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<td>Author(s)</td>
<td>Izawa, Takeshi</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 1501: 124-131</td>
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<tr>
<td>Issue Date</td>
<td>2006-07</td>
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<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/58425">http://hdl.handle.net/2433/58425</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
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Kyoto University
RESIDUE OF CODIMENSION 1 SINGULAR HOLOMORPHIC DISTRIBUTIONS

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The aim of this note is to describe the residue formula for singular holomorphic distribution in terms of the conormal sheaf \( \mathcal{G} \) in codimension 1 case.

We also prove the Baum-Bott type residue formula for singular distributions. If we define the tangent sheaf of the distribution \( \mathcal{F} \) by taking the annihilator of \( \mathcal{G} \) by the dual coupling, we will show that the residue formula for \( \mathcal{G} \) deduce the Baum-Bott type residue formula for the top Chern class of the normal sheaf \( N_\mathcal{F} \).

If we assume the Frobenius integrability condition for \( \mathcal{G} \), we have the Baum-Bott residue formula

\[
\int_X \varphi(N_\mathcal{F}) = \text{Res}_\varphi(N_\mathcal{F}, S(\mathcal{F}))
\]

for \( n \)-th symmetric polynomial \( \varphi \). In this case, the Baum-Bott residue formula for \( \varphi = c_n \) is equivalent to the formula we will prove, which means that the Bott vanishing theorem based on the involutivity of \( \mathcal{F} \) is not necessary for the top Chern class \( c_n(N_\mathcal{F}) \).

As an application of our results, we will give a residue formula for the non-transversality of a holomorphic map \( F : X \to Y \) to a non-singular distribution on \( Y \).

2. SINGULAR HOLOMORPHIC DISTRIBUTION

2.1. Singular holomorphic distribution. Let \( X \) be a complex manifold. We define a singular holomorphic distribution \( \mathcal{F} \) on \( X \) to be a coherent subsheaf of the tangent sheaf \( \Theta_X \). We call \( \mathcal{F} \) the tangent sheaf of the distribution. We say \( \mathcal{F} \) is dimension \( p \) if a generic stalk of \( \mathcal{F} \) is rank \( p \) free \( \mathcal{O}_X \)-module. We also define the normal sheaf \( N_\mathcal{F} \) of \( \mathcal{F} \) by the exact sequence

\[
0 \to \mathcal{F} \to \Theta_X \to N_\mathcal{F} \to 0.
\]

The singular set \( S(\mathcal{F}) \) of \( \mathcal{F} \) is defined by \( S(\mathcal{F}) = \{ p \in X | N_{\mathcal{F},p} \text{ is not } \mathcal{O}_p \text{-free} \} \).

We can also give a definition of a singular holomorphic distribution \( \mathcal{G} \) on \( X \) to be a coherent subsheaf of the cotangent sheaf \( \Omega_X \). We call \( \mathcal{G} \) the conormal sheaf of the distribution. We also say \( \mathcal{G} \) is codimension \( q \) if the generic rank is \( q \). We also define the cotangent sheaf \( \Omega_\mathcal{G} \) of \( \mathcal{G} \) by the exact sequence

\[
0 \to \mathcal{G} \to \Omega_X \to \Omega_\mathcal{G} \to 0.
\]

The singular set \( S(\mathcal{F}) \) of \( \mathcal{F} \) is also defined by \( S(\mathcal{G}) = \{ p \in X | \Omega_{\mathcal{G},p} \text{ is not } \mathcal{O}_p \text{-free} \} \).
2.2. Codimension 1 case. We give more simple descriptions for codimension 1 singular distributions. A codimension 1 locally free singular holomorphic distribution is given by a collection of 1-forms \( \omega = (\omega_a, U_a) \) for an open covering \( \{U_a\} \) of \( X \) which has the transition relations \( \omega_{ij} = g_{ij\alpha} \omega_{\alpha} \) on the intersection \( U_a \cap U_\beta \) with \( g_{ij\alpha} \in \mathcal{O}^*(U_a \cap U_\beta) \). Then the cocycle \( (g_{ij\alpha}) \) defines a line bundle \( G \). Generically at \( p \), the covector \( \omega_p \) gives an embedding of the fiber \( G_p \) into \( T^*_pX \) by \( f_p \in G_p \mapsto f_p \omega_p \in T^*_pX \). Thus \( G \) is regarded as a subbundle of \( T^*X \) without on the zero loci of \( \omega \). Since the map of germs of sections \( (f)_p \in \mathcal{O}_X(G)_p \mapsto (f \omega)_p \in \Omega_{X,p} \) are injective for all \( p \in X \), the sheaf \( G = \mathcal{O}_X(G) \) gives the subsheaf of \( \Omega_X \) in the above sense in 1.2. Since the quotient sheaf \( \Omega_X/G \) is not \( \mathcal{O} \)-free on the zero loci of \( \omega \) on which we can not define the quotient bundle \( T^*X/G \), we see the singular set of \( G \) is \( S(G) = \{ p \mid \omega(p) = 0 \} \).

3. Residue of Codimension 1 Distribution

3.1. Localization of the top Chern class. We determine the dual homology class of \( c_n(\Omega_X \otimes \mathcal{G}^\vee) \). Our main tool is the Čech-de Rham techniques. For generalities on the integration and the Chern-Weil theory on the Čech-de Rham cohomology, see [S3] or [IS]. We set for an analytic set \( S, U_0 = X \setminus S, U_1 \) is a regular neighbourhood of \( S \), and \( U_{01} = U_0 \cap U_1 \). For a covering \( U = \{U_0, U_1\} \) of \( X \), the Čech-de Rham cohomology group \( H^{2n}(\mathcal{A}^*(U)) \) is represented by the group of cocycles of the type \( (\sigma_0, \sigma_1, \sigma_{01}) \) for \( \sigma_0 \in Z^{2n}(U_0), \sigma_1 \in Z^{2n}(U_1) \), and \( \sigma_{01} \in A^{2n-1}(U_{01}) \) with \( d\sigma_{01} = \sigma_1 - \sigma_0 \). We note that the Čech-de Rham cohomology can be regarded as the hypercohomology of the de Rham complex \( \{\mathcal{A}^*, d\} \).

By usual spectral sequence arguments for double complexes, we see that the Čech-de Rham cohomology group is canonically isomorphic to the de Rham cohomology group. If we take the subgroup \( H^{2n}(\mathcal{A}^*(U, U_0)) \) of cocycles of the form \( (0, \sigma_1, \sigma_{01}) \), then this is also isomorphic to the relative cohomology group \( H^{2n}(X, X \setminus S; \mathcal{C}) \).

In the above settings, the top Chern class \( c_n(E) \) of a vector bundle \( E \) of rank \( n \) is given by the cocycle in \( H^{2n}(\mathcal{A}^*(U)) \) as follows. For \( i = 0, 1 \), let \( \nabla_i \) be a connection for \( E \) on \( U_i \) and \( c_n(\nabla_i) \) the \( n \)-th Chern form of \( \nabla_i \). We also write by \( c_n(\nabla_0, \nabla_1) \) the transgression form of \( c_n(\nabla_i) \)'s on \( U_{01} \). Then \( c_n(E) \) is represented by
\[
(c_n(\nabla_1), c_n(\nabla_1), c_n(\nabla_0, \nabla_1)).
\]
If \( E \) has a global section \( s \) with zero loci \( S \), then we take \( \nabla_0 \) as the \( s \)-trivial connection such that we have \( c_n(\nabla_0) = 0 \). Thus we can define the localized Chern class at \( p \) in \( H^{2n}(X, X \setminus S; \mathcal{C}) \) by a Čech-de Rham cocycle \( (0, c_n(\nabla_1), c_n(\nabla_0, \nabla_1)) \).

The integration of \( c_n(E) = (0, c_n(\nabla_1), c_n(\nabla_0, \nabla_1)) \) is defined by
\[
\int_X c_n(E) = \int_R c_n(\nabla_1) - \int_{\partial R} c_n(\nabla_0, \nabla_1)
\]
for a tubular neighbourhood \( R \subset U_1 \) of \( S \).

3.2. Residue of Codimension 1 distributions. Now we apply the above arguments to our situations. Let \( G \) be a codimension 1 locally free distribution with the zero loci \( S(G) \) and we suppose that \( S(G) \) has connected components \( S_j \). We set \( U_0 = X \setminus S(G) \) and \( U_j \) is a regular neighbourhood of \( S_j \). We consider the localized class of \( c_n(\Omega_X \otimes \mathcal{G}^\vee) \) in the Čech-de Rham cohomology group for the covering \( U = \{U_0, U_1, \ldots, U_j\} \). Since the collection \( \omega \) of 1-forms \( \omega_\alpha \) defines the global section of \( \Omega_X \otimes \mathcal{G}^\vee \), we can take \( \nabla_0 \) as the \( \omega \)-trivial connection such that \( c_n(\nabla_0) = 0 \).
as we discussed above. For all \( j = 1, \ldots, k \), we can also take \( \nabla_j \) as an arbitrary connection on \( U_j \). So we have

\[
c_n(\Omega_X \otimes \mathcal{G}^\vee) = (0, \{c_n(\nabla_j)\}_{j=1,\ldots,k}, \{c_n(\nabla_0, \nabla_j)\}_{j=1,\ldots,k}) \in H^{2n}(X, X \setminus S(\mathcal{G}); \mathbb{C}).
\]

We denote by \( R_j \) a tublar neighbourhood of \( S_j \) in \( U_j \). We give the following definition of residue.

**Definition 3.1.** The residue of \( \mathcal{G} \) at \( S_j \) is defined by

\[
\text{Res}(\mathcal{G}, S_j) = \int_{R_j} c_n(\nabla_j) - \int_{\partial R_j} c_n(\nabla_0, \nabla_j).
\]

We can describe the residue into precise form in isolated singular cases. Here we refer the result in [S3] of Theorem 5.5.

**Theorem 3.2.** Let \( s \) be a regular section of \( E \) with isolated zero \( \{p\} \) and \( s \) is locally given by \((f_1, \ldots, f_n)\) near \( p \). Then we have

\[
\text{Res}(\mathcal{G}, p) = \text{Res}_p[df_1 \wedge \cdots \wedge df_n]_{f_1 \cdots f_n}
\]

where \( \text{Res}_p[df_1 \wedge \cdots \wedge df_n] \) is the Grothendick residue of \((f_1, \ldots, f_n)\).

The dual correspondence in the Alexander duality

\[
AL : H^{2n}(X, X \setminus S(\mathcal{G}); \mathbb{C}) \cong \bigoplus_j H_0(S_j; \mathbb{C})
\]

is given by

\[
AL(c_n(\Omega_X \otimes \mathcal{G}^\vee)) = \sum_j \text{Res}(\mathcal{G}, S_j).
\]

Now we have the residue formula for isolated singular cases as,

**Theorem 3.3** (The residue formula for isolated singularities). Let \( \omega \) be a codimension 1 singular holomorphic distribution with the cotangent sheaf \( \mathcal{G} \) and \((f_1^{(j)}, \ldots, f_n^{(j)})\) a local expression of \( \omega \in H^0(X, \Omega_X \otimes \mathcal{G}^\vee) \) near \( p_j \).

\[
\int_X c_n(\Omega_X \otimes \mathcal{G}^\vee) = \sum_{j=1}^k \text{Res}_{p_j}[df_1^{(j)} \wedge \cdots \wedge df_n^{(j)}]_{f_1^{(j)} \cdots f_n^{(j)}}.
\]

4. **Baum-Bott type residue formula**

4.1. **Koszul resolution.** First let us remember the definition of the Koszul complex. (See [FG], Chapter 4 or [GH], Chapter 5.) Let \( E \) be a locally free \( \mathcal{O} \)-module of rank \( n \) and \( d : E \to \mathcal{O} \) an \( \mathcal{O} \)-homomorphism. Then the Koszul complex of sheaves

\[
0 \to \wedge^n E \to \wedge^{n-1} E \to \cdots \to \wedge^1 E \to \mathcal{O} \to 0
\]

is defined by the boundary operator

\[
d_p(\epsilon_1 \wedge \cdots \wedge \epsilon_p) = \sum_{i=1}^p (-1)^{i-1}d(\epsilon_i) \epsilon_1 \wedge \cdots \wedge \hat{\epsilon_i} \wedge \cdots \wedge \epsilon_p.
\]

This complex is exact except for the last term. If the image \( \mathcal{I}_d \) of \( d \) is regular ideal, the complex

\[
0 \to \wedge^n E \to \wedge^{n-1} E \to \cdots \to \wedge^1 E \to \mathcal{O} \to \mathcal{O}/\mathcal{I}_d \to 0
\]
is exact. We call this exact sequence the Koszul resolution of $O/I_d$.

Now let us consider our case. As observed in 2.1, $\omega$ can be regarded as a homomorphism $\omega : G \to \Omega_X$ such that it defines a global section

$$\omega \in H^0(X, \mathcal{H}om_{O}(G, \Omega_X)) \simeq H^0(X, \Omega_X \otimes \Theta^{\vee}).$$

Locally on $U_\alpha$, $\omega$ is given by $\omega_\alpha \otimes s^\vee_\alpha = \sum f_i dz_i \otimes s^\vee_{\alpha i}$ for some local coordinates of $X$ and a local frame $s^\vee_\alpha$ for $\Theta^{\vee}$. In the other words, $\omega$ acts on $(\Omega_X \otimes \Theta^{\vee})^\vee \simeq \Theta \otimes G$ as a contraction operator $\omega : \Theta \otimes G \to O$. We denote by $I_\omega$ the ideal sheaf defined by $\text{Im}(\omega : \Theta \otimes G \to O)$. We assume that $S(G) = \{ p \in X | \omega_p = 0 \}$ consists only of isolated points such that the local coefficients $(f_1, \cdots, f_n)$ of $\omega$ is regular sequence on $S(G)$. Then the complex of sheaves

$$0 \to \wedge^n(\Theta \otimes G) \to \wedge^{n-1}(\Theta \otimes G) \to \cdots \to \wedge^1(\Theta \otimes G) \to O \to O/I_\omega \to 0$$

is exact with the boundary operator

$$d_p(e_1 \wedge \cdots \wedge e_p) = \sum_{i=1}^p (-1)^{i-1} f_i e_1 \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge e_p$$

where we set $e_i = \frac{\partial}{\partial x_i} \otimes s$. Therefore this gives the Koszul resolution of $O/I_\omega$.

By using this projective resolution, we can defines the Chern character of the coherent sheaf $O/I_\omega$ by

**Proposition 4.1.**

$$ch(O/I_\omega) = c_n(\Omega_X \otimes G^{\vee}).$$

**Proof.** We use [H] of Theorem 10.1.1 and we have

$$ch(O/I_\omega) = ch(\sum_{i=0}^n (-1)^i \wedge^i (\Theta \otimes G))$$

$$= td^{-1}(\Omega_X \otimes \Theta^{\vee}) c_n(\Omega_X \otimes \Theta^{\vee})$$

$$= c_n(\Omega_X \otimes \Theta^{\vee}).$$

**4.2. Baum-Bott type residue formula.** Now we translate the above results in terms of differential system in the tangent sheaf $\Theta_X$: Let $F = \{ v \in \Theta_X | < v, \omega > = 0 \}$ be the annihilator of $G$. Then $F$ defines a $n - 1$ dimensional (possibly) singular distribution. Since $G$ is locally free, by applying $\otimes G$ to the exact sequence

(1) $$0 \to F \to \Theta_X \to N_F \to 0,$$

the following sequence

$$0 \to F \otimes G \to \Theta_X \otimes G \to N_F \otimes G \to 0.$$
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We use $\mathcal{F} = \mathcal{H}om_{\mathcal{O}}(\Omega_{\mathcal{G}}, \mathcal{O})$ and $\Theta_{\mathcal{X}} = \mathcal{H}om_{\mathcal{O}}(\Omega_{\mathcal{X}}, \mathcal{O})$ in the above. Thus we obtain

(3) \[ 0 \rightarrow N_{\mathcal{F}} \rightarrow \mathcal{G}^\vee \rightarrow \mathcal{E}xt^{1}_{\mathcal{O}}(\Omega_{\mathcal{G}}, \mathcal{O}) \rightarrow 0. \]

By taking the Chern characters of (3), we have

(4) \[ ch(N_{\mathcal{F}}) = ch(\mathcal{G}^\vee) - ch(\mathcal{E}xt^{1}_{\mathcal{O}}(\Omega_{\mathcal{G}}, \mathcal{O})). \]

By tensoring $\mathcal{G}$ for each term of (3), we also have the exact sequence

\[ 0 \rightarrow \mathcal{I}_{\omega} \rightarrow \mathcal{O} \rightarrow \mathcal{E}xt^{1}_{\mathcal{O}}(\Omega_{\mathcal{G}}, \mathcal{O}) \otimes \mathcal{G} \rightarrow 0 \]

and which gives the isomorphism $\mathcal{O}/\mathcal{I}_{\omega} \simeq \mathcal{E}xt^{1}_{\mathcal{O}}(\Omega_{\mathcal{G}}, \mathcal{O}) \otimes \mathcal{G}$. Thus the Chern characters of those sheaves satisfy

(5) \[ ch(\mathcal{E}xt^{1}_{\mathcal{O}}(\Omega_{\mathcal{G}}, \mathcal{O})) = ch(\mathcal{O}/\mathcal{I}_{\omega})ch(\mathcal{G}^\vee). \]

Therefore by combining the two equalities (4) and (5) for the Chern characters, we obtain

**Proposition 4.2.**

\[ ch(N_{\mathcal{F}}) = (1 - ch(\mathcal{O}/\mathcal{I}_{\omega}))ch(\mathcal{G}^\vee) = (1 - c_{n}(\Omega_{\mathcal{X}} \otimes \mathcal{G}^\vee))ch(\mathcal{G}^\vee). \]

Now we find the top Chern class of $N_{\mathcal{F}}$.

**Proposition 4.3.**

\[ c_{n}(N_{\mathcal{F}}) = (-1)^{n}(n-1)!c_{n}(\Omega_{\mathcal{X}} \otimes \mathcal{G}^\vee). \]

**Proof.** Let $\{\xi_{i}\}$ be the formal Chern roots of $c(N_{\mathcal{F}})$ and $ch_{i}$ the terms of $i$-th degree in $ch$. Then from proposition 3.1, we have

\[ ch_{i}(N_{\mathcal{F}}) = \frac{1}{i!}c_{1}(\mathcal{G}^\vee)^{i} \]

for $i \leq n-1$ and

\[ ch_{n}(N_{\mathcal{F}}) = \frac{1}{n!}c_{1}(\mathcal{G}^\vee)^{n} - c_{n}(\Omega_{\mathcal{X}} \otimes \mathcal{G}^\vee). \]

$ch_{1}(N_{\mathcal{F}}) = c_{1}(\mathcal{G}^\vee)$ is obvious. we also see that

\[ \frac{1}{2!}c_{1}(\mathcal{G}^\vee)^{2} = ch_{2}(N_{\mathcal{F}}) = \frac{1}{2!}(\xi_{1}^{2} + \cdots + \xi_{n}^{2}) \]

\[ = \frac{1}{2!}\{((1 \cdots \xi_{n})^{2} - 2 \sum \xi_{i}\xi_{j}\} \]

\[ = \frac{1}{2!}c_{1}(\mathcal{G}^\vee)^{2} - c_{2}(N_{\mathcal{F}}), \]

which implies $c_{2}(N_{\mathcal{F}}) = 0$. We continue the same computations for fundamental symmetric polynomials, we have

\[ c_{2}(N_{\mathcal{F}}) = \cdots = c_{n-1}(N_{\mathcal{F}}) = 0. \]
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Thus for \( n \)-th term, we have

\[
\frac{1}{n!}c_1(\mathcal{G}^\vee)^n - c_n(\Omega_X \otimes \mathcal{G}^\vee) = ch_n(N_F)
\]

\[
= \frac{1}{n!}(\xi_1^n + \cdots + \xi_n^n)
\]

\[
= \frac{1}{n!}\{ (\xi_1 + \cdots + \xi_n)^n - (-1)^n n \xi_1 \cdots \xi_n \}
\]

\[
= \frac{1}{n!}c_1(\mathcal{G}^\vee)^n - \frac{(-1)^n}{(n-1)!}c_n(N_F)
\]

from which the result follows.

We combine the results in (2.3), we can derive the formula for the normal sheaf \( N_F \), which is the Baum-Bott type residue formula.

**Theorem 4.4** (Baum-Bott type residue formula). Let \( \omega \) be a codimension 1 distribution with conormal sheaf \( \mathcal{G} \), and \( F \) the annihilator of \( \mathcal{G} \). We suppose that \( S(\mathcal{G}) = \{p_1, \cdots, p_k\} \) and we write \( \omega = \sum f_i^{(j)}(dx_i \otimes s^\vee) \) near \( p_j \). Then we have

\[
\int_X c_n(N_F) = (-1)^n(n-1)! \sum_j \text{Res} \left[ \frac{df_1^{(j)} \wedge \cdots \wedge df_n^{(j)}}{f_1^{(j)} \cdots f_n^{(j)}} \right].
\]

**proof.** This is simply given by

\[
\int_X c_n(N_F) = (-1)^n(n-1)! \int_X c_n(\Omega_X \otimes \mathcal{G}^\vee)
\]

\[
= (-1)^n(n-1)! \sum_j \text{Res} \left[ \frac{df_1^{(j)} \wedge \cdots \wedge df_n^{(j)}}{f_1^{(j)} \cdots f_n^{(j)}} \right].
\]

**Remarks.** If we assume the integrability condition on \( \mathcal{G} \), the above formula implies the Baum-Bott residue formula for singular holomorphic foliations. Since the Baum-Bott residue for \( c_n(N_F) \) is given by

\[
(-1)^n(n-1)! \dim Ext^1_{\mathcal{O}_p}(\Omega_{\mathcal{G},p}, \mathcal{O}_p) = (-1)^n(n-1)! \dim \mathcal{O}_p/\mathcal{I}_{\omega,p},
\]

the right hand side of 3.4 coincides the Baum-Bott residue.

5. APPLICATIONS

5.1. Residue for the non-transversal loci of a holomorphic map. Let \( F : X^n \rightarrow Y^m \) be a holomorphic map between \( n \) and \( m \) dimensional compact complex manifolds. If \( Y \) has a non-singular distribution \( \tilde{\mathcal{G}} = \mathcal{O}_Y(\mathcal{G}) \), then the inverse image \( \mathcal{G} = F^{-1}\tilde{\mathcal{G}} \) gives a distribution of \( X \) which is possibly singular. In codimension 1 case, if a distribution \( \tilde{\mathcal{G}} \) on \( Y \) is given by a collection of 1-forms \( \tilde{\omega} = (\tilde{\omega}_\alpha) \), then the inverse image \( \mathcal{G} = F^{-1}\tilde{\mathcal{G}} \) of the invertible sheaf \( \tilde{\mathcal{G}} \) is given by the collection of 1-forms \( \omega = (F^*\tilde{\omega}_\alpha) \). If the image of the differential \( DF_p \) does not contain the normal space \( G_p^* \), we see that covector \( \omega_p \) is zero. Thus the non-transversal loci of \( F \) to \( \tilde{\mathcal{G}} \) is given by

\[
S(\mathcal{G}) = \{ p \in X : F^*\tilde{\omega}_\alpha(p) = 0 \}
\]

Now we give the residue formula for the non-transversality of \( F \) to \( \tilde{\mathcal{G}} \). We assume that \( S(\mathcal{G}) \) consists of isolated points \( \{p_1, \cdots, p_k\} \). We set that, near \( p_j \), \( f_i^{(j)} \) are
the coefficients of $F^{*}\tilde{\omega}_{\alpha}^{(j)}$ such that we write $F^{*}\tilde{\omega}_{\alpha}^{(j)} = f_{1}^{(j)}dx_{1} + \cdots + f_{n}^{(j)}dx_{n}$. Then we have

$$\int_{X} c_{n}(\Omega_{X} \otimes \mathcal{G}^\vee) = \sum_{l=0}^{n} \int_{X} c_{n-l}(\Theta_{X})c_{1}(\mathcal{G})^{l}$$

$$= \sum_{j=1}^{k} \text{Res}_{p_{j}} \left[ df_{1}^{(j)} \wedge \cdots \wedge df_{n}^{(j)} \right].$$

Now we have the result.

**Theorem 5.1 (Residue formula for non-transversality).** Let $F : X^{n} \longrightarrow Y^{m}$ be a holomorphic map of generic rank $r$ and $\tilde{\mathcal{G}}$ a codimension 1 non-singular distribution of $Y$. We assume that the non-transversal points of $F$ to $\tilde{\mathcal{G}}$ are $\{p_{1}, \ldots, p_{k}\}$, then we have

$$\chi(X) + \sum_{l=1}^{r} \int_{F_{*}(c_{n-1}(\Theta_{X})-\{X\})} c_{1}(\mathcal{G})^{l} = \sum_{j=1}^{k} \text{Res}_{p_{j}} \left[ df_{1}^{(j)} \wedge \cdots \wedge df_{n}^{(j)} \right].$$

**Proof.** We denote by $X^{*}$ the set of generic points where $F$ has rank $k$. By using projection formula,

$$\int_{X} c_{n-l}(\Theta_{X})c_{1}(\mathcal{G})^{l} = \int_{X^{*}} c_{n-l}(\Theta_{X})F^{*}(c_{1}(\tilde{\mathcal{G}})^{l})$$

$$= \int_{F_{*}(c_{n-1}(\Theta_{X})-\{X\})} c_{1}(\tilde{\mathcal{G}})^{l}.$$

It is obvious that the above terms are zero for $k \leq l$.

Here let $F : X^{2} \longrightarrow Y^{m}$ be a map from compact complex surface. In this case we write down the above general form of the formula into geometric forms. We set that $y_{m} = F_{m}^{(j)}(x_{1}, x_{2})$ is the $m$-th entry of a local representation of $F$ near $p_{j}$ and also write $dF_{m}^{(j)} = f_{1}^{(j)}dx_{1} + f_{2}^{(j)}dx_{2}$. Then the above formula is

$$\chi(X) + \int_{F_{*}(c_{1}(\mathcal{G})^{j})} c_{1}(\tilde{\mathcal{G}}) + \int_{F_{*}[X]} c_{1}(\tilde{\mathcal{G}})^{2} = \sum_{j=1}^{k} \text{Res}_{p_{j}} \left[ df_{1}^{(j)} \wedge f_{2}^{(j)} \right].$$

We remark that if the generic rank of $F$ is 1, the last term in the left-hand side of the above vanishes and we have

$$\chi(X) + \chi(M_{F}) \int_{F_{*}[X]} c_{1}(\tilde{\mathcal{G}}) = \sum_{j=1}^{k} \text{Res}_{p_{j}} \left[ df_{1}^{(j)} \otimes f_{2}^{(j)} \right].$$

In the above, $M_{F}$ is the generic fiber of $F$.

As an other example let us consider the case that $F : X^{n} \longrightarrow C$ is a map for a curve $C$ and $\tilde{\mathcal{G}} = \Omega_{C}$ is the point distribution. Then the above formula implies the multiplicity formula. (See [IS], [F].)

**Theorem 5.2 (The multiplicity formula).** Let $F : X^{n} \longrightarrow C$ be a holomorphic function for a compact complex curve $C$ with the generic fiber $M_{F}$. If $F$ has finite
number of isolated critical points \( \{ p_1, \ldots, p_k \} \), then we have
\[
\chi(X) - \chi(M_F) \chi(C) = (-1)^n \sum_{j=1}^{k} \mu(F, p_j)
\]
where \( \mu(F, p_j) \) is the Milnor number of \( F \) at \( p_j \).

**Remarks.** The one dimensional cases of theorem 4.1 is the classical Riemann-Hurwitz formula for a morphism of Riemann surfaces \( F : C \to \overline{C} \). We note that it cannot be deduced from the Baum-Bott type formula for \( c_1(N_F) \) in the above settings, however we can still apply the residue formula for \( \mathcal{G} \) in theorem 2.4. By taking the anihilator of the inverse image \( \mathcal{G} \) of \( \Omega_{\overline{C}} \), the given tangent sheaf \( \mathcal{F} \) of the lifted foliation turn out to be reduced. Since 1 dimensional manifolds only admits point foliations, the zero schemes of singularities are the points with multiplicities. Thus those kinds of singularities become non-singular by taking reduction. Therefore in our pull-back situation, the normal sheaf \( N_F \) is always locally free and only \( \mathcal{G} \) itself keeps the informations of singularities of \( F \).

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