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Topological Radon Transforms and Projective Duality

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Abstract

We study various topological properties of projective duality in algebraic geometry by using the microlocal theory of sheaves developed by Kashiwara-Schapira [21]. In particular, in the real algebraic case we obtain some results similar to Ernström's ones [9] obtained in the complex case. For this purpose, we use constructible functions and their topological Radon transforms. We also generalize a class formula (i.e. a formula which expresses the degrees of dual varieties) in [10] to the case of associated varieties studied by Gelfand-Kapranov-Zelevinsky [12] etc. For the detail, see [26] and [27].

1 Introduction

We denote the projective space of dimension $n$ over $\mathbb{K} (=\mathbb{R}$ or $\mathbb{C})$ by $\mathbb{P}_n$ and its dual space by $\mathbb{P}_n^\ast$. These spaces are naturally identified with the following sets.

$$
\mathbb{P}_n = \{l \mid l \text{ is a line in } \mathbb{K}^{n+1} \text{ through the origin}\}, \quad (1.1)
$$

$$
\mathbb{P}_n^\ast = \{H' \mid H' \text{ is a hyperplane in } \mathbb{K}^{n+1} \text{ through the origin}\}. \quad (1.2)
$$

Note that if we projectivize a hyperplane $H'$ in $\mathbb{K}^{n+1}$ we obtain a hyperplane $H$ in $\mathbb{P}_n$. Therefore in what follows we identify the dual projective space $\mathbb{P}_n^\ast$ with the set

$$
\{H \mid H \text{ is a hyperplane in } \mathbb{P}_n\}. \quad (1.3)
$$
Definition 1.1 Let $V$ be a projective variety in $\mathbb{P}_n$. We define the dual variety $V^*$ of $V$ by

$$V^* := \left\{ H \in \mathbb{P}_n^* \mid \exists x \in V_{\text{reg}} \cap H \text{ s.t. } T_xV \subseteq T_xH \right\} \subset \mathbb{P}_n^*.$$

(1.4)

When $V$ is smooth, $V^*$ is the set of hyperplanes tangent to $V$. As we see in the example below, even if $V$ is smooth, $V^*$ may be very singular in general.

Example 1.2 (i) Let $\iota_n: \mathbb{P}_1 \hookrightarrow \mathbb{P}_n$ be the Veronese embedding given by $[x : y] \mapsto [x^n : x^{n-1}y : \cdots : y^n]$ and set $V = \iota_n(\mathbb{P}_1) \subset \mathbb{P}_n$. Then the dual $V^* \subset \mathbb{P}_n^*$ is a hypersurface defined by the classical discriminant for polynomials of degree $n$.

(ii) For $n \geq m$, consider the Segre embedding $\iota_{n,m}: \mathbb{P}_n \times \mathbb{P}_m \hookrightarrow \mathbb{P}_{(n+1)\times(m+1)}$ given by $([x_0 : \cdots : x_n], [y_0 : \cdots : y_m]) \mapsto [\cdots : x_i y_j : \cdots]$. Set $W = \iota_{n,m}(\mathbb{P}_n \times \mathbb{P}_m) \subset \mathbb{P}_{(n+1)\times(m+1)-1}$. Then the dual variety $W^* \subset \mathbb{P}_{(n+1)\times(m+1)-1}^*$ has very complicated singularities and the dual defect $\delta^*(W)$ of $W$ (see (2.2) below) is $n - m$. Indeed, let $M_{(n+1),(m+1)}$ be the space of $(n + 1) \times (m + 1)$ matrices and identify the dual projective space $\mathbb{P}^*_{(n+1)\times(m+1)-1}$ with its projectivization $\mathbb{P}(M_{(n+1),(m+1)})$. Then the dual $W^* \subset \mathbb{P}^*_{(n+1)\times(m+1)-1}$ is explicitly written by

$$W^* = \mathbb{P}\{A \in M_{(n+1),(m+1)} \mid \text{rank}A \leq m\}.$$  

(1.5)

Therefore the dual $W^*$ admits a stratification defined by the ranks of matrices.

Many mathematicians were interested in the mysterious relations between projective varieties and their duals. Above all, they observed that the tangency of a hyperplane $H \in V^*$ with $V$ is related to the singularity of the dual $V^*$ at $H$. For example, consider the case of a plane curve $C \subset \mathbb{P}_2$ over $\mathbb{C}$. Then a tangent line $l$ at an inflection point of $C$ corresponds to a cusp of the dual curve $C^*$, and a bitangent (double tangent) line $l$ of $C$ corresponds to an ordinary double point of $C^*$. The most general results for complex plane curves were found in the 19th century by Klein, Plücker and Clebsch etc. (see for example, [34, Theorem 1.6] and [38, Chapter 7] etc.).

In the last two decades, this beautiful correspondence was extended to higher-dimensional complex projective varieties from the viewpoint of the geometry of hyperplane sections. In particular, after some important contributions by Viro [37] and Dimca [8] etc., Ernström proved the following remarkable result in 1994.
Theorem 1.3 [9, Corollary 3.9] Let $V \subset \mathbb{P}_n$ be a smooth projective variety over $\mathbb{C}$. Take a generic hyperplane $H$ in $\mathbb{P}_n$ such that $H \notin V^*$. Then for any hyperplane $L \in V^*$, we have
\[ \chi(V \cap L) - \chi(V \cap H) = (-1)^{n-1+\dim V^* - \dim V} \text{Eu}_{V^*}(L), \] (1.6)
where $\chi$ stands for the topological Euler characteristic and $\text{Eu}_{V^*} : V^* \rightarrow \mathbb{Z}$ is the Euler obstruction of $V^*$ (introduced by Kashiwara [17] and MacPherson [24] independently).

Recall that the Euler obstruction $\text{Eu}_{V^*}$ of $V^*$ is a $\mathbb{Z}$-valued function on $V^*$ which measures the singularity of $V^*$ at each point of $V^*$. For example, $\text{Eu}_{V^*}$ takes the value 1 on the regular part of $V^*$. Moreover, if we take a Whitney stratification $\bigcup_{\alpha \in A} V^*_{\alpha}$ of $V^*$ consisting of connected strata, then $\text{Eu}_{V^*}$ is constant on each stratum $V^*_{\alpha}$. The values of $\text{Eu}_{V^*}$ on a stratum $V^*_{\alpha}$ is determined by those on $V^*_{\beta}$'s satisfying the condition $V^*_{\alpha} \subset V^*_{\beta}$ (for more detail, see e.g. [18]).

Hence Ernström's result says that the jumping number of the topological Euler characteristics of hyperplane sections of $V$ at $L$ is expressed by $\text{Eu}_{V^*}(L)$, that is, the singularity of the dual variety $V^*$ at $L$.

The aim of this article is to introduce our results in the real algebraic case similar to this Ernström's one and to survey its theoretical background.

2 Main results

Consider a real projective space $X = \mathbb{R} \mathbb{P}_n$ of dimension $n$ and its dual $Y = \mathbb{R} \mathbb{P}_n^*$. Let $M \subset X$ be a smooth real projective variety and $M^* \subset Y$ its dual variety.

We fix a $\mu$-stratification $Y = \bigcup_{\alpha \in A} Y_{\alpha}$ of $Y = \mathbb{R} \mathbb{P}_n^*$ consisting of connected strata and adapted to $M^*$. Note that Trotman [35] proved that this $\mu$-condition is equivalent to famous Verdier's w-regularity condition.

Definition 2.1 We define a $\mathbb{Z}$-valued function $\varphi_M : Y \rightarrow \mathbb{Z}$ on $Y = \mathbb{R} \mathbb{P}_n^*$ by
\[ \varphi_M(H) = \chi(M \cap H) \quad (H \in Y). \] (2.1)

Since the function $\varphi_M$ is defined by the topological Euler characteristics of hyperplane sections $M \cap H$ of $M$, to obtain results similar to Ernström's formula (1.6) it suffices to describe the function $\varphi_M$ in terms of the singularities of $M^*$. We will show that the whole function $\varphi_M$ can be reconstructed
from one value \( \varphi_M(y) \) at a point \( y \in Y \setminus M^* \) and the singularities of \( M^* \).

First of all, for the above \( \mu \)-stratification \( Y = \bigsqcup_{\alpha \in A} Y_{\alpha} \) of \( Y = \mathbb{R}\mathbb{P}_n^* \) we can prove the following basic result.

**Proposition 2.2** The function \( \varphi_M \) is constant on each stratum \( Y_{\alpha} \).

We denote the value of \( \varphi_M \) on \( Y_{\alpha} \) by \( \varphi_{\alpha} \). Our main results are reconstruction theorems of \( \varphi_M \). Namely, we can determine all the values \( \varphi_{\alpha}'s \) of \( \varphi_M \) from only one value \( \varphi_M(y) \) at a point \( y \in Y \setminus M^* \) and the topology of \( M^* \).

To state the first theorem, we introduce two notations concerning dual varieties. Recall that the dual variety \( M^* \) is usually a hypersurface in \( Y = \mathbb{R}\mathbb{P}_n^* \).

**Definition 2.3** (i) We denote the dual defect of \( M \) by

\[
\delta^*(M) = (n - 1) - \dim M^*. \tag{2.2}
\]

(ii) For a conormal vector \( \vec{p} \in T_{M_{\text{re}}}^* Y \) at \( y \in M_{\text{reg}}^* \), consider the second fundamental form

\[
h_{M^*;\vec{p}}: T_y M^* \times T_y M^* \rightarrow \mathbb{R}, \tag{2.3}
\]

with respect to the canonical (Fubini-Study) metric of \( Y = \mathbb{R}\mathbb{P}_n^* \) and set

\[
J_{\vec{p}} := \# \{ \text{positive eigenvalues of } h_{M^*;\vec{p}} \} + \delta^*(M). \tag{2.4}
\]

Now, let us state our first main theorem which describes the values of \( \varphi_M \) on \( Y \setminus M_{\text{sing}}^* \).

**Theorem 2.4** ([26])

(i) Assume that \( \delta^*(M) > 0 \). Then on \( Y \setminus M^* \) the function \( \varphi_M \) is constant. Moreover for any \( y \in M_{\text{reg}}^* \) there exists an neighborhood \( U \) of \( y \) such that we have

\[
\varphi_M = d \cdot 1_Y + (-1)^{J_{\vec{p}}} 1_{M_{\text{reg}}^*}. \tag{2.5}
\]

on \( U \), where \( d \) is the value of \( \varphi_M \) on \( Y \setminus M^* \) and \( \vec{p} \in T_{M_{\text{reg}}^*} Y \) is a conormal vector at \( y \in M_{\text{reg}}^* \).
(ii) Assume that \( \delta^*(M) = 0 \), that is \( M^* \) is a hypersurface in \( Y = \mathbb{R}P^n \), and consider the following local situation. Let \( Y_{\alpha_1} \) and \( Y_{\alpha_2} \) be two strata in \( Y \setminus M^* \), \( Y_{\beta} \) an open stratum in \( M_{\text{reg}}^* \) such that \( Y_{\beta} \subset Y_{\alpha_i} \) for \( i = 1, 2 \) and \( \vec{p} \in T_{M_{\text{reg}}}^* Y \) a conormal vector at a point \( y \in Y_{\beta} \) pointing from \( Y_{\alpha_i} \) to \( Y_{\alpha_2} \) (see Figure 2.4.1 below). Then we have

\[
\begin{align*}
\varphi_{\alpha_2} - \varphi_{\alpha_1} &= (-1)^{J_{\overline{p}}} - (-1)^{J_{-\overline{p}}} \quad (= 0, \pm 2), \quad (2.6) \\
\varphi_{\beta} &= \begin{cases} 
\frac{1}{2}(\varphi_{\alpha_1} + \varphi_{\alpha_2}) & \text{if } \varphi_{\alpha_1} \neq \varphi_{\alpha_2}, \\
\varphi_{\alpha_1} + (-1)^{J_{\overline{p}}} & \text{if } \varphi_{\alpha_1} = \varphi_{\alpha_2},
\end{cases} \quad (2.7)
\end{align*}
\]

Figure 2.4.1

Remark 2.5  
(i) If we rewrite (2.5) by Euler characteristics, we obtain an equality analogous to Ernström's formula (1.6). Namely, we have

\[
\chi(M \cap L) - \chi(M \cap H) = (-1)^{J_{\overline{p}}} \quad (2.8)
\]

for any \( L \in M_{\text{reg}}^* \), where \( H \in Y \setminus M^* \) is a generic hyperplane in \( Y = \mathbb{R}P^n \).

(ii) In the case where \( M^* \) is a hypersurface, the complement of \( M^* \) is divided into several connected components in general. So when we cross the hypersurface \( M^* \), the value of the function \( \varphi_M \) may jump. Our formula (2.6) describes this jumping number in terms of the principal curvature \( J_{\overline{p}} \) of \( M_{\text{reg}}^* \).

Next, we state our second main theorem which reconstructs the values of \( \varphi_M \) on \( M_{\text{sing}}^* \).

Theorem 2.6 ([26]) Let \( k \geq \text{codim}_Y M^* \). Suppose that the values \( \varphi_{\alpha}'s \) on \( Y_{\alpha} \)'s satisfying \( \text{codim}_Y Y_{\alpha} \leq k \) are already determined. Then the value \( \varphi_{\beta} \) on \( Y_{\beta} \) satisfying \( \text{codim}_Y Y_{\beta} = k + 1 \) is given by

\[
\varphi_{\beta} = \sum_{\alpha : Y_{\alpha} \cap B \neq \emptyset} \varphi_{\alpha} \cdot \{\chi(Y_{\alpha} \cap B) - \chi(\partial Y_{\alpha} \cap B)\}. \quad (2.9)
\]
Here we set \( B = B(y, \varepsilon) \cap \{ \psi < 0 \} \) by taking a small enough open ball \( B(y, \varepsilon) \) centered at a point \( y \in Y_\beta \) and and a real-valued real analytic function \( \psi \) defined in a neighborhood of \( y \) satisfying \( \psi^{-1}(0) \supset Y_\beta \) and

\[
(y, \text{grad } \psi(y)) \in \overline{T_{Y_\beta}^r Y} \setminus \bigcup_{\alpha \neq \beta} \overline{T_{Y_\alpha}^r Y}
\]  

(2.10)

(see Figure 2.5.1 below).

By Theorem 2.6, we can recursively determine the values \( \varphi_\alpha \)'s of \( \varphi_M \) by induction on the codimensions of strata \( Y_\alpha \)'s. Note that this representation of the function \( \varphi_M \) is completely analogous to that of the Euler obstructions in [18].

**Example 2.7** Consider a smooth projective curve \( M \) defined by the homogeneous equation \( x^4 + x^3 z + z^4 - y^3 z = 0 \) in \( \mathbb{RP}_2 \) (see Figure 2.6.1 below).

![Figure 2.6.1](image)

Then the dual curve \( M^* \subset \mathbb{RP}_2^* \) has a shape as in Figure 2.6.2 below. More precisely, as a \( \mu \)-stratification of \( Y = \mathbb{RP}_2^* \) adapted to \( M^* \), we can take \( Y = \bigsqcup_{i=0}^{11} Y_i \) in Figure 2.6.2. Since the last strata \( Y_{11} \) is contained in the line at infinity \((\simeq \mathbb{RP}_1)\) of \( \mathbb{RP}_2 \) it does not appear in the figure.
Now let us apply our two main theorems to this case. Denote by $\varphi_i$ the value of the function $\varphi_M$ on $Y_i$. Then we can easily see that $\varphi_0 = 0$. Starting from this value $\varphi_0 = 0$, we can recursively determine all the values $\varphi_i$'s of $\varphi_M$ as follows.

For example, by Theorem 2.4 the values $\varphi_1$ and $\varphi_3$ on $Y_1$ and $Y_3$ respectively can be calculated in the following way.

$$\varphi_1 = \varphi_0 + (-1)^2 - (-1)^1 = 2,$$

$$\varphi_3 = \frac{1}{2}(\varphi_0 + \varphi_1) = 1.$$  \hfill (2.11) \hfill (2.12)

Moreover, by Theorem 2.6 the value $\varphi_{10}$ on $Y_{10}$ is determined by $\varphi_1$, $\varphi_2$, $\varphi_5$ and $\varphi_6$ as follows.

$$\varphi_{10} = \varphi_1 \cdot 0 + \varphi_6 \cdot 1 + \varphi_2 \cdot (-1) + \varphi_6 \cdot 1 + \varphi_1 \cdot 0 = 2.$$  \hfill (2.13)
In this case, we can easily check these results simply by counting the intersection numbers of $M$ and lines in $\mathbb{RP}_2$. Namely, our results are the generalization of this very simple example to higher dimensional cases.

3 Theoretical background

Since the function $\varphi_M$ in our main theorems is constant on each stratum of $Y$, we consider a class of such functions to study $\varphi_M$, which are called constructible functions.

**Definition 3.1** Let $X$ be a real analytic manifold. We say that a function $\varphi: X \to \mathbb{Z}$ is constructible if there exists a locally finite family $\{X_i\}$ of compact subanalytic subsets $X_i$ of $X$ such that $\varphi$ is expressed by

$$\varphi = \sum_i c_i 1_{X_i} \quad (c_i \in \mathbb{Z}).$$

(3.1)

We denote the abelian group of constructible functions on $X$ by $CF(X)$.

We define the operations of constructible functions in the following way.

**Definition 3.2** ([21] and [37]) Let $f: Y \to X$ be a morphism of real analytic manifolds.

(i) (The inverse image) For $\varphi \in CF(X)$, we define a function $f^* \varphi \in CF(Y)$ by

$$f^* \varphi(y) := \varphi(f(y)).$$

(3.2)

(ii) (The integral) Let $\varphi = \sum_i c_i 1_{X_i} \in CF(X)$ such that $\text{supp}(\varphi)$ is compact. Then we define a topological (Euler) integral $\int_X \varphi \in \mathbb{Z}$ of $\varphi$ by

$$\int_X \varphi := \sum_i c_i \cdot \chi(X_i).$$

(3.3)

(iii) (The direct image) Let $\psi \in CF(Y)$ such that $f|_{\text{supp}(\psi)}: \text{supp}(\psi) \to X$ is proper. Then we define a function $\int_f \psi \in CF(X)$ by

$$(\int_f \psi)(x) := \int_Y (\psi \cdot 1_{f^{-1}(x)}).$$

(3.4)
From now on, we shall use various notions concerning derived categories of constructible sheaves. For the detail of these notions, see [21] etc. We denote by $D^b(X)$ the derived category of bounded complexes of $C_X$-modules on $X$. Its full subcategory consisting of complexes whose cohomology sheaves are $\mathbb{R}$-constructible is denoted by $D_{R-c}^b(X)$.

Recall also that the Grothendieck group $K_{R-c}(X)$ of $D_{R-c}^b(X)$ is a quotient group of the free abelian group generated by objects of $D_{R-c}^b(X)$ by the subgroup generated by

$$[F] - [F'] - [F''] \quad (F' \rightarrow F \rightarrow F'' \xrightarrow{+1} \text{is a distinguished triangle}).$$

Then the natural morphism

$$\chi: K_{R-c}(X) \rightarrow CF(X) \quad \text{(3.6)}$$

defined by $\chi([F])(x) = \sum_{j \in \mathbb{Z}} (-1)^j \dim H^j(F)_x \ (x \in X)$ is an isomorphism.

Moreover, by the isomorphism $\chi: K_{R-c}(X) \sim CF(X)$ the operations of constructible functions that we introduced in Definition 3.2 correspond to those for $\mathbb{R}$-constructible sheaves. For example, let $f: Y \rightarrow X$ be a morphism of real analytic manifolds and for $\psi \in CF(Y)$ take an object $G \in D_{R-c}^b(Y)$ such that $\psi = \chi(G)$. Then we have $\chi(Rf_*G) = \int f \psi$ in $CF(X)$. In the same way, we can slightly generalize the notion of topological integrals of constructible functions as follows.

**Definition 3.3** ([26]) Let $U$ be a relatively compact subanalytic open subset of $X$ and $\varphi \in CF(X)$. Take an object $F \in D_{R-c}^b(X)$ such that $\varphi = \chi(F)$ and set

$$\int_U \varphi := \chi(R\Gamma(U; F)). \quad \text{(3.7)}$$

We can easily check that the definition above does not depend on the choice of $F$ such that $\varphi = \chi(F)$. Note that we do not have to assume that the support of $\varphi$ is compact in $U$ as in the usual definition (Definition 3.2 (ii)). Using this slight modification of the notion of topological integrals, we can express the R.H.S of (2.9) simply by $\int_B \varphi_M$. In fact, we used this fact in the proof of Theorem 2.6.

Now, let $X$ be a real analytic manifold and denote by $\mathcal{L}_X$ the sheaf of conic ($\mathbb{R}_{>0}$-invariant) subanalytic Lagrangian cycles in the cotangent bundle $T^*X$ of $X$. Its global section $H^0(T^*X; \mathcal{L}_X)$ is the abelian group of conic
subanalytic Lagrangian cycles in $T^*X$. In 1985, Kashiwara [19] constructed a group homomorphism $CC: K_{R-c}(X) \rightarrow H^0(T^*X;\mathcal{L}_X)$ and associated with each $[F] \in K_{R-c}(X)$ a Lagrangian cycle $CC([F])$ in $T^*X$. This Lagrangian cycle $CC([F])$ is called the characteristic cycle of $[F] \in K_{R-c}(X)$. The following very important theorem was proved also in [19] (see [21] for the detail).

**Theorem 3.4** [21, Theorem 9.7.11] There exists a commutative diagram

\[ \begin{array}{ccc}
K_{R-c}(X) & \overset{CC}{\sim} & CF(X) \\
\sim & \sim & \sim \\
H^0(T^*X;\mathcal{L}_X) & \rightarrow & CF(X)
\end{array} \]  

in which all arrows are isomorphisms.

By this theorem, we can reduce the problem of constructible functions (sheaves) to that of Lagrangian cycles.

### 4 Outline of the proof of main theorems

In this section, we give an outline of the proof of our main theorems. Let $X = \mathbb{R}P_n$ and $Y = \mathbb{R}P_n^*$ as before. Consider the incidence submanifold $S = \{(x, H) \in X \times Y | x \in H\}$ of $X \times Y$ and the diagram

\[ \begin{array}{ccc}
X \times Y & \overset{f}{\rightarrow} & Y \\
\downarrow p_1 & & \downarrow p_2 \\
X & \overset{g}{\rightarrow} & Y
\end{array} \]

where $p_1$ and $p_2$ are natural projections and $f$ and $g$ are restrictions of $p_1$ and $p_2$ to $X$ and $Y$ respectively.

**Definition 4.1** Let $\varphi \in CF(X)$. We define the topological Radon transform $\mathcal{R}_S(\varphi) \in CF(Y)$ of $\varphi$ by

\[ \mathcal{R}_S(\varphi) := \int f^*\varphi. \]

In particular, for a real analytic submanifold $M$ of $X = \mathbb{R}P_n$ and a hyperplane $H$ in $X = \mathbb{R}P_n$ ($\iff H \in Y = \mathbb{R}P_n^*$) we have

\[ \mathcal{R}_S(1_M)(H) = \chi(M \cap H) \quad \left(= \varphi_M(H) \right). \]
Therefore for the study of the function $\varphi_M \in CF(Y)$ it suffices to study the topological Radon transform $R_S(1_M)$. Using the isomorphisms in Theorem 3.4, instead of the topological Radon transform $R_S: CF(X) \rightarrow CF(Y)$ itself, we studied the corresponding operation for Lagrangian cycles (characteristic cycles). Then we found an isomorphism

$$\Psi: H^0(\hat{T}^*X; L_X) \rightarrow H^0(\hat{T}^*Y; L_Y),$$

where we set $\hat{T}^*X = T^*X \setminus T^*_X X$ and $\hat{T}^*Y = T^*Y \setminus T^*_Y Y$ (the zero-sections are removed). Moreover this operation $\Psi$ is (up to some sign $\varepsilon = \pm 1$) the isomorphism of Lagrangian cycles induced by the canonical diffeomorphism $\Phi: \hat{T}^*X \rightarrow \hat{T}^*Y$ which coincides with the classical Legendre transform in the standard affine charts of $X = \mathbb{RP}_n$ and $Y = \mathbb{RP}_n$. Since the characteristic cycle $CC(1_M)$ of $1_M \in CF(X)$ is the conormal cycle $[\bar{T}_N^* X]$ in $T^* X$, the characteristic cycle $CC(R_S(1_M))$ of the topological Radon transform $R_S(1_M) = \varphi_M$ is $\varepsilon T^*_M X$. Set $\pi_Y: T^*Y \rightarrow Y$ and $N = (\pi_Y \circ \Phi)(\bar{T}_M^* X) \subset Y$. Then we can easily prove that $N$ coincides with the dual variety $M^*$ of $M$, which is a closed subanalytic subset of $Y = \mathbb{RP}_n$ (in classical terminology we call it a caustic or Legendre singularity). Moreover it turns out that the closure $\bar{T}_N^* Y$ of the conormal bundle $T_N^* Y$ in $T^* Y$ is nothing but $\Phi(\bar{T}_M^* X)$ (see [16] for a similar argument). Then by using this very nice property of the characteristic cycle $CC(\varphi_M)$ we can reconstruct the function $\varphi_M$ from the geometry of the dual variety $M^* = N$. Theorem 2.6 was proved in this way.

To prove Theorem 2.4, we have to determine the sign $\varepsilon = \pm 1$, which is the most difficult part of our study. We could determine it by employing the theory of pure sheaves in [21]. More precisely, we expressed the Maslov indices of the Lagrangian submanifolds $\bar{T}_M^* X$ and $\bar{T}_N^* Y$ by the principal curvatures of $M$ and $N_{\text{reg}}$ respectively with the help of results in [11].

**Remark 4.2** By the same argument as above, we can give a more transparent proof to the main results of Ernström [9] in the complex case.

## 5 Grassmann cases and class formulas

### 5.1 $k$-dual varieties

We shall generalize the situation considered in the previous sections to Grassmann cases and obtain similar results. Let $0 \leq k \leq n - 1$ be an integer.

Recall that the Grassmann manifold consisting of $k$-dimensional planes in $\mathbb{P}_n$ is defined by

$$G_{n,k} = \{ L' \mid L' \text{ is a } (k + 1)\text{-dimensional linear subspace in } \mathbb{K}^{n+1} \},$$

$$\{ L \mid L \text{ is a } k\text{-dimensional linear subspace in } \mathbb{P}_n \}.$$
Note that $G_{n,0} = \mathbb{P}_n$ and $G_{n,n-1} = \mathbb{P}_n^*$. Then the notion of dual varieties is generalized to Grassmann cases as follows.

**Definition 5.1** Let $V \subset \mathbb{P}_n$ be a projective variety. We define the $k$-dual variety $V^{(k)}$ of $V$ by

$$V^{(k)} := \{ L \in G_{n,k} \mid \exists x \in V_{\text{reg}} \cap L \text{ s.t. } V \not\subset L \text{ at } x \} \subset G_{n,k}.$$  

(5.3)

If $k = n - 1$ the $k$-dual $V^{(k)} \subset G_{n,k} \simeq \mathbb{P}_n^*$ is nothing but the classical dual variety of $V$. In [12], Gelfand-Kapranov-Zelevinsky called $V^{(k)}$ the associated variety of $V$ and showed that $V^{(n - \dim V - 1)}$ is a hypersurface.

### 5.2 Topological class formulas

From now on, we always assume that the ground field $\mathbb{K}$ is $\mathbb{C}$. Let $V \subset \mathbb{P}_n$ be a projective variety over $\mathbb{C}$ and $0 \leq k \leq n - 1$ an integer. Assume that $V^{(k)}$ is a hypersurface in $G_{n,k}$.

**Definition 5.2** [12, Proposition 2.1 of Chapter 3] Consider the Plücker embedding:

$$V^{(k)} \subset G_{n,k} \subset \mathbb{P}_{\binom{n+1}{k+1} - 1}.$$  

(5.4)

We call the degree of the defining polynomial of $V^{(k)}$ in $\mathbb{P}_{\binom{n+1}{k+1} - 1}$ the degree of $V^{(k)}$ and denote it by $\deg V^{(k)}$.

In [27], we proved the following topological class formula (i.e. a formula which expresses the degrees of dual varieties) for $k$-dual varieties by using Ernström's result [9] and some elementary formulas on constructible functions.

**Theorem 5.3** ([27]) In the situation as above, for generic linear subspaces $L_1 \simeq \mathbb{P}_{k-1}$, $L_2 \simeq \mathbb{P}_k$ and $L_3 \simeq \mathbb{P}_{k+1}$ of $\mathbb{P}_n$ we have

$$\deg V^{(k)} = (-1)^{(n-k)+\dim V+1} \left\{ \int_{L_1} \text{Ev}_V - 2 \int_{L_2} \text{Ev}_V + \int_{L_3} \text{Ev}_V \right\}.$$  

(5.5)

**Corollary 5.4** Let $L \simeq \mathbb{P}_{k+1}$ be a generic $(k+1)$-dimensional linear subspace of $\mathbb{P}_n$ and consider the usual dual variety $(V \cap L)^* \subset \mathbb{P}_{k+1}^*$ of $V \cap L \subset L \simeq \mathbb{P}_{k+1}$. Then we have

$$\deg V^{(k)} = \deg (V \cap L)^*.$$  

(5.6)
The formula in Theorem 5.3 expresses the algebraic invariant $\deg V^{(k)}$ of $V^{(k)}$ by the topological data of $V$. In the case where $k = n - 1$, we thus reobtain the topological class formulas obtained by Ernström [10], Parusinski and Kleiman [22] etc. See [34, Section 10.1] for an excellent review on this subject. In a forthcoming paper [28], from these topological class formulas we derive various more computable class formulas which extend the previous results obtained by Teissier and Kleiman [23] etc.

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