Topological Radon Transforms and Projective Duality

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Abstract

We study various topological properties of projective duality in algebraic geometry by using the microlocal theory of sheaves developed by Kashiwara-Schapira [21]. In particular, in the real algebraic case we obtain some results similar to Ernström's ones [9] obtained in the complex case. For this purpose, we use constructible functions and their topological Radon transforms. We also generalize a class formula (i.e. a formula which expresses the degrees of dual varieties) in [10] to the case of associated varieties studied by Gelfand-Kapranov-Zelevinsky [12] etc. For the detail, see [26] and [27].

1 Introduction

We denote the projective space of dimension $n$ over $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$) by $\mathbb{P}_n$ and its dual space by $\mathbb{P}^*_n$. These spaces are naturally identified with the following sets.

$\mathbb{P}_n = \{ l \mid l \text{ is a line in } \mathbb{K}^{n+1} \text{ through the origin} \}$, \hspace{1cm} (1.1)

$\mathbb{P}^*_n = \{ H' \mid H' \text{ is a hyperplane in } \mathbb{K}^{n+1} \text{ through the origin} \}$. \hspace{1cm} (1.2)

Note that if we projectivize a hyperplane $H'$ in $\mathbb{K}^{n+1}$ we obtain a hyperplane $H$ in $\mathbb{P}_n$. Therefore in what follows we identify the dual projective space $\mathbb{P}_n^*$ with the set

$\{ H \mid H \text{ is a hyperplane in } \mathbb{P}_n \}$. \hspace{1cm} (1.3)
Definition 1.1 Let $V$ be a projective variety in $\mathbb{P}_n$. We define the dual variety $V^*$ of $V$ by

$$V^* := \{ H \in \mathbb{P}_n^n \mid \exists x \in V_{\text{reg}} \cap H \text{ s.t. } T_x V \subset T_x H \} \subset \mathbb{P}_n^n. \quad (1.4)$$

When $V$ is smooth, $V^*$ is the set of hyperplanes tangent to $V$. As we see in the example below, even if $V$ is smooth, $V^*$ may be very singular in general.

Example 1.2 (i) Let $\iota_n : \mathbb{P}_1 \hookrightarrow \mathbb{P}_n$ be the Veronese embedding given by $[x : y] \mapsto [x^n : x^{n-1}y : \ldots : y^n]$ and set $V = \iota_n(\mathbb{P}_1) \subset \mathbb{P}_n$. Then the dual $V^* \subset \mathbb{P}_n^n$ is a hypersurface defined by the classical discriminant for polynomials of degree $n$.

(ii) For $n \geq m$, consider the Segre embedding $\iota_{n,m} : \mathbb{P}_n \times \mathbb{P}_m \twoheadrightarrow \mathbb{P}((n+1)(m+1)-1)$ given by $([x_0 : \ldots : x_n], [y_0 : \ldots : y_m]) \mapsto [\ldots : x_i y_j : \ldots]$. Set $W = \iota_{n,m}(\mathbb{P}_n \times \mathbb{P}_m) \subset \mathbb{P}((n+1)(m+1)-1)$. Then the dual variety $W^* \subset \mathbb{P}((n+1)(m+1)-1)$ has very complicated singularities and the dual defect $\delta^*(W)$ of $W$ (see (2.2) below) is $n - m$. Indeed, let $M_{(n+1),(m+1)}$ be the space of $(n+1) \times (m+1)$ matrices and identify the dual projective space $\mathbb{P}(M_{(n+1),(m+1)})$ with its projectivization $\mathbb{P}(M_{(n+1),(m+1)})$. Then the dual $W^* \subset \mathbb{P}((n+1)(m+1)-1)$ is explicitly written by

$$W^* = \mathbb{P}(\{ A \in M_{(n+1),(m+1)} \mid \text{rank} A \leq m \}). \quad (1.5)$$

Therefore the dual $W^*$ admits a stratification defined by the ranks of matrices.

Many mathematicians were interested in the mysterious relations between projective varieties and their duals. Above all, they observed that the tangency of a hyperplane $H \in V^*$ with $V$ is related to the singularity of the dual $V^*$ at $H$. For example, consider the case of a plane curve $C \subset \mathbb{P}_2$ over $\mathbb{C}$. Then a tangent line $l$ at an inflection point of $C$ corresponds to a cusp of the dual curve $C^*$, and a bitangent (double tangent) line $l$ of $C$ corresponds to an ordinary double point of $C^*$. The most general results for complex plane curves were found in the 19th century by Klein, Plücker and Clebsch etc. (see for example, [34, Theorem 1.6] and [38, Chapter 7] etc.).

In the last two decades, this beautiful correspondence was extended to higher-dimensional complex projective varieties from the viewpoint of the geometry of hyperplane sections. In particular, after some important contributions by Viro [37] and Dimca [8] etc., Ernström proved the following remarkable result in 1994.
Theorem 1.3 [9, Corollary 3.9] Let $V \subset \mathbb{P}_n$ be a smooth projective variety over $\mathbb{C}$. Take a generic hyperplane $H$ in $\mathbb{P}_n$ such that $H \notin V^*$. Then for any hyperplane $L \in V^*$, we have
\[
\chi(V \cap L) - \chi(V \cap H) = (-1)^{n-1+\dim V^*-\dim V^*} \text{Eul}_{V^*}(L),
\]
where $\chi$ stands for the topological Euler characteristic and $\text{Eul}_{V^*} : V^* \to \mathbb{Z}$ is the Euler obstruction of $V^*$ (introduced by Kashiwara [17] and MacPherson [24] independently).

Recall that the Euler obstruction $\text{Eul}_{V^*}$ of $V^*$ is a $\mathbb{Z}$-valued function on $V^*$ which measures the singularity of $V^*$ at each point of $V^*$. For example, $\text{Eul}_{V^*}$ takes the value 1 on the regular part of $V^*$. Moreover, if we take a Whitney stratification $\bigcup_{\alpha \in A} V^*_\alpha$ of $V^*$ consisting of connected strata, then $\text{Eul}_{V^*}$ is constant on each stratum $V^*_\alpha$. The values of $\text{Eul}_{V^*}$ on a stratum $V^*_\alpha$ is determined by those on $V^*_\beta$'s satisfying the condition $V^*_\alpha \subset \overline{V^*_\beta}$ (for more detail, see e.g. [18]).

Hence Ernström's result says that the jumping number of the topological Euler characteristics of hyperplane sections of $V$ at $L$ is expressed by $\text{Eul}_{V^*}(L)$, that is, the singularity of the dual variety $V^*$ at $L$.

The aim of this article is to introduce our results in the real algebraic case similar to this Ernström's one and to survey its theoretical background.

2 Main results

Consider a real projective space $X = \mathbb{R} \mathbb{P}_n$ of dimension $n$ and its dual $Y = \mathbb{R} \mathbb{P}_n^*$. Let $M \subset X$ be a smooth real projective variety and $M^* \subset Y$ its dual variety.

We fix a $\mu$-stratification $Y = \bigsqcup_{\alpha \in A} Y_\alpha$ of $Y = \mathbb{R} \mathbb{P}_n^*$ consisting of connected strata and adapted to $M^*$. Note that Trotman [35] proved that this $\mu$-condition is equivalent to famous Verdier's w-regularity condition.

Definition 2.1 We define a $\mathbb{Z}$-valued function $\varphi_M : Y \to \mathbb{Z}$ on $Y = \mathbb{R} \mathbb{P}_n^*$ by
\[
\varphi_M(H) = \chi(M \cap H) \quad (H \in Y).
\]

Since the function $\varphi_M$ is defined by the topological Euler characteristics of hyperplane sections $M \cap H$ of $M$, to obtain results similar to Ernström's formula (1.6) it suffices to describe the function $\varphi_M$ in terms of the singularities of $M^*$. We will show that the whole function $\varphi_M$ can be reconstructed
from one value $\varphi_M(y)$ at a point $y \in Y \setminus M^*$ and the singularities of $M^*$. First of all, for the above $\mu$-stratification $Y = \bigsqcup_{\alpha \in A} Y_{\alpha}$ of $Y = \mathbb{R}\mathbb{P}_n^*$ we can prove the following basic result.

**Proposition 2.2** The function $\varphi_M$ is constant on each stratum $Y_\alpha$.

We denote the value of $\varphi_M$ on $Y_\alpha$ by $\varphi_\alpha$. Our main results are reconstruction theorems of $\varphi_M$. Namely, we can determine all the values $\varphi_\alpha$'s of $\varphi_M$ from only one value $\varphi_M(y)$ at a point $y \in Y \setminus M^*$ and the topology of $M^*$.

To state the first theorem, we introduce two notations concerning dual varieties. Recall that the dual variety $M^*$ is usually a hypersurface in $Y = \mathbb{R}\mathbb{P}_n^*$.

**Definition 2.3**

(i) We denote the dual defect of $M$ by

$$\delta^*(M) = (n - 1) - \dim M^*.$$  \hfill (2.2)

(ii) For a conormal vector $\vec{p} \in T^*_{M^*Y}$ at $y \in M^*_{\text{reg}}$, consider the second fundamental form

$$h_{M^*,\vec{p}}: T_y M^* \times T_y M^* \rightarrow \mathbb{R},$$  \hfill (2.3)

with respect to the canonical (Fubini-Study) metric of $Y = \mathbb{R}\mathbb{P}_n^*$ and set

$$J_{\vec{p}} := \#\{\text{positive eigenvalues of } h_{M^*,\vec{p}}\} + \delta^*(M).$$  \hfill (2.4)

Now, let us state our first main theorem which describes the values of $\varphi_M$ on $Y \setminus M^*_{\text{sing}}$.

**Theorem 2.4** ([26])

(i) Assume that $\delta^*(M) > 0$. Then on $Y \setminus M^*$ the function $\varphi_M$ is constant. Moreover for any $y \in M^*_{\text{reg}}$ there exists an neighborhood $U$ of $y$ such that we have

$$\varphi_M = d \cdot 1_U + (-1)^{J_{\vec{p}}} 1_{M^*_{\text{reg}}},$$  \hfill (2.5)

on $U$, where $d$ is the value of $\varphi_M$ on $Y \setminus M^*$ and $\vec{p} \in T^*_{M^*Y}$ is a conormal vector at $y \in M^*_{\text{reg}}$. 


(ii) Assume that $\delta^*(M) = 0$, that is $M^*$ is a hypersurface in $Y = \mathbb{RP}^n$, and consider the following local situation. Let $Y_{\alpha_1}$ and $Y_{\alpha_2}$ be two strata in $Y \setminus M^*$, $Y_\beta$ an open stratum in $M^*_\text{reg}$ such that $Y_\beta \subset Y_{\alpha_i}$ for $i = 1, 2$ and $\vec{p} \in T_{M^*_\text{reg}} Y$ a conormal vector at a point $y \in Y_\beta$ pointing from $Y_{\alpha_1}$ to $Y_{\alpha_2}$ (see Figure 2.4.1 below). Then we have

\[
\varphi_{\alpha_2} - \varphi_{\alpha_1} = (-1)^{J_\vec{p}} - (-1)^{J_{-\vec{p}}} = 0, \pm 2)
\]

(2.6)

\[
\varphi_\beta = \begin{cases} 
\frac{1}{2}(\varphi_{\alpha_1} + \varphi_{\alpha_2}) & \text{if } \varphi_{\alpha_1} \neq \varphi_{\alpha_2}, \\
\varphi_{\alpha_1} + (-1)^{J_\vec{p}} & \text{if } \varphi_{\alpha_1} = \varphi_{\alpha_2}. 
\end{cases}
\]

(2.7)

\[\begin{array}{c}
Y_\beta \\
\vec{p}
\end{array}\]

\[\begin{array}{c}
\vec{-p}
\end{array}\]

\[\begin{array}{c}
\rightarrow
\end{array}\]

\[\begin{array}{c}
Y_{\alpha_2}
\end{array}\]

\[\begin{array}{c}
Y_{\alpha_1}
\end{array}\]

\[\begin{array}{c}
M^*
\end{array}\]

Figure 2.4.1

**Remark 2.5**

(i) If we rewrite (2.5) by Euler characteristics, we obtain an equality analogous to Ernström's formula (1.6). Namely, we have

\[
\chi(M \cap L) - \chi(M \cap H) = (-1)^{J_\vec{p}}
\]

(2.8)

for any $L \in M^*_\text{reg}$, where $H \in Y \setminus M^*$ is a generic hyperplane in $Y = \mathbb{RP}^n$.

(ii) In the case where $M^*$ is a hypersurface, the complement of $M^*$ is divided into several connected components in general. So when we cross the hypersurface $M^*$, the value of the function $\varphi_M$ may jump. Our formula (2.6) describes this jumping number in terms of the principal curvature $J_\vec{p}$ of $M^*_\text{reg}$.

Next, we state our second main theorem which reconstructs the values of $\varphi_M$ on $M^*_\text{sing}$.

**Theorem 2.6** ([26]) Let $k \geq \text{codim}_YM^*$. Suppose that the values $\varphi_\alpha$'s on $Y_\alpha$'s satisfying $\text{codim}_YM_\alpha \leq k$ are already determined. Then the value $\varphi_\beta$ on $Y_\beta$ satisfying $\text{codim}_YM_\beta = k + 1$ is given by

\[
\varphi_\beta = \sum_{\alpha: Y_\alpha \cap B \neq \emptyset} \varphi_\alpha \cdot \{\chi(Y_\alpha \cap B) - \chi(\partial Y_\alpha \cap B)\}.
\]

(2.9)
Here we set $B = B(y, \varepsilon) \cap \{\psi < 0\}$ by taking a small enough open ball $B(y, \varepsilon)$ centered at a point $y \in Y_\beta$ and and a real-valued real analytic function $\psi$ defined in a neighborhood of $y$ satisfying $\psi^{-1}(0) \supset Y_\beta$ and

\[
(y; \text{grad}\psi(y)) \in \overset{\circ}{T}_{Y_\beta}Y \setminus \bigcup_{\alpha \neq \beta} \overline{T}_{Y_\alpha}Y \tag{2.10}
\]

(see Figure 2.5.1 below).

By Theorem 2.6, we can recursively determine the values $\varphi_\alpha$'s of $\varphi_M$ by induction on the codimensions of strata $Y_\alpha$'s. Note that this representation of the function $\varphi_M$ is completely analogous to that of the Euler obstructions in [18].

**Example 2.7** Consider a smooth projective curve $M$ defined by the homogeneous equation $x^4 + x^3 z + x^4 - y^3 z = 0$ in $\mathbb{RP}_2$ (see Figure 2.6.1 below).

![Figure 2.6.1](image)

Then the dual curve $M^* \subset \mathbb{RP}_2$ has a shape as in Figure 2.6.2 below. More precisely, as a $\mu$-stratification of $Y = \mathbb{RP}_\infty^*$ adapted to $M^*$, we can take $Y = \bigcup_{i=0}^{11} Y_i$ in Figure 2.6.2. Since the last strata $Y_{11}$ is contained in the line at infinity ($\simeq \mathbb{RP}_1$) of $\mathbb{RP}_2^*$ it does not appear in the figure.
Now let us apply our two main theorems to this case. Denote by $\varphi_i$ the value of the function $\varphi_M$ on $Y_i$. Then we can easily see that $\varphi_0$ is 0. Starting from this value $\varphi_0 = 0$, we can recursively determine all the values $\varphi_i$'s of $\varphi_M$ as follows.

For example, by Theorem 2.4 the values $\varphi_1$ and $\varphi_3$ on $Y_1$ and $Y_3$ respectively can be calculated in the following way.

$$
\varphi_1 = \varphi_0 + (-1)^2 - (-1)^1 = 2, \quad (2.11)
$$
$$
\varphi_3 = \frac{1}{2}(\varphi_0 + \varphi_1) = 1. \quad (2.12)
$$

Moreover, by Theorem 2.6 the value $\varphi_{10}$ on $Y_{10}$ is determined by $\varphi_1$, $\varphi_2$, $\varphi_5$ and $\varphi_6$ as follows.

$$
\varphi_{10} = \varphi_1 \cdot 0 + \varphi_5 \cdot 1 + \varphi_2 \cdot (-1) + \varphi_6 \cdot 1 + \varphi_1 \cdot 0 = 2. \quad (2.13)
$$
In this case, we can easily check these results simply by counting the intersection numbers of M and lines in \( \mathbb{RP}_2 \). Namely, our results are the generalization of this very simple example to higher dimensional cases.

### 3 Theoretical background

Since the function \( \varphi_M \) in our main theorems is constant on each stratum of Y, we consider a class of such functions to study \( \varphi_M \), which are called constructible functions.

**Definition 3.1** Let \( X \) be a real analytic manifold. We say that a function \( \varphi: X \to \mathbb{Z} \) is constructible if there exists a locally finite family \( \{X_i\} \) of compact subanalytic subsets \( X_i \) of \( X \) such that \( \varphi \) is expressed by

\[
\varphi = \sum_i c_i 1_{X_i}, \quad (c_i \in \mathbb{Z}).
\]

We denote the abelian group of constructible functions on \( X \) by \( CF(X) \).

We define the operations of constructible functions in the following way.

**Definition 3.2** ([21] and [37]) Let \( f: Y \to X \) be a morphism of real analytic manifolds.

(i) (The inverse image) For \( \varphi \in CF(X) \), we define a function \( f^*\varphi \in CF(Y) \) by

\[
f^*\varphi(y) := \varphi(f(y)).
\]

(ii) (The integral) Let \( \varphi = \sum_i c_i 1_{X_i} \in CF(X) \) such that \( \text{supp}(\varphi) \) is compact. Then we define a topological (Euler) integral \( \int_X \varphi \in \mathbb{Z} \) of \( \varphi \) by

\[
\int_X \varphi := \sum_i c_i \cdot \chi(X_i).
\]

(iii) (The direct image) Let \( \psi \in CF(Y) \) such that \( f|_{\text{supp}(\psi)}: \text{supp}(\psi) \to X \) is proper. Then we define a function \( \int_f \psi \in CF(X) \) by

\[
\left( \int_f \psi \right)(x) := \int_Y (\psi \cdot 1_{f^{-1}(x)}).
\]
From now on, we shall use various notions concerning derived categories of constructible sheaves. For the detail of these notions, see [21] etc. We denote by $D^b(X)$ the derived category of bounded complexes of $\mathbb{C}_X$-modules on $X$. Its full subcategory consisting of complexes whose cohomology sheaves are $\mathbb{R}$-constructible is denoted by $D^b_{\mathbb{R}-c}(X)$.

Recall also that the Grothendieck group $K_{\mathbb{R}-c}(X)$ of $D^b_{\mathbb{R}-c}(X)$ is a quotient group of the free abelian group generated by objects of $D^b_{\mathbb{R}-c}(X)$ by the subgroup generated by

$$[F] - [F'] - [F''] \quad (F' \rightarrow F \rightarrow F'' \rightarrow 1 \text{ is a distinguished triangle}).$$

(3.5)

Then the natural morphism

$$\chi: K_{\mathbb{R}-c}(X) \rightarrow CF(X)$$

defined by $\chi([F])(x) = \sum_{j \in \mathbb{Z}} (-1)^j \dim H^j(F)_x \ (x \in X)$ is an isomorphism. Moreover, by the isomorphism $\chi: K_{\mathbb{R}-c}(X) \rightarrow CF(X)$ the operations of constructible functions that we introduced in Definition 3.2 correspond to those for $\mathbb{R}$-constructible sheaves. For example, let $f: Y \rightarrow X$ be a morphism of real analytic manifolds and for $\psi \in CF(Y)$ take an object $G \in D^b_{\mathbb{R}-c}(Y)$ such that $\psi = \chi(G)$. Then we have $\chi(Rf_*G) = \int_Y \psi \in CF(X)$. In the same way, we can slightly generalize the notion of topological integrals of constructible functions as follows.

**Definition 3.3** ([26]) Let $U$ be a relatively compact subanalytic open subset of $X$ and $\varphi \in CF(X)$. Take an object $F \in D^b_{\mathbb{R}-c}(X)$ such that $\varphi = \chi(F)$ and set

$$\int_U \varphi := \chi(R\Gamma(U; F)).$$

(3.7)

We can easily check that the definition above does not depend on the choice of $F$ such that $\varphi = \chi(F)$. Note that we do not have to assume that the support of $\varphi$ is compact in $U$ as in the usual definition (Definition 3.2 (ii)). Using this slight modification of the notion of topological integrals, we can express the R.H.S of (2.9) simply by $\int_B \varphi_M$. In fact, we used this fact in the proof of Theorem 2.6.

Now, let $X$ be a real analytic manifold and denote by $\mathcal{L}_X$ the sheaf of conic ($\mathbb{R}_{>0}$-invariant) subanalytic Lagrangian cycles in the cotangent bundle $T^*X$ of $X$. Its global section $H^0(T^*X; \mathcal{L}_X)$ is the abelian group of conic
group homomorphism $CC: K_{R-c}(X) \rightarrow H^0(T^*X; \mathcal{L}_X)$ and associated with
each $[F] \in K_{R-c}(X)$ a Lagrangian cycle $CC([F])$ in $T^*X$. This Lagrangian
cycle $CC([F])$ is called the characteristic cycle of $[F] \in K_{R-c}(X)$. The
following very important theorem was proved also in [19] (see [21] for the
detail).

**Theorem 3.4** [21, Theorem 9.7.11] There exists a commutative diagram

\[
\begin{array}{ccc}
H^0(T^*X; \mathcal{L}_X) & \xrightarrow{\sim} & CF(X) \\
\sim & & \sim \\
K_{R-c}(X) & \xrightarrow{\sim} & CF(X)
\end{array}
\]

in which all arrows are isomorphisms.

By this theorem, we can reduce the problem of constructible functions
(sheaves) to that of Lagrangian cycles.

## 4 Outline of the proof of main theorems

In this section, we give an outline of the proof of our main theorems. Let
$X = \mathbb{R}P^n$ and $Y = \mathbb{R}P^n$ as before. Consider the incidence submanifold
$S = \{(x, H) \in X \times Y \mid x \in H\}$ of $X \times Y$ and the diagram

\[
\begin{array}{ccc}
X \times Y & \xrightarrow{f} & Y \\
\downarrow{p_2} & & \downarrow{g} \\
X & \xrightarrow{f} & Y,
\end{array}
\]

where $p_1$ and $p_2$ are natural projections and $f$ and $g$ are restrictions of $p_1$
and $p_2$ to $X$ and $Y$ respectively.

**Definition 4.1** Let $\varphi \in CF(X)$. We define the topological Radon transform
$\mathcal{R}_S(\varphi) \in CF(Y)$ of $\varphi$ by

\[
\mathcal{R}_S(\varphi) := \int_g f^* \varphi.
\]

In particular, for a real analytic submanifold $M$ of $X = \mathbb{R}P^n$ and a
hyperplane $H$ in $X = \mathbb{R}P^n \iff H \in Y = \mathbb{R}P^n$ we have

\[
\mathcal{R}_S(1_M)(H) = \chi(M \cap H) \quad (= \varphi_M(H)).
\]
Therefore for the study of the function \( \varphi_M \in \text{CF}(Y) \) it suffices to study the topological Radon transform \( \mathcal{R}_S(1_M) \). Using the isomorphisms in Theorem 3.4, instead of the topological Radon transform \( \mathcal{R}_S: \text{CF}(X) \rightarrow \text{CF}(Y) \) itself, we studied the corresponding operation for Lagrangian cycles (characteristic cycles). Then we found an isomorphism
\[
\Psi: H^0(\dot{T}^*X; \mathscr{L}_X) \sim H^0(\dot{T}^*Y; \mathscr{L}_Y),
\]
where we set \( \dot{T}^*X = T^*X \setminus T_X^*X \) and \( \dot{T}^*Y = T^*Y \setminus T_Y^*Y \) (the zero-sections are removed). Moreover this operation \( \Psi \) is (up to some sign \( \epsilon = \pm 1 \)) the isomorphism of Lagrangian cycles induced by the canonical diffeomorphism \( \Phi: \dot{T}^*X \rightarrow \dot{T}^*Y \) which coincides with the classical Legendre transform in the standard affine charts of \( X = \mathbb{R}P^n \) and \( Y = \mathbb{R}P^n \). Since the characteristic cycle \( \text{CC}(1_M) \) of \( 1_M \in \text{CF}(X) \) is the conormal cycle \( T_M^*X \) in \( \dot{T}^*X \), the characteristic cycle \( \text{CC}(\mathcal{R}_S(1_M)) \) of the topological Radon transform \( \mathcal{R}_S(1_M) = \varphi_M \) is \( \epsilon(\Phi(T_M^*X)) \). Set \( \pi_Y: \dot{T}^*Y \twoheadrightarrow Y \) and \( N = (\pi_Y \circ \Phi)(T_M^*X) \subset Y \). Then we can easily prove that \( N \) coincides with the dual variety \( M^* \) of \( M \), which is a closed subanalytic subset of \( Y = \mathbb{R}P^n \) (in classical terminology we call it a caustic or Legendre singularity). Moreover it turns out that the closure \( \overline{\dot{T}_{N_{\text{reg}}}^*Y} \) of the conormal bundle \( \dot{T}_{N_{\text{reg}}}^*Y \) in \( \dot{T}^*Y \) is nothing but \( \Phi(T_M^*X) \) (see [16] for a similar argument). Then by using this very nice property of the characteristic cycle \( \text{CC}(\varphi_M) \) we can reconstruct the function \( \varphi_M \) from the geometry of the dual variety \( M^* = N \). Theorem 2.6 was proved in this way. To prove Theorem 2.4, we have to determine the sign \( \epsilon = \pm 1 \), which is the most difficult part of our study. We could determine it by employing the theory of pure sheaves in [21]. More precisely, we expressed the Maslov indices of the Lagrangian submanifolds \( \dot{T}_M^*X \) and \( \dot{T}_{N_{\text{reg}}}^*Y \) by the principal curvatures of \( M \) and \( N_{\text{reg}} \) respectively with the help of results in [11].

**Remark 4.2** By the same argument as above, we can give a more transparent proof to the main results of Ernström [9] in the complex case.

### 5 Grassmann cases and class formulas

#### 5.1 \( k \)-dual varieties

We shall generalize the situation considered in the previous sections to Grassmann cases and obtain similar results. Let \( 0 \leq k \leq n - 1 \) be an integer.

Recall that the Grassmann manifold consisting of \( k \)-dimensional planes in \( \mathbb{P}^n \) is defined by

\[
\mathbb{G}_{n,k} = \{ L' | L' \text{ is a } (k + 1)\text{-dimensional linear subspace in } \mathbb{K}^{n+1} \} \quad (5.1)
\]

\[
= \{ L | L \text{ is a } k\text{-dimensional linear subspace in } \mathbb{P}_n \}. \quad (5.2)
\]
Note that $G_{n,0} = \mathbb{P}_n$ and $G_{n,n-1} = \mathbb{P}_n^*$. Then the notion of dual varieties is generalized to Grassmann cases as follows.

**Definition 5.1** Let $V \subset \mathbb{P}_n$ be a projective variety. We define the $k$-dual variety $V^{(k)}$ of $V$ by

$$V^{(k)} := \{ L \in G_{n,k} \mid \exists x \in V_{\text{reg}} \cap L \text{ s.t. } V \not\ni L \text{ at } x \} \subset G_{n,k}. \quad (5.3)$$

If $k = n - 1$ the $k$-dual $V^{(k)} \subset G_{n,k} \simeq \mathbb{P}_n^*$ is nothing but the classical dual variety of $V$. In [12], Gelfand-Kapranov-Zelevinsky called $V^{(k)}$ the associated variety of $V$ and showed that $V^{(n - \dim V - 1)}$ is a hypersurface.

### 5.2 Topological class formulas

From now on, we always assume that the ground field $\mathbb{K}$ is $\mathbb{C}$. Let $V \subset \mathbb{P}_n$ be a projective variety over $\mathbb{C}$ and $0 \leq k \leq n - 1$ an integer. Assume that $V^{(k)}$ is a hypersurface in $G_{n,k}$.

**Definition 5.2** [12, Proposition 2.1 of Chapter 3] Consider the Plücker embedding:

$$V^{(k)} \subset G_{n,k} \subset \mathbb{P}_{\binom{n+1}{k+1}}. \quad (5.4)$$

We call the degree of the defining polynomial of $V^{(k)}$ in $\mathbb{P}_{\binom{n+1}{k+1}}$ the degree of $V^{(k)}$ and denote it by $\deg V^{(k)}$.

In [27], we proved the following topological class formula (i.e., a formula which expresses the degrees of dual varieties) for $k$-dual varieties by using Ernström's result [9] and some elementary formulas on constructible functions.

**Theorem 5.3** ([27]) In the situation as above, for generic linear subspaces $L_1 \simeq \mathbb{P}_{k-1}$, $L_2 \simeq \mathbb{P}_k$ and $L_3 \simeq \mathbb{P}_{k+1}$ of $\mathbb{P}_n$ we have

$$\deg V^{(k)} = (-1)^{(n-k)+\dim V + 1} \left\{ \int_{L_1} \text{Eu}_V - 2 \int_{L_2} \text{Eu}_V + \int_{L_3} \text{Eu}_V \right\}. \quad (5.5)$$

**Corollary 5.4** Let $L \simeq \mathbb{P}_{k+1}$ be a generic $(k+1)$-dimensional linear subspace of $\mathbb{P}_n$ and consider the usual dual variety $(V \cap L)^* \subset \mathbb{P}_{k+1}^*$ of $V \cap L \subset L \simeq \mathbb{P}_{k+1}$. Then we have

$$\deg V^{(k)} = \deg (V \cap L)^*. \quad (5.6)$$
The formula in Theorem 5.3 expresses the algebraic invariant $\deg V^{(k)}$ of $V^{(k)}$ by the topological data of $V$. In the case where $k = n - 1$, we thus reobtain the topological class formulas obtained by Ernström [10], Parusinski and Kleiman [22] etc. See [34, Section 10.1] for an excellent review on this subject. In a forthcoming paper [28], from these topological class formulas we derive various more computable class formulas which extend the previous results obtained by Teissier and Kleiman [23] etc.

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