CR immersions from $S^{2n+1}$ to $S^{4n+1}$

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Introduction

Elie Cartan's contribution to CR geometry is well known. In his fundamental papers, he solved the problem of equivalence for a piece of real hypersurface in $\mathbb{C}^2$ up to biholomorphism [Ca1]. It was Chern and Moser, among others, who vastly generalized this work to higher dimensions [ChM].

Recently, Huang and Ji gave a classification of the proper holomorphic maps from the unit ball $B^{n+1} \subset \mathbb{C}^{n+1}$ to $B^{2n+1}$, $n \geq 2$, [HJ]. We wish to show in this note that the corresponding local CR analogue is true. Let us denote $\Sigma^n = \partial B^{n+1} = S^{2n+1}$.

**Theorem.** Let $f : \Sigma^n \rightarrow \Sigma^{2n}$ be a $C^3$, local CR immersion, $n \geq 2$. Then up to automorphisms of the spheres, either $f$ is linear, or $f$ is locally equivalent to the boundary of Whitney map.

When the codimension is small, CR Gauß equation puts the second fundamental form of a CR immersion into a simple normal form. Our idea is to explore the successive derivatives of this relation.

The computation involved is reminiscent of Cartan's local isometric embedding of Hyperbolic space $\mathbb{H}^n$ in Euclidean space $\mathbb{E}^{2n-1}$ via exteriorly orthogonal symmetric bilinear forms [Ca2]. Overdetermined nature of CR geometry forces the structure equation to close up rather than become involutive.

Local CR immersions $\Sigma^1 \rightarrow \Sigma^2$ have been classified by Faran [Fa]. In contrast to $n \geq 2$ cases, there exist four inequivalent such immersions.
Corollary [HJ]. Let $F : B^{n+1} \rightarrow B^{2n+1}$ be a proper holomorphic map which is $C^3$ up to the boundary, $n \geq 2$. Then up to automorphisms of the unit balls, either $F$ is linear, or $F$ is equivalent to Whitney map.

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1 CR immersion

We first set up the basic structure equations for CR immersions in spheres. For general reference in CR geometry, [ChM][EHZ].

Let $\mathbb{C}^{N+1,1}$ be the complex vector space with coordinates $z = (z^0, z^A, z^{N+1})$, $1 \leq A \leq N$, and a Hermitian scalar product

$$\langle z, \bar{z} \rangle = z^A \bar{z}^A + i (z^0 \bar{z}^{N+1} - z^{N+1} \bar{z}^0).$$

Let $\Sigma^N$ be the set of equivalence classes up to scale of null vectors with respect to this product. Let $\text{SU}(N+1,1)$ be the group of unimodular linear transformations that leave the form $\langle z, \bar{z} \rangle$ invariant. Then $\text{SU}(N+1,1)$ acts transitively on $\Sigma^N$, and

$$p : \text{SU}(N+1,1) \rightarrow \Sigma^N = \text{SU}(N+1,1)/P$$

for an appropriate subgroup $P$ [ChM].

Explicitly, consider an element $Z = (Z_0, Z_A, Z_{N+1}) \in \text{SU}(N+1,1)$ as an ordered set of $(N+2)$-column vectors in $\mathbb{C}^{N+1,1}$ such that $\det(Z) = 1$, and that

$$\langle Z_A, \bar{Z}_B \rangle = \delta_{AB}, \quad \langle Z_0, \bar{Z}_{N+1} \rangle = -\langle Z_{N+1}, \bar{Z}_0 \rangle = i,$$

while all other scalar products are zero. We define $p(Z) = [Z_0]$, where $[Z_0]$ is the equivalence class of null vectors represented by $Z_0$. The left invariant Maurer-Cartan form $\pi$ of $\text{SU}(N+1,1)$ is defined by the equation

$$dZ = Z \pi,$$
which is in coordinates
\[
d( Z_0, Z_A, Z_{N+1} ) = ( Z_0, Z_B, Z_{N+1} ) \begin{pmatrix} \pi_0^0 & \pi_0^A & \pi_0^{N+1} \\ \pi_0^B & \pi_A^B & \pi_A^{N+1} \\ \pi_0^{N+1} & \pi_A^{N+1} & \pi_A^{N+1} \end{pmatrix}.
\tag{2}
\]

Coefficients of $\pi$ are subject to the relations obtained from differentiating (1) which are
\[
\begin{align*}
\pi_0^0 + \pi_{N+1}^{N+1} & = 0 \\
\pi_0^{N+1} & = \bar{\pi}_0^{N+1}, \quad \pi_{N+1}^0 = \bar{\pi}_{N+1}^0 \\
\pi_A^{N+1} & = -i \bar{\pi}_0^A, \quad \pi_A^A = i \bar{\pi}_A^0 \\
\pi_B^A + \bar{\pi}_B^A & = 0
\end{align*}
\]

and $\pi$ satisfies the structure equation
\[
- d\pi = \pi \wedge \pi.
\tag{3}
\]

It is well known that the SU($N + 1, 1$)-invariant CR structure on $\Sigma^N \subset \mathbb{C}P^{N+1}$ as a real hypersurface is biholomorphically equivalent to the standard CR structure on $S^{2N+1} = \partial \mathbb{B}^{N+1}$, where $\mathbb{B}^{N+1} \subset \mathbb{C}^{N+1}$ is the unit ball. The structure equation (2) shows that for any local section $s : \Sigma^N \rightarrow \text{SU}(N + 1, 1)$, this CR structure is defined by the hyperplane fields $(s^* \pi_0^{N+1})^\perp = \mathcal{H}$ and the set of (1,0)-forms $\{ s^* \pi_A^i \}$.

**Definition.** Let $M$ be a CR manifold of hypersurface type with CR hyperplane fields $\mathcal{H}^M$ equipped with a complex structure. An immersion $f : M \hookrightarrow \Sigma^N$ is a **CR immersion** if $f_* : \mathcal{H}^M \rightarrow \mathcal{H}$ is complex linear.

Consider $f^* \text{SU}(N + 1, 1) \rightarrow M$. From the definition, we may arrange so that $\pi_0^a = 0$ for $n + 1 \leq a \leq m = N - n$ on this bundle. Differentiating this, we get
\[
\pi_i^a \wedge \pi_i^i + \pi_{N+1}^a \wedge \pi_0^{N+1} = 0.
\]
By Cartan's lemma,
\[ \pi^i_i \equiv h^a_{ij} \pi^j_0 \pmod{\pi^{N+1}_0}, \] (4)
for coefficients \( h^a_{ij} = h^a_{ji} \). \( h^a_{ij} \) represents the second fundamental form of \( f \) [EHZ].

**Example** [Whitney immersion]. Let \( \xi = (\xi^0, \xi^i, \xi^{n+1}) \), \( \mu = (\mu^0, \mu^i, \mu^{n+i} \mu^{2n+1}) \), 1 \( \leq i \leq n \), be the coordinates of \( \mathbb{C}^{n+1,1} \) and \( \mathbb{C}^{2n+1,1} \) respectively. Whitney immersion \( \Gamma_n : \Sigma^n \rightarrow \Sigma^{2n} \) is induced from the quadratic map \( \hat{\Gamma}_n : \mathbb{C}^{n+1,1} \rightarrow \mathbb{C}^{2n+1,1} \) defined as
\[
\begin{align*}
\mu^0 &= 2 \xi^0 \xi^{n+1}, \\
\mu^{2n+1} &= (\xi^{n+1})^2 - (\xi^0)^2, \\
\mu^i &= \xi^i (i \xi^0 + \xi^{n+1}), \\
\mu^{n+i} &= \xi^i (-i \xi^0 + \xi^{n+1}).
\end{align*}
\]
\( \hat{\Gamma}_n (\mu, \overline{\mu}) = 2 (\xi^0 \overline{\xi^0} + \xi^{n+1} \overline{\xi^{n+1}}) (\xi, \overline{\xi}) \), and the induced map \( \Gamma_n \) is well defined.

It is easy to check \( \Gamma_n \) is CR-equivalent to the boundary map \( \partial \mathcal{W}_n : S^{2n+1} \rightarrow S^{4n+1} \) of the following Whitney map \( \mathcal{W}_n : B^{n+1} \rightarrow B^{2n+1} \). Here \((z^0, z^i)\) is a coordinate of \( \mathbb{C}^{n+1} \).
\[
\mathcal{W}_n(z^0, z^i) = ((z^0)^2, z^0 z^i, z^i).
\] (5)
This equivalence is via the isomorphism \( \Sigma^n \simeq S^{2n+1} \) given in coordinates
\[
\begin{align*}
z^0 &= \frac{i \xi^0 + \xi^{n+1}}{-i \xi^0 + \xi^{n+1}}, \\
z^i &= \frac{\sqrt{2} \xi^i}{-i \xi^0 + \xi^{n+1}}
\end{align*}
\]
Set \( \Sigma_0^n = \{ [\xi] \in \Sigma^n | \xi^i = 0, \forall i \} \) and \( \Sigma_\delta^n = \{ [\xi] \in \Sigma^n | i \xi^0 + \xi^{n+1} = 0 \} \). Then \( \Gamma_n \) is an immersion which is 1 to 1 on \( \Sigma^n - \Sigma_0^n \), 2 to 1 on \( \Sigma_0^n \), and the second fundamental form vanishes along \( \Sigma_\delta^n \).
2 Proof of theorem

Our proof of Theorem is based on the following algebraic Lemma due to Iwatani on the asymptotic subspace of the second fundamental form of a Bochner-Kähler submanifold [Iw][Br]. Let $V = \mathbb{C}^n$, $W = \mathbb{C}^n$ with the standard Hermitian scalar product. Let $\{ z^i \}$ be a unitary $(1,0)$-basis for $V^*$, and $\{ w_a \}$ be a unitary basis for $W$. Let $S^{p,q}$ denote the space of polynomials of type $(p,q)$ on $V$.

Lemma [Iw]. Suppose $H = h^a_{ij} z^i z^j \otimes w_a \in S^{2,0} \otimes W$ satisfies
\[
\gamma(H, H) = h^a_{ij} z^i z^j \otimes \bar{z}^k \bar{z}^l = (z^k \bar{z}^k) h \in S^{2,2}, \quad h \in S^{1,1},
\]
or simply $\gamma(H, H)$ is Bochner-flat [Br]. Then the asymptotic vectors $\{ v \in V \mid H(v, v) = 0 \}$ form a subspace of $V$ of codimension at most one.

Up to a unitary transformation on $V$, we may thus arrange
\[
h^a_{ij} = h^a_i \delta_{jn} + h^a_j \delta_{in},
\]
for coefficients $h^a_i$. Set $\nu_i = h^a_i w_a \in W$. A computation shows $\gamma(H, H)$ is Bochner-flat when $\langle \nu_i, \nu_j \rangle = 0$ for $i \neq j$, and $\langle \nu_i, \nu_i \rangle = \langle \nu_j, \nu_j \rangle$ for all $i, j$. Up to a unitary transformation of $W$, we may set
\[
\nu_i = rw_i,
\]
for some $r \geq 0$.

Let $f : \Sigma^n \hookrightarrow \Sigma^{2n}$ be a local CR immersion. Let $\mathcal{H}$ be the CR hyperplane fields on $\Sigma^{2n}$. Since $\Sigma^n$ is CR flat, after identifying $V = f_* T\Sigma^n \cap \mathcal{H} \cong \mathbb{C}^n$ and $W = V^\perp \cong \mathbb{C}^n \subset \mathcal{H}$, the second fundamental form of $f$ is Bochner-flat [EHZ]. From (4) and the argument above, we may write
\[
\pi_q^{n+i} \equiv r \delta_{iq} \pi^0_i \mod \pi_0^{2n+1}, \quad \text{for } q < n,
\]
\[
\pi_n^{n+i} \equiv r (1 + \delta_{in}) \pi^0_i \mod \pi_0^{2n+1}.
\]
Assume $f$ is not linear, $H \neq 0$, and we may scale $\tau = 1$ using the group action by $	ext{Re} \pi_0^0$. We obtain the following normalized structure equation for a nonlinear local CR immersion $f: \Sigma^n \hookrightarrow \Sigma^{2n}$:

$$\begin{align*}
\pi_n^{n+i} &= (1 + \delta_{in})\omega_i + h_i^{n+1}\pi_0^{2n+1}, \\
\pi_q^{n+i} &= \delta_{iq}\omega^n + h_q^{n+1}\pi_0^{2n+1}
\end{align*}$$

for coefficients $h_j^i$.

**Theorem** is now obtained by successive application of Maurer-Cartan equation (3) to this structure equation. We assume $n \geq 3$ for simplicity for the rest of this section, as $n = 2$ case can be treated in a similar way. We shall agree on the index range $1 \leq p, q, s, t \leq n - 1$, and denote $p' = n + p$, $n' = n + n$. We denote $\theta = \pi_0^{2n+1}$, $-d\theta \equiv i\pi_0^k \wedge \overline{\pi_0^k} = i\mod \theta$, and $\pi_0^1 = \omega^i$ for the sake of notation.

***Step 1.*** Differentiating $\pi_{s}^{n'} = h_{s}^{n}\theta \mod \theta$, we get

$$i h_{s}^{n} \omega \equiv (\pi_{s}^{n'} - 2\pi_{s}^{n}) \wedge \omega^n + \pi_{2n+1}^{n'} \wedge (-i\overline{\omega}^s) \mod \theta.$$ 

Since $n - 1 \geq 2$, this implies $h_{s}^{n} = 0$, and by Cartan's lemma

$$\begin{pmatrix}
\pi_{s}^{n'} - 2\pi_{s}^{n} \\
\pi_{2n+1}^{n'}
\end{pmatrix} \equiv 
\begin{pmatrix}
2c_s & u \\
u & 0
\end{pmatrix}
\begin{pmatrix}
\omega^n \\
-i\overline{\omega}^s
\end{pmatrix} \mod \theta$$

for coefficients $c_s, u$.

***Step 2.*** Differentiating $\pi_{s}^{t'} = h_{s}^{t}\theta \mod \theta$ for $t \neq s$, we get

$$i h_{s}^{t} \omega \equiv (\pi_{s}^{t'} - \pi_{s}^{t}) \wedge \omega^n - \pi_{s}^{n} \wedge \omega^t + \pi_{2n+1}^{t'} \wedge (-i\overline{\omega}^s) \mod \theta.$$ 

Since $n - 1 \geq 3$, this implies $h_{s}^{t} = 0$ for $t \neq s$, and by Cartan's lemma

$$\begin{pmatrix}
\pi_{s}^{t'} - \pi_{s}^{t} \\
-\pi_{s}^{n} \\
\pi_{2n+1}^{t'}
\end{pmatrix} \equiv 
\begin{pmatrix}
0 & b_s & -i\overline{b}_t \\
0 & b_s & 0 \\
-i\overline{b}_t & e & 0
\end{pmatrix}
\begin{pmatrix}
\omega^n \\
\omega^t \\
-i\overline{\omega}^s
\end{pmatrix} \mod \theta.$$
for coefficients $b_t, e$. Since $\pi_{s'}^t - \pi_s^t$ is skew Hermitian, it cannot have any $\omega^n$-term.

**Step 3.** Differentiating $\pi_{s'}^t = \omega^n + i h_t^t \theta \mod \theta$ and collecting terms, we get

$$h_t^t \omega \equiv (\pi_{s'}^t - \pi_s^t + \pi_0^t - \pi_n^t) \land \omega^n + (b_p \omega^n - i e \overline{\omega}^p) \land \omega^p + (-b_t \omega^t + \overline{b}_t \overline{\omega}^t) \land \omega^n \mod \theta.$$ 

Since $n - 1 \geq 2$, this implies $h_t^t = e$, and

$$\Delta_t = \pi_{s'}^t - \pi_s^t + \pi_0^t - \pi_n^t = a_t \omega^n - i e \overline{\omega}^n + (b_t \omega^t - \overline{b}_t \overline{\omega}^t) + \sum_p b_p \omega^p - A_t \theta$$

for coefficients $a_t, A_t$.

**Step 4.** From Step 2, we may use the group action by $\pi_{2n+1}^t$ to translate $e = 0$, which we assume from now on. We also translate $h_n^t = 0$ by $\pi_{2n+1}^t$. Differentiating $\pi_{n'}^n = \omega^n \mod \theta$ with these relations, we get

$$0 \equiv \bar{b}_t \left( \sum_p \omega^p \land \overline{\omega}^p \right) + (2 \bar{c}_t - 3 \bar{b}_t) \omega^n \land \overline{\omega}^n + (a_t + 2i \bar{u}) \omega^n \land \omega^t \mod \theta.$$ 

Thus $b_t = c_t = 0$, $a_t = -2i \bar{u}$.

**Step 5.** Differentiating $\pi_{n'}^n = 2 \omega^n + h_n^n \theta \mod \theta$ and collecting terms, we get

$$i h_n^t \omega \equiv 2 (\pi_{n'}^n - \pi_n^0 + \pi_0^0 - \pi_n^0) \land \omega^n + i u \omega^p \land \overline{\omega}^p - i u \omega^n \land \overline{\omega}^n \mod \theta.$$ 

This implies $h_n^n = u$, and

$$\Delta_n = \pi_{n'}^n - \pi_n^0 + \pi_0^0 - \pi_n^0$$

$$= a_n \omega^n - i u \overline{\omega}^n - A_n \theta$$

for coefficients $a_n, A_n$. But $\Delta_t - \Delta_n$ is purely imaginary, and comparing with Step 3, $a_n = -3i \bar{u}$.

**Step 6.** Now by considering $\theta$-terms in Step 1, 2, 3, 4, 5 and the fact $\pi_t^t \land \omega^t + \pi_{2n+1}^t \land \theta = 0$, we obtain the following simple structure equations. We omit the details of computations.
\[
\left( \begin{array}{c}
\pi'_n - 2\pi^n \\
\pi'_{2n+1}
\end{array} \right) = \left( \begin{array}{cc}
0 & u \\
u & 0
\end{array} \right) \left( \begin{array}{c}
\omega^n \\
-i\omega^s
\end{array} \right),
\]

\[\pi'_{t'} = \pi^t_s, \quad t \neq s,\]

\[\Delta_t = -2i\bar{u}\omega^n - A\theta,\]

\[
\left( \begin{array}{c}
-\pi^n_s \\
\pi'_{2n+1}
\end{array} \right) = \left( \begin{array}{c}
0 \\
0
\end{array} \right),
\]

\[\pi^t_{2n+1} = (A - iu\bar{u})\omega^t + B_t\theta\]

\[\pi^n_{2n+1} = A\omega^n + B_n\theta\]

**Step 7.** Differentiating \(\pi'_n - \pi^n_s = 0, t \neq s, \pi^n_s = 0\), we get first \(B_s = 0, B = 0\), and

\[A - \bar{A} = i(u\bar{u} - 1).\]  

(7)

**Step 8.** Differentiating \(\pi'_n = 2\omega^n + u\theta\) and \(\pi'_{2n+1} = u\omega^n\),

\[du = u(\pi'_{2n+1} - \pi^0_0 + \pi^n_n - \pi'_{n'}) + 2(A - A_n)\omega^n - 2uA\theta.\]  

(8)

**Step 9.** Differentiating \(\pi'_n = -iu\bar{\omega}^s\) using (8) and collecting terms in \(\theta \wedge \bar{\omega}^s\),

\[A_n = 2A - \bar{A}.\]

**Step 10.** Differentiating \(\pi^t_{2n+1} = (\bar{A} - i)\omega^t, \pi^n_{2n+1} = A\omega^n\), we get

\[dA = A(\pi'_{2n+1} - \pi^0_0) + \pi^n_{2n+1} + 2(u\bar{\omega}^n - \bar{u}\omega^n) + (u\bar{u} - A^2)\theta.\]  

(9)

We normalize \(A = i\alpha\) for a real number \(\alpha\) using group action by \(\pi^0_{2n+1}\). Since \(\pi^t + \pi^n_t = 0\), \(\Delta_t + \bar{\Delta}_t = \pi^0 + \bar{\pi}^0\), and (9) is now reduced to

\[d\alpha = 2i(\alpha + 1)(\bar{u}\omega^n - u\bar{\omega}^n)\]  

(10)

\[\pi^0_{2n+1} = -(\alpha + 1)^2\theta.\]
When \( u \neq 0 \), we may also rotate \( u \) to be a positive number, in which case it is determined by (7)

\[
2 \alpha + 1 = u \bar{u}.
\] (11)

At this stage, note that the only independent coefficients in the structure equations are \( \alpha \), \( u \), and that the expression for their derivatives does not involve any new coefficients. The structure equations for local CR immersion \( f : \Sigma^n \hookrightarrow \Sigma^{2n} \) thus close up as follows.

\[
\begin{pmatrix}
\pi_q'^p & \pi_n'^p \\
\pi_q'^n & \pi_n'^n
\end{pmatrix} =
\begin{pmatrix}
\delta_{pq} \omega^n & \omega^p \\
0 & 2 \omega^n + u \theta
\end{pmatrix}
\]

\[
\begin{pmatrix}
\pi_q'^n - 2 \pi_n'^n \\
\pi_{2n+1}'^n
\end{pmatrix} =
\begin{pmatrix}
0 & u \\
u & 0
\end{pmatrix}
\begin{pmatrix}
\omega^n \\
-i \bar{\omega}^n
\end{pmatrix}
\]

\[
\pi_{q'}^t = \pi_s^t, \quad t \neq s,
\]

\[
\begin{pmatrix}
-\pi_s^n \\
\pi_{2n+1}^t
\end{pmatrix} =
\begin{pmatrix}
0 \\
0
\end{pmatrix},
\]

\[
\pi_{2n+1}^t = (A - i u \bar{u}) \omega^t
\]

\[
\pi_{2n+1}^n = A \omega^n
\]

\[
\Delta_t = -2 i \bar{u} \omega^n - A \theta
\]

\[
\Delta_n = -3 i \bar{u} \omega^n - i u \bar{\omega}^n - A_n \theta
\]

\[
du = u (\pi_{2n+1}^{n+1} - \pi_0^n + \pi_n^n - \pi_n^n') + 2(A - A_n) \omega^n - 2 u A \theta,
\]

\[
d\alpha = 2 i (\alpha + 1) (\bar{u} \omega^n - \bar{u} \bar{\omega}^n)
\]

\[
\pi_{2n+1}^0 = -(\alpha + 1)^2 \theta
\]

where \( A = i \alpha \), \( A_n = 3 i \alpha \), and \( \alpha, u \) satisfy the relation (11). Moreover, a long but direct computation shows that this structure equation is compatible, i.e., \( d^2 = 0 \) is a formal identity of the structure equation.
Remark. The structure equation closes up at order 3. This implies a $C^3$, nonlinear CR immersion $f : \Sigma^n \to \Sigma^{2n}$ is real analytic.

The remaining step in the proof of Theorem consists of identifying this as the structure equation of Whitney immersion. We make one useful observation. Let $\Sigma^* = \Sigma^n - \Sigma'^n$, which is a connected set. Note also the structure equation (12) implies that the set of points where $u = 0$, or equivalently $\alpha = -\frac{1}{2}$, cannot have any interior on $\Sigma^*$. We claim the invariant $\alpha$ takes any value $> -\frac{1}{2}$ on $\Sigma^*$.

Suppose $\alpha_+ = \sup_{\Sigma^*} \alpha > -\frac{1}{2}$ is finite. Applying the existence part of Cartan's generalization of Lie's third fundamental theorem on closed structure equations, [Br3], there exists for any $p_0 \in \Sigma^n$ a neighborhood $U \subset \Sigma^n$ of $p_0$ and a CR immersion $g : U \to \Sigma^{2n}$ with invariant $\alpha|_{p_0} = \alpha_+$, hence necessarily $u_{p_0} = \sqrt{2\alpha_0 + 1}$. From (12), $d\alpha|_{p_0} \neq 0$ and let $p_- \in U$ be a point with $-\frac{1}{2} < \alpha|_{p_-} < \alpha_+$. Then by uniqueness part of Cartan's theorem, there exists a neighborhood $U' \subset U$ of $p_-$ on which $g$ agrees with Whitney immersion $\Gamma_n$ up to automorphisms of $\Sigma^n$ and $\Sigma^{2n}$. Since $g$ and $\Gamma_n$ satisfy the closed set of structure equations, they are real analytic. Thus $g$ is a part of $\Gamma_n$. But $d\alpha|_{p_0} \neq 0$, and there exists a point $p_+ \in U \subset \Sigma^*$ such that $\alpha|_{p_+} > \alpha_+$, a contradiction. By similar argument, $\inf_{\Sigma^*} \alpha = -\frac{1}{2}$, and the claim follows for $\Sigma^*$ is connected.

Proof of Theorem. Since the set of points $\alpha = -\frac{1}{2}$ cannot have any interior, let $p \in \Sigma$ be a point with $\alpha|_p > -\frac{1}{2}$. From the results above and the uniqueness part of aforementioned Cartan's theorem, $f = \Gamma_n$ on a neighborhood $U$ of $p$ up to automorphisms of the spheres. The theorem follows for both $f$ and $\Gamma_n$ are real analytic. □

Proof of Corollary. By the regularity theorem [HJX], $F$ is real analytic up to $\partial F$. Since the CR structure on $S^{2n+1} = \Sigma^n$ is definite, the set of points where $\partial F$ has holomorphic rank $n$ is a dense open subset. Assume $F$ is not linear. There exists a point $p \in \Sigma^n$ where the second fundamental form does not vanish either. By Theorem, $\partial F$ agrees with Whitney immersion $\Gamma_n$ in a neighborhood of $p$ up to automorphisms of
the spheres. The real analyticity then implies $\partial F = \Gamma_n$ on $\Sigma^n$, and hence $F = \mathcal{W}_n$ on $B^{n+1}$. □

We may apply the existence part of Cartan’s generalization of Lie’s third fundamental theorem and show that Whitney immersion gives an example of a deformable CR-submanifold. Take a point $p \in \Sigma^n$, and an analytic one parameter family of real numbers $\alpha_t > -\frac{1}{2}$, and set $u_t = \sqrt{2\alpha_t + 1}$. Then by the existence part of Cartan’s theorem, there exists a neighborhood $U$ of $p$ and a one parameter family of CR immersions $f_t : U \to \Sigma^{2n}$ with the induced structure equations (12) such that the invariants $\alpha, u$ have the prescribed values $\alpha_t, u_t$ at $p$. This deformation of course is tangential and does not actually deform the submanifold. It is due to an intrinsic CR symmetry of $\Sigma^n$ that cannot be extended to a symmetry of the ambient $\Sigma^{2n}$ along Whitney immersion.

References


[ChM] Chern, S. S.; Moser, J. K., Real hypersurfaces in complex manifolds, S. S. Chern Selected papers vol III, 209–262


[HJX] , , Xu, D., *Several Results for Holomorphic Mappings from \( B^n \) into \( B^N \),* Geometric analysis of PDE and several complex variables, Contemp. Math., 368, AMS, (2005) 267–292


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