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Some Periodicity of Words and Marcus Contextual Grammars [1]

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Abstract

In this paper, first we define a periodic (semi-periodic, quasi-periodic) word and then we define a primitive (strongly primitive, hyper primitive) word. After we define several Marcus contextual grammars, we show that the set of all primitive (strongly primitive, hyper primitive) words can be generated by some Marcus contextual grammar.

1 Introduction

Let X^* denote the free monoid generated by a nonempty finite alphabet X and let $X^+ = X^* \setminus \{\lambda\}$ where λ denotes the empty word of X^*. For the sake of simplicity, if $X = \{a\}$, then we write a^+ and a^* instead of $\{a\}^+$ and $\{a\}^*$, respectively. Let $L \subseteq X^*$. Then L is called a *language* over X. By $|L|$, we denote the cardinality of L. If $L \subseteq X^*$, then L^+ denotes the set of all concatenations of words in L and $L^* = L^+ \cup \{\lambda\}$. In particular, if $L = \{w\}$, then we write w^+ and w^* instead of $\{w\}^+$ and $\{w\}^*$, respectively. Let $u \in X^*$. Then u is called a *word* over X.

Definition 1.1 A word $u \in X^+$ is said to be *periodic* if u can be represented as $u = v^n, v \in X^+, n \geq 2$. If u is not periodic, then it is said to be *primitive*. By Q we denote the set of all primitive words.

[1] This is an abstract and the details will be published elsewhere.

Definition 1.2 A word $u \in X^+$ is said to be *semi-periodic* if u can be represented as $u = v^n v', v \in X^+, n \geq 2$ and $v' \in Pr(v)$ where $Pr(v)$ denotes the set of all prefixes of v. If u is not semi-periodic, then it is said to be *strongly primitive.* By SQ we denote the set of all strongly primitive words.

Definition 1.3 A word $u \in X^+$ is said to be *quasi-periodic* if a letter in any position in u can be covered by some $v \in X^+$ with $|v| \leq |u|$. More precisely, if $u = wax, w, x \in X^*$ and $a \in X$, then $v \in Suf(w)aPr(x)$ where $Suf(w)$ denotes the set of all suffixes of w. If u is not quasi-periodic, then it is said to be *hyper primitive.* By HQ we denote the set of all hyper primitive words.

Then we have the following inclusion relations.

Fact 1.1 $HQ \subset SQ \subset Q$.

2 Marcus Contextual Grammars

We begin this section by the following definition.

Definition 2.1 *(Marcus) contextual grammar with choice* is a structure $G = (X, A, C, \varphi)$ where X is an alphabet, A is a finite subset of X^*, i.e. the set of axioms, C is a finite subset of $X^* \times X^*$, i.e. the set of contexts, and $\varphi : X^* \to 2^C$ is the choice function. If $\varphi(x) = C$ holds for every $x \in X^*$ then we say that G is a *(Marcus) contextual grammar without choice* .

Definition 2.2 We define two relations on X^*: for any $x \in X^*$, we write $x \Rightarrow_{ex} y$ if and only if $y = uxv$ for a context $(u, v) \in \varphi(x), x \Rightarrow_{in} y$ if and only if $x = x_1x_2x_3, y = x_1ux_2vx_3$ for some $(u, v) \in \varphi(x_2)$. By $\Rightarrow^*_{ex}$ and $\Rightarrow^*_{in}$, we denote the reflexive and transitive closure of each relation and let $L_\alpha(G) = \{x \in X^* \mid w \Rightarrow^*_\alpha x, w \in A\}$ for $\alpha \in \{ex, in\}$. Then $L_{ex}(G)$ is the *(Marcus) external contextual language (with or without choice) generated by* G, and similarly, $L_{in}(G)$ is the *(Marcus) internal contextual language (with or without choice) generated by* G.

Example 2.1 Let $X = \{a, b\}$ and let $G = (X, A, C, \varphi)$ be a Marcus contextual grammar where $A = \{a\}, C = \{(\lambda, \lambda), (\lambda, a), (\lambda, b)\}, \varphi(\lambda) = \{(\lambda, \lambda)\}, \varphi(ua) = \{(\lambda, b)\}$ for $u \in X^*$ and $\varphi(ub) = \{(\lambda, a)\}$ for $u \in X^*$. Then $L_{ex}(G) = a(ab)^* \cup a(ba)^+b$ and $L_{in}(G) = aX^* \cup X^*a^2X^*$.

As for more details on Marcus contextual grammars and languages, see [3].

3 Set of Primitive Words

In this section, we deal with the set of all primitive words. First we provide the following three lemmas. The proofs of the lemmas are based on the results in [2] and [4].

Lemma 3.1 *For any $u \in X^+$, there exist unique $q \in Q$ and $i \geq 1$ such that $u = q^i$.*

Lemma 3.2 *Let $i \geq 1$ let $u, v \in X^*$ and $uv \in \{q^i \mid q \in Q\}$. Then $vu \in \{q^i \mid q \in Q\}$.*

Lemma 3.3 *Let X be an alphabet with $|X| \geq 2$. If $w, wa \notin Q$ where $w \in X^+$ and $a \in X$, then $w \in a^+$.*

Using the above lemmas, we can prove the following. The proof can be seen in [1].

Proposition 3.1 *The language Q is a Marcus external contextual language with choice.*

However, in the case of $|X| \geq 2$ we can prove that the other types of Marcus contextual grammars cannot generate Q.

4 Set of Strongly Primitive Words

In this section, we deal with the set of all primitive words. First we provide the follpwing three lemmas. All results in this section can be seen in [1].

Lemma 4.1 *Let X be an alphabet with $|X| \geq 2$. If $awb \in SQ$ where $w \in X^*$ and $a, b \in X$, then $aw \in SQ$ or $wb \in SQ$.*

Using the above lemma, we can prove the following.

Proposition 4.1 *The language SQ is a Marcus external contextual language with choice.*

However, we can prove that the other types of Marcus contextual grammars cannot generate SQ.

5 Set of Hyper Primitive Words

In this section, first we characterize a quasi-periodic word.

Definition 5.1 Let $u \in X^+$ be a quasi-periodic word and let any letter in u be covered by a word v. Then we denote $u = v \otimes v \otimes \cdots \otimes v$.

Proposition 5.1 *Let $u \in X^+$. Then there exists a hyper primitive word $v \in HQ$ such that $u = v \otimes v \otimes \cdots \otimes v$. In fact, v and each position of v are uniquely determined*

Lemma 5.1 *Let X be an alphabet with $|X| \geq 2$. If $awb \in HQ$ where $w \in X^*$ and $a, b \in X$, then $aw \in HQ$ or $wb \in HQ$.*

Using the above lemma, we can prove the following.

Proposition 5.2 *The language HQ is a Marcus external contextual language with choice.*

However, we can prove that the other types of Marcus contextual grammars cannot generate HQ.

References

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