{(w)}$ are Special Case. Then we have the pole λ_Q^* whose order θ in Main Theorem 1.*

4.1 Proof of Theorem 2 : Step 1

Let ζ_1 be the total number of $s_\tau = 1$ and ζ_2 the total number of $s_\tau = 1$ with $a_\tau^{**} \neq 0$ among $1 \le \tau \le \tilde{r}$. We can assume that $s_1 = \cdots = s_{\zeta_2} = 1$, $B_\tau^{(w)} = \{\tau\}$ for $1 \le \tau \le \zeta_2$ and $s_{\tilde{r}+1} = \cdots = s_{\tilde{r}+\zeta_1-\zeta_2} = 1$, $B_\tau^{(w)} = \{\tau\}$ for $\tilde{r}+1 \le \tau \le \zeta_1 - \zeta_2 + \tilde{r}$. Also, we can assume that $a_{r+1} = a_1^{**}$, $\ldots$, $a_{r+\zeta_2} = a_{\zeta_2}^{**}$ in Eq.(3). Then $C_{r+1} = \sum_{m=1}^{\zeta_2} g(r+1, m+r) a_m^{**} b_m + \sum_{\substack{r+\zeta_2+1 \le m \le p', \\ J_m^{(1)}=0}} g(r+1,m) a_m b_m + \sum_{\substack{r+\zeta_2+1 \le m \le p', \\ J_m^{(1)} \neq 0, a_m^* = 0}} g(r+1,m) a_m b_m + \sum_{\substack{r+\zeta_2+1 \le m \le p', \\ J_m^{(1)} \neq 0, a_m^* \neq 0}} g(r+1,m) a_m \prod_{\substack{\zeta_2+1 \le i' \le \tilde{r}, \\ J_{i'}^{(1)} = J_m^{(1)}}} (b_m - b_{i'})$. We can change the variable from b_1 to d_{r+1} by $d_{r+1} = C_{r+1}$. Also we have $C_{r+2} = \sum_{m=2}^{\zeta_2} g(r+2, m+r) a_m^{**} b_m + \sum_{\substack{r+\zeta_2+1 \le m \le p', \\ J_m^{(1)}=0}} g(r+2,m) a_m b_m + \sum_{\substack{r+\zeta_2+1 \le m \le p', \\ J_m^{(1)} \neq 0, a_m^* = 0}} g(r+2,m) a_m b_m + \sum_{\substack{r+\zeta_2+1 \le m \le p', \\ J_m^{(1)} \neq 0, a_m^* \neq 0}} g(r+2,m) a_m \prod_{\substack{\zeta_2+1 \le i' \le \tilde{r}, \\ J_{i'}^{(1)} = J_m^{(1)}}} (b_m - b_{i'})$. So, change the variable from b_2 to d_{r+2} by $d_{r+2} = C_{r+2}$ and so on. Therefore we have

$$\Psi' = \{d_1^2 + \cdots + d_{r+\zeta_2}^2 + C_{r+\zeta_2+1}^2 + \cdots + C_{\tilde{r}+p}^2\}^z \prod_{m=1}^{r+\zeta_2} \mathrm{d}d_m \prod_{m=r+1}^{p} \mathrm{d}a_m \prod_{m=\tilde{r}+1}^{r} \mathrm{d}b_m^{(w)} \prod_{m=1+\zeta_2}^{\tilde{r}} \mathrm{d}b_m \prod_{m=r+1}^{p} \mathrm{d}b_m. \tag{9}$$

Note that b_τ with $s_\tau = 1$ disappear in the expressions C_i because $g(i,m)$'s include b_τ. Put $Y' = \#\Theta$. For simplicity, number the set

$$\Theta = \{\tau_1', \cdots, \tau_{Y'}'\} = \left\{ \tau_0, \tau_1, \tau_2 \;\middle|\; \begin{array}{ll} Q(\tilde{n}_0^2 - \tilde{n}_0) + 2\tilde{n}_0 = 2s_{\tau_0}, & s_{\tau_0} \ge 1, \tau_0 = 0, \\ (n_{\tau_1} - 1)^2 + n_{\tau_1} - 1 = 2s_{\tau_1}, & s_{\tau_1} > 1, 1 \le \tau_1 \le \tilde{r}, \\ (n_{\tau_2} - 1)^2 + n_{\tau_2} - 1 = 2(s_{\tau_2} - 1), & s_{\tau_2} > 1, \tilde{r} < \tau_2 \le r \end{array} \right\}, \text{ and}$$

$\{\tau_1', \ldots, \tau_{Y'}', \tau_{Y'+1}', \ldots, \tau_Y'\} = \{\tau_0, \tau_1 \,|\, s_0 \ge 1, s_{\tau_1} > 1, 1 \le \tau_1 \le r\}$. Set $n_i' = \begin{cases} Q\tilde{n}_0 + 1, & \text{if } \tau_i' = 0, \\ n_{\tau_i'}, & \text{if } \tau_i' \neq 0. \end{cases}$ for $1 \le i \le Y$. Then we have $n_i' \le s_{\tau_i'}$, if $1 \le \tau_i' \le r$. Let I be the lowest common multiple of $n_1', \ldots, n_Y'$ and let $\ell_i = \frac{I}{n_i'}$ for $1 \le i \le Y$. The greatest common divisor of ℓ_i's is clearly 1. Using induction, we can construct a $Y \times Y$ matrix $L_1 = (l_{ij}^{(1)}) \in \mathcal{M}_Y(\mathbb{Z})$ such that (1) $l_{i1}^{(1)} = \ell_i$ for $i = 1, \ldots, Y$, (2) $\det L_1 = \pm 1$, (3) $I_j^{(1)} := n_i' l_{ij}^{(1)}$ for any $i \ge j$, (4) $n_{j-1}' l_{j-1,j}^{(1)} > I_j^{(1)}$. Assume that $Y = 2$. Since the greatest common divisor of ℓ_1 and ℓ_2 is 1, there exist $l_{12}^{(1)}$ and $l_{22}^{(1)}$ such that $\ell_1 l_{22}^{(1)} - \ell_2 l_{12}^{(1)} = -1$. Then $\ell_1(n_1' l_{12}^{(1)} - n_2' l_{22}^{(1)}) = \ell_1 n_1' l_{12}^{(1)} - n_2' \ell_2 l_{12}^{(1)} + n_2' = n_2' > 0$. So $\begin{pmatrix} \ell_1 & l_{12}^{(1)} \\ \ell_2 & l_{22}^{(1)} \end{pmatrix}$ satisfies the above conditions. Let ξ be the greatest common divisor of $\ell_2, \ldots, \ell_Y$ and $\ell_i = \xi \ell_i'$. It is clear that the greatest common divisor of ℓ_i''s is 1. We can assume that we have a matrix $L_2 = (l_{ij}')$, $2 \le i, j \le Y$ satisfying (1)$\sim$(4) for $\ell_2', \ldots, \ell_Y'$. Since the greatest common divisor of ℓ_1 and ξ is 1, there exist $l_{12}^{(1)}$ and ξ_2 such that $\ell_1 \xi_2 - l_{12}^{(1)} \xi = -1$. We then have $n_1' l_{12}^{(1)} - n_2' \ell_2' \xi_2 = n_1' l_{12}^{(1)} - n_2' \ell_2 \xi_2 / \xi = n_1' l_{12}^{(1)} - n_1' \ell_1 \xi_2 / \xi = n_1' l_{12}^{(1)} - n_1' l_{12}^{(1)} + n_1'/\xi > 0$. Put $L_1 = \begin{pmatrix} \ell_1 & l_{12}^{(1)} & 0 & \cdots & 0 \\ \ell_2 & \ell_2' \xi_2 & l_{23}' & \cdots & l_{2Y}' \\ & \vdots & & & \\ \ell_Y & \ell_Y' \xi_2 & l_{Y3}' & \cdots & l_{YY}' \end{pmatrix}$. We have $\det L_1 = (\ell_1 - l_{12}^{(1)} \xi / \xi_2) \xi_2 \det L_2 = (\ell_1 \xi_2 - l_{12}^{(1)} \xi) \det L_2 = -\det L_2$. So, L_1 satisfies the conditions (1)$\sim$(4). Next construct a $Y \times Y$ matrix

$$L = (l_{ij}) \in \mathcal{M}_Y(\mathbb{Z}) \tag{10}$$

such that (i) $l_{i1}^{(1)} = \ell_i$ for $i = 1, \ldots, Y$, (ii) $\det L = \pm 1$, (iii) for every pair of $(\beta_i, \beta_{i'})$ with $1 \le \beta_i \le n_i'$, $1 \le \beta_{i'} \le n_{i'}'$, $1 \le i, i' \le Y$, there exists $0 \le j \le Y$ such that $\beta_i l_{i1} = \beta_{i'} l_{i'1}, \ldots, \beta_i l_{ij} = \beta_{i'} l_{i'j}, \beta_i l_{i,j+1} >$

$\beta_{i'} l_{i',j+1}, \ldots, \beta_i l_{iY} > \beta_{i'} l_{i'Y}$, or $\beta_i l_{i1} = \beta_{i'} l_{i'1}, \ldots, \beta_i l_{ij} = \beta_{i'} l_{i'j}, \beta_i l_{i,j+1} < \beta_{i'} l_{i',j+1}, \ldots, \beta_i l_{iY} < \beta_{i'} l_{i'Y}$. (iv) $I_j := n'_i l_{ij}$ for $i \geq j$, (v) $n'_i l_{ij} > I_j$, $i < j$.

For any positive integers M_{1j}, set $L^{(2)} = (l_{ij}^{(2)}) = L_1 \begin{pmatrix} 1 & M_{12} & \cdots & M_{1Y} \\ 0 & 1 & 0 \cdots & 0 \\ & & \vdots & \\ 0 & 0 & \cdots & 1 \end{pmatrix}$. If $\beta_i l_{i1}^{(1)} > \beta_{i'} l_{i'1}^{(1)}$ and M_{1j}'s are large enough, then $\beta_i l_{ij}^{(2)} > \beta_{i'} l_{i'j}^{(2)}$ for all j. Also we have $I_j^{(2)} := I_j^{(1)} + IM_{1j} = n'_i(l_{ij}^{(1)} + l_{i1}^{(1)} M_{1j})$, $(i \geq j)$ and $n'_{j-1}(l_{j-1,j}^{(1)} + l_{j-1,1}^{(1)} M_{1j}) > I_j^{(1)} + IM_{1j} = I_j^{(2)}$, $(j \geq 2)$. That is, $I_j^{(2)} = n'_i l_{ij}^{(2)}$, $(i \geq j)$, and $n'_{j-1} l_{j-1,j}^{(2)} > I_j^{(2)}$, $(j \geq 2)$, where $I_1^{(2)} := I_1^{(1)} = I$, $I_j^{(2)} := I_j^{(1)} + IM_{1j}$ for $j \geq 2$. Again by using large integers M_{2j} and by setting $L^{(3)} = (l_{ij}^{(3)}) = L^{(2)} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & M_{23} & \cdots & M_{2Y} \\ 0 & 0 & 1 & 0 \cdots & 0 \\ & & & \vdots & \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$, we have $\beta_i l_{ij}^{(3)} > \beta_{i'} l_{i'j}^{(3)}$ for all j when $\beta_i l_{i1}^{(3)} > \beta_{i'} l_{i'1}^{(3)}$. Also we have $\beta_i l_{ij}^{(3)} > \beta_{i'} l_{i'j}^{(3)}$ for $j \geq 2$ when $\beta_i l_{i1}^{(3)} = \beta_{i'} l_{i'1}^{(3)}$, $\beta_i l_{i2}^{(3)} > \beta_{i'} l_{i'2}^{(3)}$. From $I_j^{(3)} := I_j^{(2)} + I_2^{(2)} M_{2j} = n'_i(l_{ij}^{(2)} + l_{i2}^{(2)} M_{2j})$, $(i \geq j \geq 3)$, $n'_1(l_{1j}^{(2)} + l_{12}^{(2)} M_{2j}) > I_j^{(2)} + I_2^{(2)} M_{2j} = I_j^{(3)}$, $(j \geq 3)$, and $n'_{j-1}(l_{j-1,j}^{(2)} + l_{j-1,2}^{(2)} M_{2j}) > I_j^{(2)} + I_2^{(2)} M_{2j} = I_j^{(3)}$, $(j \geq 3)$, we have $n'_i l_{ij}^{(3)} = I_j^{(3)}$, $(i \geq j)$, $n'_1 l_{1j}^{(3)} > I_j^{(3)}$, $(j \geq 2)$, and $n'_{j-1} l_{j-1,j}^{(3)} > I_j^{(3)}$, $(j \geq 2)$, where $I_1^{(3)} := I_1^{(2)} = I$, $I_2^{(3)} := I_2^{(2)}$, $I_j^{(3)} := I_j^{(2)} + IM_{1j}$ for $j \geq 3$. By repeating the process, $L = (l_{ij}) = L^{(Y-1)} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 \cdots & 0 \\ & & \vdots & \\ 0 & 0 & \cdots 1 & M_{Y-1,Y} \\ 0 & 0 & \cdots & 1 \end{pmatrix}$ can be constructed satisfying conditions (i)$\sim$(v). Note that for any $i \neq i'$, there is no case $\beta_i l_{ij} = \beta_{i'} l_{i'j}$ for all j, since $\det L \neq 0$.

4.2 Proof of Theorem 2 : Step 2

Put $B_{\tau'_j} = \{b_k \mid$ non-constant, $b_k^{*Q} = b_{\tau'_j}^{**Q}\}$, $s'_j = \begin{cases} s_{\tau'_j} & \text{if } 0 \leq \tau'_j \leq \tilde{r}, \\ s_{\tau'_j} - 1 & \text{if } \tilde{r}+1 \leq \tau'_j \leq r, \end{cases}$ and $s(i,j) = \#\{m \mid 1 \leq m \leq i-1, b_m \in B_{\tau'_j}\} - \begin{cases} 0, & \tau'_j = 0, \tilde{r}+1 \leq \tau'_j \leq r, \\ 1, & 1 \leq \tau'_j \leq \tilde{r}, \end{cases}$ for $1 \leq j \leq Y$. By numbering elements in $B_{\tau'_j}$, let $B_{\tau'_j} = \{b_{1j}, \ldots, b_{s'_j j}\}$ for $1 \leq j \leq Y$. Set $b_{1j} = u_{1j}$, $b_{kj} = u_{1j} b_{kj}$ for $2 \leq k \leq s'_j$. Again set $b_{2j} = u_{2j}$, $b_{kj} = u_{2j} b_{kj}$ for $3 \leq k \leq s'_j$. Continue these settings until $b_{s'_j j} = u_{s'_j j}$. Let $Q_j = \begin{cases} Q & \text{if } \tau'_j = 0, \\ 1 & \text{if } 1 \leq \tau'_j \leq r. \end{cases}$

Then, substituting those valuables into Eq.(3) for $r + \zeta_2 < i \leq p'$, we have

$$C_i = \sum_{j=1}^{Y} u_{1j}^{Q_j s(i,j)+1} u_{2j}^{Q_j(s(i,j)-1)+1} \cdots u_{s(i,j)+1,j}^{1} \sum_{\substack{m=i, \\ b_m \in B_{\tau'_j}}}^{p'} a_m g(i,m), \tag{11}$$

where $g(i,m) \neq 0$ are changed properly, by taking into account that the neighborhood of all $u_{ij} = 0$. Also we have $\prod_{i=1}^{p} \mathrm{d}a_i^{(w)} \prod_{i=1}^{p} \mathrm{d}b_i^{(w)} = \prod_{i=1}^{r+\zeta_2} \mathrm{d}d_i \prod_{i=r+1}^{p} \mathrm{d}a_i \prod_{m=\tilde{r}+1}^{r} \mathrm{d}b_m^{(w)} \prod_{j=1}^{Y} (u_{1j}^{s'_j-1} u_{2j}^{s'_j-2} \cdots u_{s'_j j}^{0} \mathrm{d}u_{1j} \cdots \mathrm{d}u_{s'_j j})$.

Set $SL_j = \sum_{i=1}^{Y} l_{ji}$, where $L = (l_{ji})$ in Eq.(10) and $\mathrm{l}_j = (Q_j(s'_j - 2) + 1, Q_j(s'_j - 3) + 1, \cdots, 1)$ for $j = 1, \ldots, Y$. Put $p'' = p - \zeta_1 = s'_1 + \cdots + s'_Y$. Define a $Y \times Y$ matrix L_4 and a $Y \times (s'_1 - 1 + s'_2 - 1 + \cdots + s'_Y - 1)$ matrix L_5 by

$$L_4 = \begin{pmatrix} SL_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & SL_Y \end{pmatrix}, L_5 = \begin{pmatrix} \mathrm{l}_1 & \mathrm{l}_2 & \cdots & \mathrm{l}_Y \\ \vdots & \vdots & \vdots & \vdots \\ \mathrm{l}_1 & \mathrm{l}_2 & \cdots & \mathrm{l}_Y \end{pmatrix}.$$

Let $\mathbf{E}$ be the $(s'_1 - 1 + s'_2 - 1 + \cdots + s'_Y - 1) \times (s'_1 - 1 + s'_2 - 1 + \cdots + s'_Y - 1)$ unit marix. Let $B_L = \begin{pmatrix} L & L_4 L_5 \\ 0 & \mathbf{E} \end{pmatrix}$, which is a $p'' \times p''$ matrix. Consider the coordinate $\mathbf{u} = (u_{11}, u_{12}, \ldots, u_{1Y}) \times (u_{21}, u_{31}, \ldots, u_{s'_1 1}, u_{22}, u_{32}, \ldots, u_{s'_2 2}, \cdots, u_{2Y}, u_{3Y}, \ldots, u_{s'_Y Y})$ in this order. Set $\sigma_{B_L} = \sum_{i=1}^{p''} \mathbb{R}_+ \mathbf{Bl}_i = \{\sum_{i=1}^{p''} r_i \mathbf{Bl}_i \mid r_i \geq 0\} \subset \mathbb{R}^{p''}$, where $B_L = (\mathbf{Bl}_1, \cdots, \mathbf{Bl}_{p''})$. There is a refinement fan Δ of the first quadrant $\tilde{\Delta}$ such that $\sigma_{B_L} \in \Delta$ [5]. Furthermore, we have a refinement fan Δ' of Δ so that Δ' is non-singular. Since $\det B_L = 1$, a set of B_L's column vectors is a basis of $\mathbb{Z}^{p''}$. So we can assume that

$\sigma_{B_L} \in \Delta'$. Δ' is clearly a refinement of $\tilde{\Delta}$. Since the toric variety $X(\Delta')$ defined by Δ' is non-singular, we have the proper map $\pi : X(\Delta') \to X(\tilde{\Delta}) \cong \mathbb{R}^{p''}$ such that $\pi_\sigma : \mathbf{v}^\sigma = (v_1^\sigma, \cdots, v_{p''}^\sigma) \in U_\sigma \mapsto {}^{A_\sigma}\mathbf{v}^\sigma \in \mathbb{R}^{p''}$, where $A_\sigma = (\mathbf{a}_1, \cdots, \mathbf{a}_{p''})$, $\sigma = \sum_{i=1}^{p''} \mathbb{R}_+ \mathbf{a}_i \in \Delta'$ and U_σ is the related subset of $X(\Delta')$ to σ. In particular, $\mathbf{u} = \pi_{\sigma_{B_L}}(\mathbf{v}) = {}^{B_L}\mathbf{v}$ on $U_{\sigma_{B_L}}$. Using $\det B_L = \pm 1$, we have

$$\begin{aligned} d\pi_{\sigma_{B_L}}(\mathbf{v}) = \pm & v_{11}^{l_{11}+\cdots+l_{Y1}-1} v_{12}^{l_{12}+l_{22}+\cdots+l_{Y2}-1} \cdots v_{1Y}^{l_{1Y}+l_{2Y}+\cdots+l_{YY}-1} \{ v_{21}^{Q_1(s'_1-2)+1} v_{31}^{Q_1(s'_1-3)+1} \cdots v_{s'_1 1}^1 \\ & v_{22}^{Q_2(s'_2-2)+1} v_{32}^{Q_2(s'_2-3)+1} \cdots v_{s'_2 2}^1 \cdots v_{2Y}^{Q_Y(s'_Y-2)+1} v_{3Y}^{Q_Y(s'_Y-3)+1} \cdots v_{s'_Y Y}^1 \}^{SL_1+\cdots+SL_Y} d\mathbf{v}. \end{aligned}$$

The differential form $\prod_{j=1}^Y (u_{1j}^{s'_j-1} u_{2j}^{s'_j-2} \cdots u_{s'_j j}^0 du_{1j} \cdots du_{s'_j j})$ is

$$\begin{aligned} & \prod_{j=1}^Y \{ \{ v_{11}^{l_{j1}} v_{12}^{l_{j2}} \cdots v_{1Y}^{l_{jY}} (v_{21}^{Q_1(s'_1-2)+1} v_{31}^{Q_1(s'_1-3)+1} \cdots v_{s'_1 1}^1 v_{22}^{Q_2(s'_2-2)+1} v_{32}^{Q_2(s'_2-3)+1} \cdots v_{s'_2 2}^1 \\ & \quad \cdots v_{2Y}^{Q_Y(s'_Y-2)+1} v_{3Y}^{Q_Y(s'_Y-3)+1} \cdots v_{s'_Y Y}^1)^{SL_j} \}^{s'_j-1} (v_{2j}^{s'_j-2} \cdots v_{s'_j j}^0) \} d\pi_{\sigma_{B_L}}(\mathbf{v}) \\ = & \pm v_{11}^{l_{11}s'_1+\cdots+l_{Y1}s'_Y-1} v_{12}^{l_{12}s'_1+\cdots+l_{Y2}s'_Y-1} \cdots v_{1Y}^{l_{1Y}s'_1+\cdots+l_{YY}s'_Y-1} (v_{21}^{Q_1(s'_1-2)+1} v_{31}^{Q_1(s'_1-3)+1} \cdots v_{s'_1 1}^1 \\ & v_{22}^{Q_2(s'_2-2)+1} v_{32}^{Q_2(s'_2-3)+1} \cdots v_{s'_2 2}^1 \cdots v_{2Y}^{Q_Y(s'_Y-2)+1} v_{3Y}^{Q_Y(s'_Y-3)+1} \cdots v_{s'_Y Y}^1)^{SL_1 s'_1+\cdots+SL_Y s'_Y} \prod_{j=1}^Y \{ v_{2j}^{s'_j-2} \cdots v_{s'_j j}^0 \} d\mathbf{v}. \end{aligned}$$

Set $\prod_{j=1}^Y (u_{1j}^{s'_j-1} u_{2j}^{s'_j-2} \cdots u_{s'_j j}^0 du_{1j} \cdots du_{s'_j j}) = \prod_{j=1}^Y (v_{1j}^{l_{1j}s'_1+\cdots+l_{Yj}s'_Y-1}) G(\mathbf{v}') d\mathbf{v}$,
where $\mathbf{v}' = (v_{21}, v_{31}, \ldots, v_{s'_1 1}, v_{22}, v_{32}, \ldots, v_{s'_2 2}, \cdots, v_{2Y}, v_{3Y}, \ldots, v_{s'_Y Y})$, and

$$\begin{aligned} G(\mathbf{v}') = \pm & (v_{21}^{Q_1(s'_1-2)+1} v_{31}^{Q_1(s'_1-3)+1} \cdots v_{s'_1 1}^1 v_{22}^{Q_2(s'_2-2)+1} v_{32}^{Q_2(s'_2-3)+1} \cdots v_{s'_2 2}^1 \\ & \cdots v_{2Y}^{Q_Y(s'_Y-2)+1} v_{3Y}^{Q_Y(s'_Y-3)+1} \cdots v_{s'_Y Y}^1)^{SL_1 s'_1+\cdots+SL_Y s'_Y} \prod_{j=1}^Y \{ v_{2j}^{s'_j-2} \cdots v_{s'_j j}^0 \}. \end{aligned}$$

Therefore, the differential form (9) is $\Psi' = \{ d_1^2 + \cdots + d_{r+\zeta_2}^2 + C_{r+\zeta_2+1}^2 + \cdots + C_{p'}^2 \}^z$

$$v_{11}^{l_{11}s'_1+\cdots+l_{Y1}s'_Y-1} \cdots v_{1Y}^{l_{1Y}s'_1+\cdots+l_{YY}s'_Y-1} G(\mathbf{v}') \prod_{m=1}^{r+\zeta_2} dd_m \prod_{m=r+1}^{p} da_m \prod_{m=\tilde{r}+1}^{r} db_m^{(w)} d\mathbf{v}, \qquad (12)$$

where

$$\begin{aligned} C_i = & \sum_{j=1}^Y \{ v_{11}^{l_{j1}} v_{12}^{l_{j2}} \cdots v_{1Y}^{l_{jY}} (v_{21}^{Q_1(s'_1-2)+1} v_{31}^{Q_1(s'_1-3)+1} \cdots v_{s'_1 1}^1 \cdots v_{2Y}^{Q_Y(s'_Y-2)+1} v_{3Y}^{Q_Y(s'_Y-3)+1} \\ & \cdots v_{s'_Y Y}^1)^{SL_j} \}^{Q_j s(i,j)+1} (v_{2j}^{Q_j(s(i,j)-1)+1} \cdots v_{s(i,j)+1,j}^1) \sum_{\substack{m=i, \\ b_m \in B_{\tau'_j}}}^{p'} a_m g(i,m) \text{ on } U_{\sigma_{B_L}}. \end{aligned}$$

Put $V_j = v_{1j} v_{21}^{Q_1(s'_1-2)+1} v_{31}^{Q_1(s'_1-3)+1} \cdots v_{s'_1 1}^1 \cdots v_{2Y}^{Q_Y(s'_Y-2)+1} v_{3Y}^{Q_Y(s'_Y-3)+1} \cdots v_{s'_Y Y}^1$, then we have

$$C_i = \sum_{j=1}^Y \{ V_1^{l_{j1}} V_2^{l_{j2}} \cdots V_Y^{l_{jY}} \}^{Q_j s(i,j)+1} (v_{2j}^{Q_j(s(i,j)-1)+1} \cdots v_{s(i,j)+1,j}^1) \sum_{\substack{m=i, \\ b_m \in B_{\tau'_j}}}^{p'} a_m g(i,m).$$

By constructing a blowing-up of Ψ' on $U_{\sigma_{B_L}}$, let us show inductively that we obtain

$$\begin{aligned} \Psi' \quad = \quad & \{ v_{11}^{2t_1} v_{12}^{2t_2} \cdots v_{1Y}^{2t_Y} (d_1^2 + \cdots + d_{r+\zeta_2}^2 + G_{r+\zeta_2+1}(\mathbf{v}') d_{r+\zeta_2+1}^2 + \cdots + G_K(\mathbf{v}') d_K^2) \\ & + C_{K+1}^2 + \cdots + C_{p'}^2 \}^z v_{11}^{q_1} v_{12}^{q_2} \cdots v_{1Y}^{q_Y} G(\mathbf{v}') \prod_{m=1}^{K} dd_m \prod_{m=r+1}^{r+\zeta_2} da_m \prod_{m=K+1}^{p} da_m \prod_{m=\tilde{r}+1}^{r} db_m^{(w)} d\mathbf{v}, \quad (13) \end{aligned}$$

where $k_i = s(K+1,i) = \begin{cases} K_i, & \tau'_i = 0, \\ K_i - 1, & \tau'_i \neq 0, \end{cases}$ $K = K_1 + \cdots + K_Y + \zeta_1 + \zeta_2$, $t_i \; = \; \min_{j=1,\ldots,Y} \{ l_{ji}(Q_j k_j + 1) \}$,

$$C_{K+1} \quad = \quad \sum_{j=1}^Y \{ V_1^{l_{j1}} V_2^{l_{j2}} \cdots V_Y^{l_{jY}} \}^{Q_j k_j+1} (v_{2j}^{Q_j(k_j-1)+1} \cdots v_{k_j+1,j}^1) \sum_{\substack{m=i, \\ b_m \in B_{\tau'_j}}}^{p'} a_m g(i,m),$$

and $q_i = \sum_{j=1}^{\zeta_2} 2t_i + \sum_{j=1}^{\zeta_1-\zeta_2} t_i + \sum_{j=1}^{Y} \{ l_{ji}(s'_j + Q_j K_j(1+K_j)/2) - K_j(l_{ji}(Q_j k_j + 1) - t_i) \} - 1$ for $1 \leq i \leq Y$.

First, set $t_i = \min\{ l_{1i}, l_{2i}, \ldots, l_{Yi} \}$. Substituting $d_i = v_{11}^{t_1} \cdots v_{1Y}^{t_Y} d_i$ into Ψ' in Eq.(12), we have Eq.(13) with $k_i = 0$ and $q_i = \sum_{j=1}^Y l_{ji} s'_j + (K + \zeta_1 + \zeta_2) t_i - 1 = (\zeta_1 + \zeta_2) t_i + \sum_{j=1}^Y (l_{ji} s'_j + K_j t_i) - 1$

$$= -1 + \sum_{j=1}^{\zeta_2} 2t_i + \sum_{j=1}^{\zeta_1-\zeta_2} t_i + \sum_{j=1}^{Y} \begin{cases} l_{ji}s'_j & \text{if } \tau'_j = 0, \\ l_{ji}(s'_j+1) - (l_{ji} - t_i) & \text{if } \tau'_j \neq 0, \end{cases}$$ for $1 \le i \le Y$. Now assume Eq.(13). Put $\beta_1 = Q_1k_1 + 1$, $\beta_2 = Q_2k_2 + 1$, ..., $\beta_Y = Q_Yk_Y + 1$. By the condition (iii) of L in Eq.(10), for each pair of (i, i'), there exists $0 \le j \le Y$ such that $\beta_i l_{i1} = \beta_{i'} l_{i'1}$, ..., $\beta_i l_{ij} = \beta_{i'} l_{i'j}$, $\beta_i l_{i,j+1} > \beta_{i'} l_{i',j+1}$, ..., $\beta_i l_{iY} > \beta_{i'} l_{i'Y}$, or $\beta_i l_{i1} = \beta_{i'} l_{i'1}$, ..., $\beta_i l_{ij} = \beta_{i'} l_{i'j}$, $\beta_i l_{i,j+1} < \beta_{i'} l_{i',j+1}$, ..., $\beta_i l_{iY} < \beta_{i'} l_{i'Y}$. Therefore there exists i_0 such that $\beta_i l_{ik} \ge \beta_{i_0} l_{i_0 k}$ for all i and k. Set $t_k = \beta_{i_0} l_{i_0 k}$. Taking the fact $\beta_j l_{jk} > t_k$ for some k if $j \neq i_0$ into account, C_{K+1} can be devided by $\{V_1^{l_{i_01}} V_2^{l_{i_02}} \cdots V_Y^{l_{i_0Y}}\}^{Q_{i_0}k_{i_0}+1} v_{2i_0}^{Q_{i_0}(k_{i_0}-1)+1} \cdots v^1_{k_{i_0}+1,i_0}$. We can assume $b_{K+1} \in B_{\tau'_{i_0}}$. Then we have $C_{K+1} = \{V_1^{l_{i_01}} V_2^{l_{i_02}} \cdots V_Y^{l_{i_0Y}}\}^{Q_{i_0}k_{i_0}+1} v_{2i_0}^{Q_{i_0}(k_{i_0}-1)+1} \cdots v^1_{k_{i_0}+1,i_0}(g(K+1, K+1)a_{K+1} + \cdots)$. Change the variable from a_{K+1} to d_{K+1} by $d_{K+1} = a_{K+1}g(K+1, K+1) + \cdots$. We obtain

$$C_{K+1} = \{V_1^{l_{i_01}} V_2^{l_{i_02}} \cdots V_Y^{l_{i_0Y}}\}^{Q_{i_0}k_{i_0}+1} v_{2i_0}^{Q_{i_0}(k_{i_0}-1)+1} \cdots v^1_{k_{i_0}+1,i_0} d_{K+1}.$$

Set the function $G_{K+1}(\mathbf{v}') = C_{K+1}/(v_{11}^{l_{i_01}(Q_{i_0}k_{i_0}+1)} v_{12}^{l_{i_02}(Q_{i_0}k_{i_0}+1)} \cdots v_{1Y}^{l_{i_0Y}(Q_{i_0}k_{i_0}+1)} d_{K+1})$.

Put $t'_i = \min\{l_{ji}(Q_jk_j + 1), l_{i_0i}(Q_{i_0}(k_{i_0} + 1) + 1) | 1 \le j \le Y, j \neq i_0.\} \ge t_i$ for $1 \le i \le Y$.

Substituting $d_i = v_{11}^{t'_1-t_1} v_{12}^{t'_2-t_2} \cdots v_{1Y}^{t'_Y-t_Y} d_i$ for $1 \le i \le K+1$ into Eq.(13), we have

$$\Psi' = \{v_{11}^{2t'_1} v_{12}^{2t'_2} \cdots v_{1Y}^{2t'_Y} (d_1^2 + \cdots + d_{r+\zeta_2}^2 + G_{r+\zeta_2+1}(\mathbf{v}')d_{r+\zeta_2+1}^2 + \cdots + G_{K+1}(\mathbf{v}')d_{K+1}^2) + C_{K+2}^2 + \cdots + C_{p'}^2\}^z$$
$$v_{11}^{q'_1} v_{12}^{q'_2} \cdots v_{1Y}^{q'_Y} G(\mathbf{v}') \prod_{m=1}^{K+1} \mathrm{d}d_m \prod_{m=r+1}^{r+\zeta_2} \mathrm{d}a_m \prod_{m=K+2}^{p} \mathrm{d}a_m \prod_{m=\bar{r}+1}^{r} \mathrm{d}b_m^{(w)} \mathrm{d}\mathbf{v},$$

where $q'_i = q_i + (t'_i - t_i)(K+1) = \sum_{j=1}^{\zeta_2} 2t_i + \sum_{j=1}^{\zeta_1-\zeta_2} t_i + \sum_{j=1}^{Y}\{l_{ji}(s'_j + Q_jK_j(1+K_j)/2) - K_j(l_{ji}(Q_jk_j+1) - t_i)\} + (t'_i - t_i)(K+1) - 1$ for $1 \le j \le Y$.

If $j \neq i_0$, then $l_{ji}(s'_j + Q_jK_j(1+K_j)/2) - K_j(l_{ji}(Q_jk_j+1) - t_i) + (t'_i - t_i)K_j = l_{ji}(s'_j + Q_jK_j(1+K_j)/2) - K_j(l_{ji}(Q_jk_j+1) - t'_i)$. Also we have $l_{i_0i}(s'_{i_0} + Q_{i_0}K_{i_0}(1+K_{i_0})/2) - K_{i_0}(l_{i_0i}(Q_{i_0}k_{i_0}+1) - t_i) + (t'_i - t_i)(K_{i_0}+1) = l_{i_0i}(s'_{i_0} + Q_{i_0}(K_{i_0}+1)(2+K_{i_0})/2) - (K_{i_0}+1)(l_{i_0i}(Q_{i_0}(k_{i_0}+1)+1) - t'_i)$. Thus, we have $q'_i = \sum_{j=1}^{\zeta_2} 2t'_i + \sum_{j=1}^{\zeta_1-\zeta_2} t'_i + \sum_{j=1, j\neq i_0}^{Y} \{l_{ji}(s'_j + Q_jK_j(1+K_j)/2) - K_j(l_{ji}(Q_jk_j+1) - t'_i)\} + l_{i_0i}(s'_{i_0} + Q_{i_0}(K_{i_0} + 1)(2+K_{i_0})/2) - (K_{i_0}+1)(l_{i_0i}(Q_{i_0}(k_{i_0}+1)+1) - t'_i)$. By putting $K_{i_0} \to K_{i_0} + 1$ and $k_{i_0} \to k_{i_0} + 1$, we have the induction with $K+1$.

Finally, from the induction, we have the followings. Set j_1 as satisfying $Q_{j_1}\ell_{j_1} = \min_{1\le j\le Y}\{Q_j\ell_j\}$. Since $I = n'_jl_{j1} = n'_j\ell_j$ and the condition (iii) of L, we have (13) with $k_j = (n'_j - 1)/Q_j$ for $j \neq j_1$, $k_{j_1} = (n'_{j_1} - 1)/Q_{j_1} - 1$, $t_1 = l_{j_11}(n'_{j_1} - 1) < I$, and $t_i = l_{j_1i}(n'_{j_1} - 1)$. Put $I_i = \min_{1\le j\le Y}\{l_{ji}n'_j\}$. After changing the variable from a_{K+1} to d_{K+1} and substituting $d_i = v_{11}^{I_1-t_1-1} v_{12}^{I_2-t_2} \cdots v_{1Y}^{I_Y-t_Y}$ for $1 \le i \le K+1$, we have

$$\Psi' = \{v_{11}^{2I_1-2} v_{12}^{2I_2} \cdots v_{1Y}^{2I_Y} (d_1^2 + \cdots + d_{r+\zeta_2}^2 + G_{r+\zeta_2+1}(\mathbf{v}')d_{r+\zeta_2+1}^2 + \cdots + G_{K+1}(\mathbf{v}')d_{K+1}^2)$$
$$+ C_{K+2}^2 + \cdots + C_{p'}^2\}^z v_{11}^{q_1-(K+1)} v_{12}^{q_2} \cdots v_{1Y}^{q_Y} G(\mathbf{v}') \prod_{m=1}^{K+1} \mathrm{d}d_m \prod_{m=r+1}^{r+\zeta_2} \mathrm{d}a_m \prod_{m=K+2}^{p} \mathrm{d}a_m \prod_{m=\bar{r}+1}^{r} \mathrm{d}b_m^{(w)} \mathrm{d}\mathbf{v} \quad (14)$$

where $C_{K+2} = \sum_{j=1}^{Y}\{V_1^{l_{j1}} V_2^{l_{j2}} \cdots V_Y^{l_{jY}}\}^{n'_j}(v_{2j}^{n'_j-Q_j} \cdots v^1_{(n'_j-1)/Q_j+1,j}) \sum_{m=i, b_m\in B_{\tau'_j}}^{p'} a_m g(i, m)$,

$K = K_1 + \cdots + K_Y + \zeta_1 + \zeta_2$ and

$$q_i = \sum_{j=1}^{\zeta_2} 2I_i + \sum_{j=1}^{\zeta_1-\zeta_2} I_i - 1 + \sum_{j=1}^{Y} \begin{cases} l_{ji}(s'_j + Q(\tilde{n}_0+1)\tilde{n}_0/2) - \tilde{n}_0(l_{ji}(Q\tilde{n}_0+1) - I_i), & \tau'_j = 0, \\ l_{ji}(s'_j + (n'_j+1)n'_j/2) - n'_j(l_{ji}n'_j - I_i), & \tau'_j \neq 0, \end{cases}$$

for $1 \le i \le Y$,

4.3 Proof of Theorem 2 : Step 3

$$\text{Set } Z_{ji}(t) = \begin{cases} \dfrac{l_{ji}(s'_j + Q\tilde{n}_0(\tilde{n}_0+1)/2) - \tilde{n}_0(l_{ji}(Q\tilde{n}_0+1) - t)}{2t} & \tau'_j = 0, \\ \dfrac{l_{ji}(s'_j + (n'_j+1)n'_j/2) - n'_j(l_{ji}n'_j - t)}{2t}, & \tau'_j \neq 0, \end{cases}$$

If $Q(\tilde{n}_0^2 - \tilde{n}_0) + 2\tilde{n}_0 = 2s_0$ then
$\frac{\tilde{n}_0}{2} = \frac{Q(\tilde{n}_0 + \tilde{n}_0^2) + 2s_0}{4Q\tilde{n}_0 + 4} = \frac{Q(\tilde{n}_0 + \tilde{n}_0^2) + 2s_0 - 4\tilde{n}_0}{4Q\tilde{n}_0 - 4} = \cdots = \frac{Q(\tilde{n}_0 - 1 + (\tilde{n}_0 - 1)^2) + 2s_0}{4Q(\tilde{n}_0 - 1) + 4}$.

So, if $Q(\tilde{n}_0^2 - \tilde{n}_0) + 2\tilde{n}_0 = 2s_0$ and $\tau'_j = 0$, then we have $Z_{ji}(t) = \frac{Q(\tilde{n}_0 + \tilde{n}_0^2) + 2s_0}{4Q\tilde{n}_0 + 4}$. (15)

If $(n_{\tau'_j} - 1)^2 + n_{\tau'_j} - 1 = 2s'_j$ then $\frac{n_{\tau'_j}}{2} = \frac{n_{\tau'_j} + n_{\tau'_j}^2 + 2s'_j}{4n_{\tau'_j}} = \frac{(n_{\tau'_j} - 1) + (n_{\tau'_j} - 1)^2 + 2s'_j}{4(n_{\tau'_j} - 1)}$.

Therefore for $\tau'_j \neq 0$, we have $n'_j = n_{\tau'_j}$, and $Z_{ji}(t) = \frac{n_{\tau'_j} + n_{\tau'_j}^2 + 2s'_j}{4n_{\tau'_j}}$. (16)

Set $d_1 = u_1$, $d_2 = u_1 d_2$, $\cdots$, $d_{K+1} = u_1 d_{K+1}$, $v_{11} = u_1 v_{11}$ in Eq.(14). Then we have

$$\begin{aligned}\Psi' &= \{u_1^{2I_1} v_{11}^{2I_1-2} v_{12}^{2I_2} \cdots v_{1Y}^{2I_Y}(1 + d_2^2 + \cdots + d_{r+\zeta_2}^2 + G_{r+\zeta_2+1}(\mathrm{v}')d_{r+\zeta_2+1}^2 + \cdots + G_{K+1}(\mathrm{v}')d_{K+1}^2) \\ &\quad + C_{K+2}^2 + \cdots + C_{p'}^2\}^z u_1^{q_1} v_{11}^{q_1-(K+1)} v_{12}^{q_2} \cdots v_{1Y}^{q_Y} G(\mathrm{v}') \prod_{m=1}^{K+1} \mathrm{d}d_m \prod_{m=r+1}^{r+\zeta_2} \mathrm{d}a_m \prod_{m=K+2}^{p} \mathrm{d}a_m \prod_{m=\bar{r}+1}^{r} \mathrm{d}b_m^{(w)} \mathrm{d}\mathrm{v}.\end{aligned}$$

So the poles $\frac{q_1 - K}{2(I_1-1)}$, $\frac{q_i+1}{2I_i}$, $1 \le i \le Y$ are obtained. By Eq.(15) and (16), we have

$$\frac{q_1 - K}{2(I_1 - 1)} = \sum_{j=1}^{\zeta_2} 1 + \sum_{j=1}^{\zeta_1-\zeta_2} \frac{1}{2} + \sum_{j=1}^{Y'} \begin{cases} \frac{2s_0 + Q(\tilde{n}_0^2 + \tilde{n}_0)}{4Q\tilde{n}_0 + 4} & \text{if } \tau'_j = 0, \\ \frac{n_{\tau'_j} + n_{\tau'_j}^2 + 2s'_j}{4n_{\tau'_j}} & \text{if } \tau'_j \neq 0, \end{cases} + \sum_{j=Y'+1}^{Y} Z_{j1}(I_1 - 1), \text{ and}$$

$$\frac{q_i + 1}{2I_i} = \sum_{j=1}^{\zeta_2} 1 + \sum_{j=1}^{\zeta_1-\zeta_2} \frac{1}{2} + \sum_{j=1}^{Y'} \begin{cases} \frac{2s_0 + Q(\tilde{n}_0^2 + \tilde{n}_0)}{4Q\tilde{n}_0 + 4} & \text{if } \tau'_j = 0, \\ \frac{n_{\tau'_j} + n_{\tau'_j}^2 + 2s'_j}{4n_{\tau'_j}}. & \text{if } \tau'_j \neq 0, \end{cases} + \sum_{j=Y'+1}^{Y} Z_{ji}(I_i).$$

By using $I_i = n'_j l_{ji}$, $i \le j$, we have $Z_{ji}(I_i) = \begin{cases} \frac{2s_0 + Q(\tilde{n}_0^2 + \tilde{n}_0)}{4Q\tilde{n}_0 + 4} & \text{if } \tau'_j = 0, \\ \frac{n_{\tau'_j} + n_{\tau'_j}^2 + 2s'_j}{4n_{\tau'_j}} & \text{if } \tau'_j \neq 0, \end{cases}$ for $Y' + 1 \le j \le Y$, $i \le j$.

Thus, we have $\frac{q_1 + 1}{2t_1} = \cdots = \frac{q_{Y'+1} + 1}{2t_{Y'+1}} = \lambda_Q^*$. Also if $Y = Y'$ then $\frac{q_1 - K}{2(I_1 - 1)} = \lambda_Q^*$.

5 Proof of Main Theorem 1: Part 3

Finally, we prove that the pole in Theorem 2 and its order are max.

Lemma 3 *If $f_1, g_1, f_2, g_2, \ldots, f_m, g_m > 0$, then* $\frac{\sum_{i=1}^m f_i}{\sum_{i=1}^m g_i} \ge \min_{i=1,\ldots,m} \{\frac{f_i}{g_i}\}$.

Lemma 4 *Let $K \in \mathbb{Z}_+$. Assume that $\eta_k \in \mathbb{Z}_+$, $k = 1, \ldots, K$ satisfy $0 \le \eta_1 \le Q, 0 \le \eta_1 + \eta_2 \le 2Q, \cdots, 0 \le \eta_1 + \eta_2 + \cdots + \eta_K \le QK$. Let $t = \eta_1 + \cdots + \eta_K = Q(i-1) + m$, $i \in \mathbb{N}$, $0 \le m \le Q - 1$, and $\varphi = \eta_1 + 2\eta_2 + \cdots + K\eta_K$. Then $\varphi \ge Qi(i-1)/2 + im$.*

Next assume that $\eta_k \in \mathbb{Z}_+$, $k = 1, \ldots, K$ satisfy $\eta_1 = 0, 0 \le \eta_2 \le 1, 0 \le \eta_2 + \eta_3 \le 2, \cdots, 0 \le \eta_2 + \eta_3 + \cdots + \eta_K \le K - 1$. Then by setting $t = 1 + \eta_1 + \cdots + \eta_K$, and $\varphi = \eta_1 + 2\eta_2 + \cdots + K\eta_K$, we have $2(1 + \varphi) \ge t^2 + t$.

Lemma 5 *For any s_0, s_1, m, i with $0 \le m \le Q - 1$, we have* $\frac{Qi(i-1)/2 + im + s_0}{2(1 + Q(i-1) + m)} \ge \frac{Q(n^2 + n) + 2s_0}{4Qn + 4}$, *where $n = \max\{i \in \mathbb{Z} \mid Q(i^2 - i) + 2i \le 2s_0\}$, or we have* $\frac{i^2 + i + 2s_1}{4i} \ge \frac{(n + n^2) + 2s_1}{4n}$, *where $n - 1 = \max\{i \in \mathbb{Z} \mid i^2 + i \le 2s_1\}$.*

Lemma 6 *If some $J_m^{(\alpha)}$ are not Special Case, then the poles appeared in Section 3 are smaller than λ_Q^* in Main Theorem 1.*

Proof Assume that $B_{\bar{\tau}}$ is not Special Case. Then by $g_{(l,\tau),\bar{\tau}} < \sum_{J_m^{(\alpha)} \ge J^{(\mu)}} D_{J^{(\mu)},(l,\tau)}$ for $b_m \in B_{\bar{\tau}}$ together with Statement (e) and (f), we have $2\frac{q_{l\tau} + 1}{t_{l\tau}} = 2\sum_{\tau'=0}^{\tau} \frac{q_{(l,\tau),\tau'}}{t_{l\tau}} > \sum_{\tau'=0}^{\tau} \frac{\sum_{\xi=1}^{g_{(l,\tau),\tau'}} \varphi_{(l,\tau),\tau'}^{(\xi)}}{\sum_{\xi=1}^{g_{(l,\tau),\tau'}} (1 + \eta_{1,(l,\tau),\tau'}^{(\xi)} + \cdots + \eta_{K_{\tau'},(l,\tau),\tau'}^{(\xi)})}$.

By Lemmas 3, 4, and 5, we obtain $2\frac{q_{l\tau} + 1}{t_{l\tau}} > \sum_{\tau'=0}^{\tau} \min_{1 \le \xi \le g_{(l,\tau),\tau'}} \frac{\varphi_{(l,\tau),\tau'}^{(\xi)}}{1 + \eta_{1,(l,\tau),\tau'}^{(\xi)} + \cdots + \eta_{K_{\tau'},(l,\tau),\tau'}^{(\xi)}} \ge \lambda_Q^*$.

Finally, let us show that the order θ is max. Q.E.D.

Theorem 7 *The order of the maximum pole* λ_Q^* *is* $\theta = \#\Theta + 1$.

Proof By Lemma 6, it is enough to consider Special Case for all $B_\tau^{(w)}$. Especially, consider the case $J_m^{(\alpha)} = ({b_m^*}^Q, 0, \ldots, 0)$, which satisfies Statements (a')∼(f'). Suppose that Y', Y, τ'_j, n'_j, $B_{\tau'_j} = \{b_{1j}, \ldots, b_{s'_j j}\}$, u_{ij}, $\mathbf{u} = (u_{11}, u_{12}, \ldots, u_{1Y}) \times (u_{21}, u_{31}, \ldots, u_{s'_1 1}, u_{22}, u_{32}, \ldots, u_{s'_2 2}, \cdots, u_{2Y}, u_{3Y}, \ldots, u_{s'_Y Y})$ and s'_j are as before. Let $N(i,j)$ be the number such that $N(i,j)$th element of $\mathbf{u}$ is u_{ij}.

Let B_L be an $(s'_1 + s'_2 + \cdots + s'_Y) \times (s'_1 + s'_2 + \cdots + s'_Y)$ matrix. For simplicity, denote the (k,l)th element of B_L by $B_L((i,j),(i',j'))$ if $k = N(i,j)$, $l = N(i',j')$.

We proceed Section 3 from Ψ' in Eq.(9).

Transformation (ii) or (iii) for $\mathbf{v}$ corresponds to ${}^{B'_L}\mathbf{v}$, where B'_L is a $(s'_1+s'_2+\cdots+s'_Y)\times(s'_1+s'_2+\cdots+s'_Y)$ matrix: $B'_L((i,j),(i',j')) =$

$$\begin{cases} 1, & \text{if } (i,j) = (i',j'), \\ 1, & \text{if } (i,\tau'_j) \in A^{(\alpha)}, (i',\tau'_{j'}) = (k',\tau'), \\ 1, & \text{if } (i,\tau'_j) = (k_\tau+1,\tau), m \in C'^{(\alpha)}, b_m \in B_\tau, \\ & (i',\tau'_{j'}) = (k',\tau'), \\ 0, & \text{others}, \end{cases} \quad \text{or} \quad \begin{cases} 1, & \text{if } (i,j) = (i',j'), \\ 1, & \text{if } (i,\tau'_j) \in A^{(\alpha)}, (i',\tau'_{j'}) = (k_{\tilde\tau}+1,\tilde\tau), \\ 1, & \text{if } (i,\tau'_j) = (k_\tau+1,\tau), m \in C'^{(\alpha)}, b_m \in B_\tau, \\ & (i',\tau'_{j'}) = (k_{\tilde\tau}+1,\tilde\tau), \\ 0, & \text{others}. \end{cases}$$

Therefore, we have $D_{J^{(\mu)}_{\tau'_j},(k,l)} = B_L((k_j^{(\mu)}+1,j),(k,l))$. If $-\frac{q_{kl}+1}{t_{kl}}$ is the maximum pole, we need that (1) $B_L((k_j^{(\mu)}+1,j),(k,l)) = 0$ for $k_j^{(\mu)} \geq 1$ and $1 \leq j \leq Y$, (2) $g_{(k,l),\tau'_j} = B_L((1,j),(k,l))$ for $1 \leq j \leq Y$, (3) $\frac{t_{kl}}{2} = \sum_{\xi=1}^{g_{(k,l),\tau'_j}}(1 + \eta^{(\xi)}_{1,(k,l),\tau'_j} + \cdots + \eta^{(\xi)}_{n_j,(k,l),\tau'_j}) = B_L((1,j),(k,l))n'_j$ and $\varphi^{(\xi)}_{(k,l),\tau'_j} = \begin{cases} \frac{Q(\tilde n_0 + \tilde n_0^2)+2s_0}{2}, & \text{if } \tau'_j = 0, \\ \frac{n_{\tau'_j} + n^2_{\tau'_j} + 2s'_j}{2}, & \text{if } \tau'_j \neq 0, \end{cases}$ for $Y'+1 \leq j \leq Y$. If there are (k_1,l_1), …, $(k_{Y'+2}, l_{Y'+2})$ which give the maximum poles, then

$$\big(B_L((1,Y'+1),(k_1,l_1)), \ \cdots, \ B_L((1,Y'+1),(k_{Y'+2},l_{Y'+2}))\big),$$
$$\big(B_L((1,Y'+2),(k_1,l_1)), \ \cdots, \ B_L((1,Y'+2),(k_{Y'+2},l_{Y'+2}))\big),$$
$$\ldots, \big(B_L((1,Y),(k_1,l_1)), \ \cdots, \ B_L((1,Y),(k_{Y'+2},l_{Y'+2}))\big)$$

are linear, since $\frac{t_{kl}}{2} = B_L((1,j),(k,l))n'_j$ for $Y'+1 \leq j \leq Y$. Then $\det B_L = 0$. This is the contradiction. So, the total number of (k,l) giving the maximum pole is less than $Y'+1$.

Next assume that $-\frac{\sum_{(k,l)\in A^{(\alpha)}} q_{kl} + K + \#A^{(\alpha)} + \#C'^{(\alpha)}}{\sum_{(kl)\in A^{(\alpha)}} t_{kl} + 2}$ is the maximum pole. We have (1) $B_L((k_j^{(\mu)} + 1,j),(k,l)) = 0$ for $k_j^{(\mu)} \geq 1$, $1 \leq j \leq Y$ and $(k,l) \in A^{(\alpha)}$, (2) $g_{(k,l),\tau'_j} = B_L((1,j),(k,l))$ for $1 \leq j \leq Y$ and $(k,l) \in A^{(\alpha)}$, (3) There exist $(k_0,l_0) \in A^{(\alpha)}$ and ξ_0 such that $\frac{t_{k_0 l_0}}{2} = B_L((1,j),(k_0,l_0))n'_j - 1$ and

$$\varphi^{(\xi_0)}_{(k_0,l_0),\tau'_j} = \begin{cases} \frac{Q(\tilde n_0 + \tilde n_0^2)+2s_0-2\tilde n_0}{2}, & \text{if } \tau'_j = 0, \\ \frac{n^2_{\tau'_j} - n_{\tau'_j} + 2s'_j}{2}, & \text{if } \tau'_j \neq 0, \end{cases}$$

for $Y'+1 \leq j \leq Y$, (4) For $(k,l) \neq (k_0,l_0)$ or $\xi \neq \xi_0$, we have $\frac{t_{kl}}{2} = \sum_{\xi=1}^{g_{(k,l),\tau'_j}}(1 + \eta^{(\xi)}_{1,(k,l),\tau'_j} + \cdots + \eta^{(\xi)}_{n_j,(k,l),\tau'_j}) = B_L((1,j),(k,l))n'_j$ and

$$\varphi^{(\xi)}_{(k,l),\tau'_j} = \begin{cases} \frac{Q(\tilde n_0 + \tilde n_0^2)+2s_0}{2} & \text{if } \tau'_j = 0, \\ \frac{n_{\tau'_j} + n^2_{\tau'_j} + 2s'_j}{2}, & \text{if } \tau'_j \neq 0, \end{cases} \quad \text{for } Y'+1 \leq j \leq Y.$$

Therefore, we similarly have the contradiction to the order $Y'+2$.

The end of the proof of Main Theorem 1

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