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Kyoto University
NAMBU-LIE GROUPS ENDOwed WITH
MULTIPlicative Tensors of TOP ORDER

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ABSTRACT. A multiplicative (Nambu-Poisson) tensor of top order on a
Lie group is characterized. As an application, we determine multiplicative
structures on 3-dimensional Lie groups.

1. INTRODUCTION

A Nambu-Lie group is defined as a natural generalization of a Poisson Lie
group. In fact, if \( \eta \) is a multiplicative \( k \)-vector field on a Lie group \( G \) which
satisfies fundamental identity, then a pair \((G, \eta)\) is called a Nambu-Lie group.
If \( k = 2 \), then \((G, \eta)\) is especially called a Poisson Lie group [1],[3]. A Nambu-
Lie group was studied by J.Grabowski and G.Marmo [2] and I.Vaisman [5]. In [2], they proved that there are no Nambu-Lie structures of order
\( k \geq 3 \) on simple Lie groups. I.Vaisman [5] gave an alternative definition of
multiplicativity by defining the \( k \)-bracket of 1-forms on \( G \). In this paper, we
characterize the properties of multiplicative Nambu-Poisson tensors of top
order (i.e., \( n = k \)). Note that the word "Nambu-Poisson" is void in this case.
As an application of these characterizations, we determine multiplicative
(Nambu-Lie) structures defined on 3-dimensional Lie groups.

2. NAMBU-LIE GROUPS

Let \( G \) be an \( n \)-dimensional connected Lie group with the Lie algebra \( \mathfrak{g} \).
We denote by \( \Gamma(\Lambda^k TG) \) the set of \( k \)-vector fields (or contravariant tensor
fields of order \( k \)) on \( G \). Let \( F \) be the set of \( C^\infty \)-functions on \( G \). Each element
\( \eta \) of \( \Gamma(\Lambda^k TG) \) defines a \( k \)-bracket of functions \( f_i \in F \) as follows.

\[ \{f_1, \ldots, f_k\} = \eta(df_1, \ldots, df_k). \]

Since this \( k \)-bracket satisfies Leibniz rule, we can define a vector field
\( X_{f_1,\ldots,f_{k-1}} \) by

\[ X_{f_1,\ldots,f_{k-1}}(g) = \{f_1, \ldots, f_{k-1}, g\}, \ g \in F. \]
Definition 2.1. An element $\eta$ of $\Gamma(\Lambda^k TG)$, $k \geq 3$, is called a Nambu-Poisson tensor of order $k$ if $\eta$ satisfies

$$\mathcal{L}_{X_{f_1}, \ldots, X_{k-1}} \eta = 0$$

for any $f_1, \ldots, f_{k-1} \in \mathcal{F}$.

Note that if $k = n$, every $\eta$ is a Nambu-Poisson tensor \[4\].

Definition 2.2. An element $\eta$ of $\Gamma(\Lambda^k TG)$ is said to be multiplicative if $\eta$ satisfies

$$\eta_{gh} = L_{g*} \eta_h + R_{h*} \eta_g$$

for any $g, h \in G$, where $L_g$ and $R_g$ denote, respectively, the left and the right translations. Let $G$ be a Lie group endowed with a multiplicative Nambu-Poisson tensor $\eta$. Then a pair $(G, \eta)$ is called a Nambu-Lie group.

For an element $\Lambda \in \Lambda^k \mathfrak{g}$, we define vector fields $\overline{\Lambda}$ and $\tilde{\Lambda}$ by

$$\overline{\Lambda}_g = L_{g*} \Lambda, \quad \tilde{\Lambda}_g = R_{g*} \Lambda,$$

for all $g \in G$.

Then it is clear that $\overline{\Lambda}$ (resp. $\tilde{\Lambda}$) is a left (resp. right) invariant vector field on $G$. Let us recall the following, which was proved by J-H Lu \[3\].

Proposition 2.1. Let $G$ be a compact (or semisimple) Lie group. Then for every multiplicative $k$-vector field $\eta \in \Gamma(\Lambda^k TG)$, there exists an element $\Lambda \in \Lambda^k \mathfrak{g}$ such that

$$\eta_g = \overline{\Lambda}_g - \tilde{\Lambda}_g$$

for all $g \in G$.

Using the above proposition, we show the following theorem.

Theorem 2.2. Let $(G, \eta)$ be an $n$-dimensional compact or semisimple Nambu-Lie group, and let $\eta$ be of top order. Then $\eta = 0$.

Proof. By Proposition 2.1, there exists an element $\Lambda$ of $\Lambda^n \mathfrak{g}$ such that $\eta = \overline{\Lambda} - \tilde{\Lambda}$. For all $g, h \in G$,

$$Ad_g \overline{\Lambda}_h = R_{g^{-1}} L_g \overline{\Lambda}_h = R_{g^{-1}} \overline{\Lambda}_{gh}.$$

On the other hand, since $G$ is a unimodular Lie group, we have

$$Ad_g \overline{\Lambda}_h = (\det Ad_g) \overline{\Lambda}_{ghg^{-1}} = \overline{\Lambda}_{ghg^{-1}}.$$

Hence we obtain that $R_{g^{-1}} \overline{\Lambda}_{gh} = \overline{\Lambda}_{ghg^{-1}}$. This means that a left invariant vector field $\overline{\Lambda}$ is also a right invariant vector field. i.e., $R_{h*} \overline{\Lambda}_g = \overline{\Lambda}_{gh}$. This equation induces

$$R_{h*} \overline{\Lambda}_g = R_{h*} L_g \Lambda = L_g R_{h*} \Lambda = \overline{\Lambda}_{gh} = L_g L_{h*} \Lambda.$$

Thus we have $R_{h*} \Lambda = L_{h*} \Lambda$ for all $h \in G$, and this means $\eta = \overline{\Lambda} - \tilde{\Lambda} = 0$. $\square$
Let $\eta$ be a Nambu-Poisson tensor of order $k$ on $G$. Then $\eta$ defines a bundle mapping

$$\#_\eta : T^*G \times \cdots \times T^*G \longrightarrow TG$$

given by

$$<\beta, \#_\eta(\alpha_1, \ldots, \alpha_{k-1})> = \eta(\alpha_1, \ldots, \alpha_{k-1}, \beta),$$

where all the arguments are covectors.

For such a tensor $\eta$, I.Vaisman [5] defined a $k$-bracket of 1-forms by

$$\{\alpha_1, \ldots, \alpha_k\} = d(\eta(\alpha_1, \ldots, \alpha_k)) + \sum_{j=1}^{k}(-1)^{k+j} i(\#_\eta(\alpha_1, \ldots, \overline{\alpha}_j, \ldots, \alpha_k))d\alpha_j,$$

where $\alpha_j$ ($j = 1, \ldots, k$) are 1-forms on $G$.

The following theorem proved by I.Vaisman [5] gives one of the characterizations of Nambu-Lie groups.

**Theorem 2.3.** If $G$ is a connected Lie group endowed with a Nambu-Poisson tensor field $\eta$ which vanishes at the unit $e$ of $G$, then $(G, \eta)$ is a Nambu-Lie group if and only if the $k$-bracket of any $k$ left (right) invariant 1-forms of $G$ is a left (right) invariant 1-form.

Using Theorem 2.3, we characterize a multiplicative tensor $\eta$ of top order. Let $\mathfrak{g}$ be a Lie algebra of $G$ with a basis $X_1, \ldots, X_n$. We also denote the extended left invariant vector fields induced from $X_i$ by the same letter. Since $\eta$ is of top order, $\eta$ has an expression $\eta = fX_1 \wedge \cdots \wedge X_n$ for some $f \in \mathcal{F}$. Let $\omega_i$ ($i = 1, \ldots, n$) be left invariant 1-forms dual to $X_i$. Under these notations we prove

**Theorem 2.4.** Let $\eta = fX_1 \wedge \cdots \wedge X_n$, $f \in \mathcal{F}$ be a tensor of top order on $G$. (Recall that such a tensor is always a Nambu-Poisson tensor.) Then $\eta$ is multiplicative if and only if $f(e) = 0$ and

$$X_i f + \left(\sum_{k=1}^{n} C_{ik}^k\right) f = q_i, \ i = 1, \ldots, n,$$

where $\{C_{ij}^k\}$ are structure constants of $\mathfrak{g}$ with respect to the basis $X_1, \ldots, X_n$, and $q_i$ ($i = 1, \ldots, n$) are some constants.

**Proof.** By Theorem 2.3, we know that $\eta$ is multiplicative if and only if $\eta_e = 0$ and

$$\{\omega_1, \ldots, \omega_n\} = d(\eta(\omega_1, \ldots, \omega_n)) + \sum_{k=1}^{n}(-1)^{n+k} i(\#_\eta(\omega_1, \ldots, \overline{\omega}_k, \ldots, \omega_n))d\omega_k$$

$$= df + f \sum_{k=1}^{n} i(X_k)d\omega_k = df + f\left(\sum_{\alpha,k=1}^{n} C_{\alpha k}^k \omega_\alpha\right)$$
is a left invariant 1-form. Since $\langle X_i, \{\omega_1, ..., \omega_n\} \rangle$ is constant for any $X_i$, we have

$$\langle X_i, \{\omega_1, ..., \omega_n\} \rangle = X_i f + \left( \sum_{k=1}^{n} C_{ik}^{k} \right) f = q_i, \quad i = 1, ..., n.$$  

\[ \square \]

3. Examples

In this section, as an application of Theorem 2.4, we calculate Nambu-Lie group structures (i.e., multiplicative Nambu-Poisson tensors) of order 3 on 3-dimensional simply connected Lie groups. Since such tensors are of top degree, we have only to see whether they are multiplicative or not.

Throughout this section, we denote by $G$ the simply connected Lie groups corresponding to Lie algebras $\mathfrak{g}$. Linearly independent three left invariant vector fields are denoted by $X, Y, Z$. Then $\eta \in \Omega(A^3TG)$ is written as $\eta = fX \wedge Y \wedge Z, \ f \in C^\infty(G)$. It is well-known that there are 9 types of 3-dimensional Lie algebras. If $\mathfrak{g}$ is not a simple Lie algebra, its corresponding simply connected Lie group has global coordinates $x, y, z$. Hence a function $f$ can be considered to be defined on $\mathbb{R}^3(x, y, z)$.

Type 1. $[\mathfrak{g}, \mathfrak{g}] = 0$. Namely $\mathfrak{g}$ is an abelian Lie algebra. The corresponding Lie group $G$ is given by

$$G = \left\{ \begin{pmatrix} e^x & 0 & 0 \\ 0 & e^y & 0 \\ 0 & 0 & e^z \end{pmatrix} \bigg| x, y, z \in \mathbb{R} \right\}.$$  

Using these coordinates $x, y, z$, left invariant vector fields are written as $X = \frac{\partial}{\partial x}, \ Y = \frac{\partial}{\partial y}, \ Z = \frac{\partial}{\partial z}$. By Theorem 2.4, a function $f(x, y, z)$ must satisfy $f(0, 0, 0) = 0$, and $\frac{\partial f}{\partial x} = a, \ \frac{\partial f}{\partial y} = b, \ \frac{\partial f}{\partial z} = c$, where $a, b, c$ are some constants. Hence $f = ax + by + cz$, and

$$\eta = (ax + by + cz) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}$$

gives a Nambu-Lie group structure on $G$.

By the similar method, we can get the results for other types.

Type 2. $\dim[\mathfrak{g}, \mathfrak{g}] = 1$. There are 2 cases as follows.

Case (1). $\mathfrak{g} = \text{Heisenberg Lie algebra}$. $\mathfrak{g}$ is characterized by the condition $[\mathfrak{g}, \mathfrak{g}] \subset 1$-dimensional center. The corresponding Lie group $G$ is given by

$$G = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \bigg| x, y, z \in \mathbb{R} \right\}.$$  

A Nambu-Lie group structure on $G$ is given by

$$\eta = (ax + by) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}.$$
Case (2). A Lie algebra \( \mathfrak{g} \) endowed with a property \([\mathfrak{g}, \mathfrak{g}] \not\subset \text{the center of } \mathfrak{g} \). The corresponding Lie group \( G \) is given by

\[
G = \left\{ \begin{pmatrix} e^{y+z} & 0 & xe^y \\ 0 & e^y & 0 \\ 0 & 0 & e^y \end{pmatrix} \middle| \right. \left. x, y, z \in \mathbb{R} \right\}.
\]

\[
\eta = \left\{ ax + c(e^z - 1) \middle| \right. \left. \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} \right\}
\]
gives a Nambu-Lie group structure on \( G \).

Type 3. \( \dim[\mathfrak{g}, \mathfrak{g}] = 2, \mathfrak{g}^{(2)} = 0 \). There are 4 cases as follows.

Case (1). Left invariant vector fields \( X, Y, Z \) satisfy \([X, Y] = 0, [X, Z] = -X, [Y, Z] = -X - Y \). The corresponding Lie group \( G \) is given by

\[
G = \left\{ \begin{pmatrix} e^{-z} & xe^{-z} & xe^{-2z} \\ 0 & e^{-z} & ye^{-2z} \\ 0 & 0 & e^{-2z} \end{pmatrix} \middle| \right. \left. x, y, z \in \mathbb{R} \right\}.
\]

We know that

\[
\eta = \frac{c(e^{2z} - 1)}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}
\]
gives a Nambu-Lie group structure on \( G \).

Case(2). Left invariant vector fields \( X, Y, Z \) satisfy \([X, Y] = 0, [X, Z] = -X, [Y, Z] = -Y \). The corresponding Lie group \( G \) is given by

\[
G = \left\{ \begin{pmatrix} e^{-z} & xe^{-2z} \\ 0 & ye^{-2z} \\ 0 & 0 \end{pmatrix} \middle| \right. \left. x, y, z \in \mathbb{R} \right\}.
\]

We have

\[
\eta = \frac{c(e^{2z} - 1)}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}.
\]

Case (3). Let \( \mathfrak{g} \) be a Lie algebra endowed with the following bracket relations. \([X, Y] = 0, [X, Z] = -X, [Y, Z] = -qY, (q \neq 0, 1) \). The corresponding Lie group \( G \) is given by

\[
G = \left\{ \begin{pmatrix} e^{-qz} & 0 & xe^{-(q+1)z} \\ 0 & e^{-z} & ye^{-(q+1)z} \\ 0 & 0 & e^{-(q+1)z} \end{pmatrix} \middle| \right. \left. x, y, z \in \mathbb{R} \right\}.
\]

We have

\[
\eta = \frac{c(e^{(q+1)z} - 1)}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}.
\]

Case (4). Let \( \mathfrak{g} \) be a Lie algebra endowed with the following bracket relations. \([X, Y] = 0, [X, Z] = -Y, [Y, Z] = X - qY, (q^2 < 4) \). The
corresponding Lie group $G$ has rather complicated expression. Put $k = q/2$, $p = \sqrt{1 - k^2} = \sqrt{4 - q^2/2}$. Then $G$ is given by

$$
G = \left\{ \begin{pmatrix}
\frac{1}{p}e^{-kz}(-k\sin(pz) + p\cos(pz)) & -\frac{1}{p}e^{-kz}\sin(pz) & xe^{-2kz} \\
\frac{1}{p}e^{-kz}\sin(pz) & \frac{1}{p}e^{-kz}(p\cos(pz) + k\sin(pz)) & ye^{-2kz} \\
0 & 0 & e^{-2kz}
\end{pmatrix} \bigg| x, y, z \in \mathbb{R} \right\}.
$$

Then

$$
\eta = \begin{cases}
\frac{1}{q}(e^{qx} - 1)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}, & q \neq 0 \\
cx\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}, & q = 0
\end{cases}
$$

gives a Nambu-Lie group structure on $G$.

Type 4. dim$[\mathfrak{g}, \mathfrak{g}] = 3$. It is well-known that such Lie algebras are simple, and there are 2 cases. The corresponding simply connected Lie groups are $G_1 = SU(2)$ and $G_2 = SL(2, \mathbb{R})$, where $SL(2, \mathbb{R})/\mathbb{Z} \cong SL(2, \mathbb{R})$. Since $G_1$ is compact, and $G_2$ is semisimple, we have $\eta = 0$ by Theorem 2.2.

**References**


