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Kyoto University
Topological Radon Transforms and Projective Duality

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Abstract

We study various topological properties of projective duality in algebraic geometry by using the microlocal theory of sheaves developed by Kashiwara-Schapira [21]. In particular, in the real algebraic case we obtain some results similar to Ernström’s ones [9] obtained in the complex case. For this purpose, we use constructible functions and their topological Radon transforms. We also generalize a class formula (i.e. a formula which expresses the degrees of dual varieties) in [10] to the case of associated varieties studied by Gelfand-Kapranov-Zelevinsky [12] etc. For the detail, see [26] and [27].

1 Introduction

We denote the projective space of dimension \( n \) over \( \mathbb{K} (= \mathbb{R} \text{ or } \mathbb{C}) \) by \( \mathbb{P}_n \) and its dual space by \( \mathbb{P}_n^* \). These spaces are naturally identified with the following sets.

\[
\mathbb{P}_n = \{ l \mid l \text{ is a line in } \mathbb{K}^{n+1} \text{ through the origin} \}, \tag{1.1}
\]

\[
\mathbb{P}_n^* = \{ H' \mid H' \text{ is a hyperplane in } \mathbb{K}^{n+1} \text{ through the origin} \}. \tag{1.2}
\]

Note that if we projectivize a hyperplane \( H' \) in \( \mathbb{K}^{n+1} \) we obtain a hyperplane \( H \) in \( \mathbb{P}_n \). Therefore in what follows we identify the dual projective space \( \mathbb{P}_n^* \) with the set

\[
\{ H \mid H \text{ is a hyperplane in } \mathbb{P}_n \}. \tag{1.3}
\]
Definition 1.1 Let $V$ be a projective variety in $\mathbb{P}_n$. We define the dual variety $V^*$ of $V$ by

$$V^* := \{ H \in \mathbb{P}^*_n \mid \exists x \in V_{\text{reg}} \cap H \text{ s.t. } T_xV \subset T_xH \} \quad (\subset \mathbb{P}^*_n). \quad (1.4)$$

When $V$ is smooth, $V^*$ is the set of hyperplanes tangent to $V$. As we see in the example below, even if $V$ is smooth, $V^*$ may be very singular in general.

Example 1.2  

(i) Let $\iota_n : \mathbb{P}_1 \hookrightarrow \mathbb{P}_n$ be the Veronese embedding given by 

$[x : y] \mapsto [x^n : x^{n-1}y : \ldots : xy^{n-1} : y^n]$ and set $V = \iota_n(\mathbb{P}_1) \subset \mathbb{P}_n$. Then the dual $V^* \subset \mathbb{P}_n^*$ is a hypersurface defined by the classical discriminant for polynomials of degree $n$.

(ii) For $n \geq m$, consider the Segre embedding $\iota_{n,m} : \mathbb{P}_n \times \mathbb{P}_m \hookrightarrow \mathbb{P}_{(n+1)(m+1)-1}$ given by 

$([x_0 : \ldots : x_n], [y_0 : \ldots : y_m]) \mapsto [\ldots : x_iy_j : \ldots]$. Set 

$W = \iota_{n,m}(\mathbb{P}_n \times \mathbb{P}_m) \subset \mathbb{P}_{(n+1)(m+1)-1}$. Then the dual variety $W^* \subset \mathbb{P}^*_{(n+1)(m+1)-1}$ has very complicated singularities and the dual defect $\delta^*(W)$ of $W$ (see (2.2) below) is $n - m$. Indeed, let $M_{(n+1),(m+1)}$ be the space of $(n + 1) \times (m + 1)$ matrices and identify the dual projective space $\mathbb{P}^*_{(n+1)(m+1)-1}$ with its projectivization $\mathbb{P}(M_{(n+1),(m+1)})$. Then the dual $W^* \subset \mathbb{P}^*_{(n+1)(m+1)-1}$ is explicitly written by

$$W^* = \mathbb{P}\{A \in M_{(n+1),(m+1)} \mid \text{rank}A \leq m\}. \quad (1.5)$$

Therefore the dual $W^*$ admits a stratification defined by the ranks of matrices.

Many mathematicians were interested in the mysterious relations between projective varieties and their duals. Above all, they observed that the tangency of a hyperplane $H \subset V^*$ with $V$ is related to the singularity of the dual $V^*$ at $H$. For example, consider the case of a plane curve $C \subset \mathbb{P}_2$ over $\mathbb{C}$. Then a tangent line $l$ at an inflection point of $C$ corresponds to a cusp of the dual curve $C^*$, and a bitangent (double tangent) line $l$ of $C$ corresponds to an ordinary double point of $C^*$. The most general results for complex plane curves were found in the 19th century by Klein, Plücker and Clebsch etc. (see for example, [34, Theorem 1.6] and [38, Chapter 7] etc.).

In the last two decades, this beautiful correspondence was extended to higher-dimensional complex projective varieties from the viewpoint of the geometry of hyperplane sections. In particular, after some important contributions by Viro [37] and Dimca [8] etc., Ernström proved the following remarkable result in 1994.
Theorem 1.3 [9, Corollary 3.9] Let $V \subset \mathbb{P}_n$ be a smooth projective variety over $\mathbb{C}$. Take a generic hyperplane $H$ in $\mathbb{P}_n$ such that $H \notin V^*$. Then for any hyperplane $L \in V^*$, we have

$$\chi(V \cap L) - \chi(V \cap H) = (-1)^{n-1 + \text{dim} V - \text{dim} V^*} \text{Eul}_{V^*}(L), \quad (1.6)$$

where $\chi$ stands for the topological Euler characteristic and $\text{Eul}_{V^*}: V^* \to \mathbb{Z}$ is the Euler obstruction of $V^*$ (introduced by Kashiwara [17] and MacPherson [24] independently).

Recall that the Euler obstruction $\text{Eul}_{V^*}$ of $V^*$ is a $\mathbb{Z}$-valued function on $V^*$ which measures the singularity of $V^*$ at each point of $V^*$. For example, $\text{Eul}_{V^*}$ takes the value 1 on the regular part of $V^*$. Moreover, if we take a Whitney stratification $\bigsqcup_{\alpha \in A} V_{\alpha}^*$ of $V^*$ consisting of connected strata, then $\text{Eul}_{V^*}$ is constant on each stratum $V_{\alpha}^*$. The values of $\text{Eul}_{V^*}$ on a stratum $V_{\alpha}^*$ is determined by those on $V_{\beta}^*$ satisfying the condition $V_{\alpha}^* \subset \overline{V_{\beta}^*}$ (for more detail, see e.g. [18]).

Hence Ernström's result says that the jumping number of the topological Euler characteristics of hyperplane sections of $V$ at $L$ is expressed by $\text{Eul}_{V^*}(L)$, that is, the singularity of the dual variety $V^*$ at $L$.

The aim of this article is to introduce our results in the real algebraic case similar to this Ernström's one and to survey its theoretical background.

2 Main results

Consider a real projective space $X = \mathbb{R}\mathbb{P}_n$ of dimension $n$ and its dual $Y = \mathbb{R}\mathbb{P}_n^*$. Let $M \subset X$ be a smooth real projective variety and $M^* \subset Y$ its dual variety.

We fix a $\mu$-stratification $Y = \bigsqcup_{\alpha \in A} Y_{\alpha}$ of $Y = \mathbb{R}\mathbb{P}_n^*$ consisting of connected strata and adapted to $M^*$. Note that Trotman [35] proved that this $\mu$-condition is equivalent to famous Verdier's w-regularity condition.

Definition 2.1 We define a $\mathbb{Z}$-valued function $\varphi_M: Y \to \mathbb{Z}$ on $Y = \mathbb{R}\mathbb{P}_n^*$ by

$$\varphi_M(H) = \chi(M \cap H) \quad (H \in Y). \quad (2.1)$$

Since the function $\varphi_M$ is defined by the topological Euler characteristics of hyperplane sections $M \cap H$ of $M$, to obtain results similar to Ernström's formula (1.6) it suffices to describe the function $\varphi_M$ in terms of the singularities of $M^*$. We will show that the whole function $\varphi_M$ can be reconstructed
from one value $\varphi_M(y)$ at a point $y \in Y \setminus M^*$ and the singularities of $M^*$. First of all, for the above $\mu$-stratification $Y = \bigsqcup_{\alpha \in A} Y_{\alpha}$ of $Y = \mathbb{RP}_n^*$ we can prove the following basic result.

**Proposition 2.2** The function $\varphi_M$ is constant on each stratum $Y_{\alpha}$.

We denote the value of $\varphi_M$ on $Y_{\alpha}$ by $\varphi_{\alpha}$. Our main results are reconstruction theorems of $\varphi_M$. Namely, we can determine all the values $\varphi_{\alpha}$'s of $\varphi_M$ from only one value $\varphi_M(y)$ at a point $y \in Y \setminus M^*$ and the topology of $M^*$.

To state the first theorem, we introduce two notations concerning dual varieties. Recall that the dual variety $M^*$ is usually a hypersurface in $Y = \mathbb{RP}_n^*$.

**Definition 2.3**

(i) We denote the dual defect of $M$ by

$$\delta^*(M) = (n - 1) - \dim M^*. \quad (2.2)$$

(ii) For a conormal vector $\vec{p} \in T^*_{ \alpha \in A} Y$ at $y \in M_{\text{reg}}^*$, consider the second fundamental form

$$h_{M^* \vec{p}.}: T_y M^* \times T_y M^* \to \mathbb{R}, \quad (2.3)$$

with respect to the canonical (Fubini-Study) metric of $Y = \mathbb{RP}_n^*$ and set

$$J_{\vec{p}} := \#\{\text{positive eigenvalues of } h_{M^* \vec{p}.}\} + \delta^*(M). \quad (2.4)$$

Now, let us state our first main theorem which describes the values of $\varphi_M$ on $Y \setminus M_{\text{sing}}^*$.

**Theorem 2.4** ([26])

(i) Assume that $\delta^*(M) > 0$. Then on $Y \setminus M^*$ the function $\varphi_M$ is constant. Moreover for any $y \in M_{\text{reg}}^*$ there exists an neighborhood $U$ of $y$ such that we have

$$\varphi_M = d \cdot 1_Y + (-1)^{J_{\vec{p}}} 1_{M_{\text{reg}}^*}, \quad (2.5)$$

on $U$, where $d$ is the value of $\varphi_M$ on $Y \setminus M^*$ and $\vec{p} \in T^*_{M_{\text{reg}}^*} Y$ is a conormal vector at $y \in M_{\text{reg}}^*$. 


(ii) Assume that $\delta^*(M) = 0$, that is $M^*$ is a hypersurface in $Y = \mathbb{R}\mathbb{P}_n$, and consider the following local situation. Let $Y_{\alpha_1}$ and $Y_{\alpha_2}$ be two strata in $Y \setminus M^*$, $Y_\beta$ an open stratum in $M_{\text{reg}}^*$ such that $Y_\beta \subset Y_{\alpha_i}$ for $i = 1, 2$ and $\vec{p} \in T_{M_{\text{reg}}^*} Y$ a conormal vector at a point $y \in Y_\beta$ pointing from $Y_{\alpha_1}$ to $Y_{\alpha_2}$ (see Figure 2.4.1 below). Then we have

$$\varphi_{\alpha_2} - \varphi_{\alpha_1} = (-1)^{J_{\overline{p}}} - (-1)^{J_{-\overline{p}}} = (0, \pm 2),$$  \hspace{1cm} (2.6)

$$\varphi_{\beta} = \begin{cases} 
\frac{1}{2}(\varphi_{\alpha_1} + \varphi_{\alpha_2}) & \text{if } \varphi_{\alpha_1} \neq \varphi_{\alpha_2}, \\
\varphi_{\alpha_1} + (-1)^{J_{\overline{p}}} & \text{if } \varphi_{\alpha_1} = \varphi_{\alpha_2}.
\end{cases}$$ \hspace{1cm} (2.7)

![Figure 2.4.1](image)

**Remark 2.5** (i) If we rewrite (2.5) by Euler characteristics, we obtain an equality analogous to Ernström's formula (1.6). Namely, we have

$$\chi(M \cap L) - \chi(M \cap H) = (-1)^{J_{\overline{p}}}$$ \hspace{1cm} (2.8)

for any $L \in M_{\text{reg}}^*$, where $H \in Y \setminus M^*$ is a generic hyperplane in $Y = \mathbb{R}\mathbb{P}_n$.

(ii) In the case where $M^*$ is a hypersurface, the complement of $M^*$ is divided into several connected components in general. So when we cross the hypersurface $M^*$, the value of the function $\varphi_M$ may jump. Our formula (2.6) describes this jumping number in terms of the principal curvature $J_\overline{p}$ of $M_{\text{reg}}^*$.

Next, we state our second main theorem which reconstructs the values of $\varphi_M$ on $M_{\text{sing}}^*$.

**Theorem 2.6** ([26]) Let $k \geq \text{codim}_Y M^*$. Suppose that the values $\varphi_{\alpha}$'s on $Y_\alpha$'s satisfying $\text{codim}_Y Y_\alpha \leq k$ are already determined. Then the value $\varphi_{\beta}$ on $Y_\beta$ satisfying $\text{codim}_Y Y_\beta = k + 1$ is given by

$$\varphi_{\beta} = \sum_{\alpha: Y_\alpha \cap B \neq \emptyset} \varphi_{\alpha} \cdot \{\chi(Y_\alpha \cap B) - \chi(\partial Y_\alpha \cap B)\}. \hspace{1cm} (2.9)$$
Here we set \( B = B(y, \varepsilon) \cap \{ \psi < 0 \} \) by taking a small enough open ball \( B(y, \varepsilon) \) centered at a point \( y \in Y_\beta \) and and a real-valued real analytic function \( \psi \) defined in a neighborhood of \( y \) satisfying \( \psi^{-1}(0) \supset Y_\beta \) and

\[
(y; \text{grad}(\psi(y))) \in T^*_Y Y \setminus \bigcup_{\alpha \neq \beta} T^*_{Y_\alpha} Y
\]  
(2.10)

(see Figure 2.5.1 below).

By Theorem 2.6, we can recursively determine the values \( \varphi_\alpha \)'s of \( \varphi_M \) by induction on the codimensions of strata \( Y_\alpha \)'s. Note that this representation of the function \( \varphi_M \) is completely analogous to that of the Euler obstructions in [18].

**Example 2.7** Consider a smooth projective curve \( M \) defined by the homogeneous equation \( x^4 + x^3z + x^3y - y^3z = 0 \) in \( \mathbb{RP}_2 \) (see Figure 2.6.1 below).

Then the dual curve \( M^* \subset \mathbb{RP}_2^* \) has a shape as in Figure 2.6.2 below. More precisely, as a \( \mu \)-stratification of \( Y = \mathbb{RP}_2^* \) adapted to \( M^* \), we can take \( Y = \bigsqcup_{i=0}^{11} Y_i \) in Figure 2.6.2. Since the last strata \( Y_{11} \) is contained in the line at infinity (\( \simeq \mathbb{RP}_1 \)) of \( \mathbb{RP}_2 \) it does not appear in the figure.
Now let us apply our two main theorems to this case. Denote by $\varphi_i$ the value of the function $\varphi_M$ on $Y_i$. Then we can easily see that $\varphi_0$ is 0. Starting from this value $\varphi_0 = 0$, we can recursively determine all the values $\varphi_i$'s of $\varphi_M$ as follows.

For example, by Theorem 2.4 the values $\varphi_1$ and $\varphi_3$ on $Y_1$ and $Y_3$ respectively can be calculated in the following way.

\[
\varphi_1 = \varphi_0 + (-1)^2 - (-1)^1 = 2, \quad (2.11)
\]
\[
\varphi_3 = \frac{1}{2}(\varphi_0 + \varphi_1) = 1. \quad (2.12)
\]

Moreover, by Theorem 2.6 the value $\varphi_{10}$ on $Y_{10}$ is determined by $\varphi_1$, $\varphi_2$, $\varphi_5$ and $\varphi_6$ as follows.

\[
\varphi_{10} = \varphi_1 \cdot 0 + \varphi_6 \cdot 1 + \varphi_2 \cdot (-1) + \varphi_6 \cdot 1 + \varphi_1 \cdot 0 = 2. \quad (2.13)
\]
In this case, we can easily check these results simply by counting the intersection numbers of \( M \) and lines in \( \mathbb{R}P_2 \). Namely, our results are the generalization of this very simple example to higher dimensional cases.

## 3 Theoretical background

Since the function \( \varphi_M \) in our main theorems is constant on each stratum of \( Y \), we consider a class of such functions to study \( \varphi_M \), which are called constructible functions.

**Definition 3.1** Let \( X \) be a real analytic manifold. We say that a function \( \varphi: X \to \mathbb{Z} \) is constructible if there exists a locally finite family \( \{X_i\} \) of compact subanalytic subsets \( X_i \) of \( X \) such that \( \varphi \) is expressed by

\[
\varphi = \sum_i c_i 1_{X_i} \quad (c_i \in \mathbb{Z}).
\]

We denote the abelian group of constructible functions on \( X \) by \( CF(X) \).

We define the operations of constructible functions in the following way.

**Definition 3.2** ([21] and [37]) Let \( f: Y \to X \) be a morphism of real analytic manifolds.

(i) (The inverse image) For \( \varphi \in CF(X) \), we define a function \( f^* \varphi \in CF(Y) \) by

\[
f^* \varphi(y) := \varphi(f(y)).
\]

(ii) (The integral) Let \( \varphi = \sum_i c_i 1_{X_i} \in CF(X) \) such that \( \text{supp}(\varphi) \) is compact. Then we define a topological (Euler) integral \( \int_X \varphi \in \mathbb{Z} \) of \( \varphi \) by

\[
\int_X \varphi := \sum_i c_i \cdot \chi(X_i).
\]

(iii) (The direct image) Let \( \psi \in CF(Y) \) such that \( f|_{\text{supp}(\psi)}: \text{supp}(\psi) \to X \) is proper. Then we define a function \( \int_f \psi \in CF(X) \) by

\[
(\int_f \psi)(x) := \int_Y (\psi \cdot 1_{f^{-1}(x)}).
\]
From now on, we shall use various notions concerning derived categories of constructible sheaves. For the detail of these notions, see [21] etc. We denote by $D^b(X)$ the derived category of bounded complexes of $\mathbb{C}_X$-modules on $X$. Its full subcategory consisting of complexes whose cohomology sheaves are $\mathbb{R}$-constructible is denoted by $D^{b}_{R-c}(X)$.

Recall also that the Grothendieck group $K_{R-c}(X)$ of $D^{b}_{R-c}(X)$ is a quotient group of the free abelian group generated by objects of $D^{b}_{R-c}(X)$ by the subgroup generated by $[F] - [F'] - [F'']$ (where $F' \to F \to F'' \to +1$ is a distinguished triangle).

Then the natural morphism

$$\chi: K_{R-c}(X) \to CF(X)$$

defined by $\chi([F])(x) = \sum_{j \in \mathbb{Z}} (-1)^j \dim H^j(F)_x$ ($x \in X$) is an isomorphism.

Moreover, by the isomorphism $\chi: K_{R-c}(X) \sim CF(X)$ the operations of constructible functions that we introduced in Definition 3.2 correspond to those for $\mathbb{R}$-constructible sheaves. For example, let $f: Y \to X$ be a morphism of real analytic manifolds and for $\psi \in CF(Y)$ take an object $G \in D^{b}_{R-c}(Y)$ such that $\psi = \chi(G)$. Then we have $\chi(Rf_*G) = \int \psi$ in $CF(X)$. In the same way, we can slightly generalize the notion of topological integrals of constructible functions as follows.

**Definition 3.3** ([26]) Let $U$ be a relatively compact subanalytic open subset of $X$ and $\varphi \in CF(X)$. Take an object $F \in D^{b}_{R-c}(X)$ such that $\varphi = \chi(F)$ and set

$$\int_U \varphi := \chi(R\Gamma(U;F)).$$

We can easily check that the definition above does not depend on the choice of $F$ such that $\varphi = \chi(F)$. Note that we do not have to assume that the support of $\varphi$ is compact in $U$ as in the usual definition (Definition 3.2 (ii)). Using this slight modification of the notion of topological integrals, we can express the R.H.S of (2.9) simply by $\int_B \varphi_M$. In fact, we used this fact in the proof of Theorem 2.6.

Now, let $X$ be a real analytic manifold and denote by $\mathcal{L}_X$ the sheaf of conic ($\mathbb{R}_{>0}$-invariant) subanalytic Lagrangian cycles in the cotangent bundle $T^*X$ of $X$. Its global section $H^0(T^*X; \mathcal{L}_X)$ is the abelian group of conic...
subanalytic Lagrangian cycles in $T^*X$. In 1985, Kashiwara [19] constructed a group homomorphism $CC: K_{\mathcal{R}-c}(X) \rightarrow H^0(T^*X; \mathcal{L}_X)$ and associated with each $[F] \in K_{\mathcal{R}-c}(X)$ a Lagrangian cycle $CC([F])$ in $T^*X$. This Lagrangian cycle $CC([F])$ is called the characteristic cycle of $[F] \in K_{\mathcal{R}-c}(X)$. The following very important theorem was proved also in [19] (see [21] for the detail).

**Theorem 3.4** [21, Theorem 9.7.11] There exists a commutative diagram

$$
\begin{array}{ccc}
H^0(T^*X; \mathcal{L}_X) & \xrightarrow{CC} & CF(X) \\
\sim & & \sim \\
K_{\mathcal{R}-c}(X) & \xrightarrow{\sim} & X
\end{array}
$$

in which all arrows are isomorphisms.

By this theorem, we can reduce the problem of constructible functions (sheaves) to that of Lagrangian cycles.

### 4 Outline of the proof of main theorems

In this section, we give an outline of the proof of our main theorems. Let $X = \mathbb{R}P_n$ and $Y = \mathbb{R}P_n^*$ as before. Consider the incidence submanifold $S = \{(x, H) \in X \times Y \mid x \in H\}$ of $X \times Y$ and the diagram

$$
\begin{array}{ccc}
X \times Y & \xrightarrow{f} & Y \\
p_1 & & \downarrow g \\
X & \xleftarrow{p_2} & \downarrow p_2
\end{array}
$$

where $p_1$ and $p_2$ are natural projections and $f$ and $g$ are restrictions of $p_1$ and $p_2$ to $X$ and $Y$ respectively.

**Definition 4.1** Let $\varphi \in CF(X)$. We define the topological Radon transform $\mathcal{R}_S(\varphi) \in CF(Y)$ of $\varphi$ by

$$
\mathcal{R}_S(\varphi) := \int_g f^* \varphi.
$$

In particular, for a real analytic submanifold $M$ of $X = \mathbb{R}P_n$ and a hyperplane $H$ in $X = \mathbb{R}P_n$ ($\iff H \in Y = \mathbb{R}P_n^*$) we have

$$
\mathcal{R}_S(1_M)(H) = \chi(M \cap H) \ (= \varphi_M(H)).
$$
Therefore for the study of the function $\varphi_M \in CF(Y)$ it suffices to study the topological Radon transform $\mathcal{R}_S(1_M)$. Using the isomorphisms in Theorem 3.4, instead of the topological Radon transform $\mathcal{R}_S : CF(X) \to CF(Y)$ itself, we studied the corresponding operation for Lagrangian cycles (characteristic cycles). Then we found an isomorphism

$$\Psi : H^0(\dot{T}^*X; \mathcal{L}_X) \stackrel{\sim}{\longrightarrow} H^0(\dot{T}^*Y; \mathcal{L}_Y),$$

(4.4)

where we set $\dot{T}^*X = T^*X \setminus T^*_X X$ and $\dot{T}^*Y = T^*Y \setminus T^*_Y Y$ (the zero-sections are removed). Moreover this operation $\Psi$ is (up to some sign $\varepsilon = \pm 1$) the isomorphism of Lagrangian cycles induced by the canonical diffeomorphism $\Phi : \dot{T}^*X \to \dot{T}^*Y$ which coincides with the classical Legendre transform in the standard affine charts of $X = \mathbb{R}P^n$ and $Y = \mathbb{R}P^n$. Since the characteristic cycle $CC(1_M)$ of $1_M \in CF(X)$ is the conormal cycle $[\dot{T}^*_X X]$ in $\dot{T}^*X$, the characteristic cycle $CC(\mathcal{R}_S(1_M))$ of the topological Radon transform $\mathcal{R}_S(1_M) = \varphi_M$ is $\varepsilon \Phi(\dot{T}^*_X X)$. Set $\pi_Y : \dot{T}^*Y \to Y$ and $N = (\pi_Y \circ \Phi)(\dot{T}^*_X X) \subset Y$. Then we can easily prove that $N$ coincides with the dual variety $M^*$ of $M$, which is a closed subanalytic subset of $Y = \mathbb{R}P^n$ (in classical terminology we call it a caustic or Legendre singularity). Moreover it turns out that the closure $\overline{T^*_N_{\text{reg}} Y}$ of the conormal bundle $T^*_N_{\text{reg}} Y$ in $\dot{T}^*Y$ is nothing but $\Phi(\dot{T}^*_X X)$ (see [16] for a similar argument). Then by using this very nice property of the characteristic cycle $CC(\varphi_M)$ we can reconstruct the function $\varphi_M$ from the geometry of the dual variety $M^* = N$. Theorem 2.6 was proved in this way. To prove Theorem 2.4, we have to determine the sign $\varepsilon = \pm 1$, which is the most difficult part of our study. We could determine it by employing the theory of pure sheaves in [21]. More precisely, we expressed the Maslov indices of the Lagrangian submanifolds $\dot{T}^*_M X$ and $\dot{T}^*_N_{\text{reg}} Y$ by the principal curvatures of $M$ and $N_{\text{reg}}$ respectively with the help of results in [11].

**Remark 4.2** By the same argument as above, we can give a more transparent proof to the main results of Ernström [9] in the complex case.

## 5 Grassmann cases and class formulas

### 5.1 $k$-dual varieties

We shall generalize the situation considered in the previous sections to Grassmann cases and obtain similar results. Let $0 \leq k \leq n - 1$ be an integer.

Recall that the Grassmann manifold consisting of $k$-dimensional planes in $\mathbb{P}_n$ is defined by

$$G_{n,k} = \{ L' \mid L' \text{ is a } (k + 1)\text{-dimensional linear subspace in } \mathbb{K}^{n+1} \}$$

(5.1)

$$= \{ L \mid L \text{ is a } k\text{-dimensional linear subspace in } \mathbb{P}_n \}.$$  

(5.2)
Note that $G_{n,0} = P_n$ and $G_{n,n-1} = P_n^*$. Then the notion of dual varieties is generalized to Grassmann cases as follows.

**Definition 5.1** Let $V \subset P_n$ be a projective variety. We define the $k$-dual variety $V^{(k)}$ of $V$ by

$$V^{(k)} := \{ L \in G_{n,k} \mid \exists x \in V_{\text{reg}} \cap L \text{ s.t. } V \not\subset L \text{ at } x \} \subset G_{n,k}. \quad (5.3)$$

If $k = n - 1$ the $k$-dual $V^{(k)} \subset G_{n,k} \simeq P_n^*$ is nothing but the classical dual variety of $V$. In [12], Gelfand-Kapranov-Zelevinsky called $V^{(k)}$ the associated variety of $V$ and showed that $V^{(n-\dim V-1)}$ is a hypersurface.

### 5.2 Topological class formulas

From now on, we always assume that the ground field $K$ is $C$. Let $V \subset P_n$ be a projective variety over $C$ and $0 \leq k \leq n - 1$ an integer. Assume that $V^{(k)}$ is a hypersurface in $G_{n,k}$.

**Definition 5.2** [12, Proposition 2.1 of Chapter 3] Consider the Plücker embedding:

$$V^{(k)} \subset G_{n,k} \subset P_{\binom{n+1}{k+1}-1}. \quad (5.4)$$

We call the degree of the defining polynomial of $V^{(k)}$ in $P_{\binom{n+1}{k+1}-1}$ the degree of $V^{(k)}$ and denote it by $\deg V^{(k)}$.

In [27], we proved the following topological class formula (i.e. a formula which expresses the degrees of dual varieties) for $k$-dual varieties by using Ernström's result [9] and some elementary formulas on constructible functions.

**Theorem 5.3** ([27]) In the situation as above, for generic linear subspaces $L_1 \simeq P_{k-1}$, $L_2 \simeq P_k$ and $L_3 \simeq P_{k+1}$ of $P_n$ we have

$$\deg V^{(k)} = (-1)^{(n-k)+\dim V+1} \left\{ \int_{L_1} E_u V - 2 \int_{L_2} E_u V + \int_{L_3} E_u V \right\}. \quad (5.5)$$

**Corollary 5.4** Let $L \simeq P_{k+1}$ be a generic $(k+1)$-dimensional linear subspace of $P_n$ and consider the usual dual variety $(V \cap L)^* \subset P_{k+1}$ of $V \cap L \subset L \simeq P_{k+1}$. Then we have

$$\deg V^{(k)} = \deg (V \cap L)^*. \quad (5.6)$$
The formula in Theorem 5.3 expresses the algebraic invariant \( \deg V^{(k)} \) of \( V^{(k)} \) by the topological data of \( V \). In the case where \( k = n - 1 \), we thus reobtain the topological class formulas obtained by Ernström [10], Parusinski and Kleiman [22] etc. See [34, Section 10.1] for an excellent review on this subject. In a forthcoming paper [28], from these topological class formulas we derive various more computable class formulas which extend the previous results obtained by Teissier and Kleiman [23] etc.

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