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Endomorphisms of a Module over a Local Ring[1]

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The matrix A of an endomorphism σ of a module M over a ring R is completely determined by the choice of a basis X for M, where A is called the matrix of σ relative to X.

Therefore, it will be natural to seek X giving a simple A, which is our primitive motivation.

Now, let R be a field. Then we have a good example of such A expressed in a nice form for a suitable X. Indeed, we know the following fact (see Lang[7,p557,Theorem 2.1] or Herstein[4,p307,Theorem 6.7.1]):

Theorem. Let R be a field. Then there are m elements $\{x_1, x_2, \cdots, x_m\}$ in M and m polynomials $\{g_1(t), g_2(t), \cdots, g_m(t)\}$ in the polynomial ring $R[t]$ over R in one indeterminate t such that A is a direct sum of m companion matrices of $\{g_1(t), g_2(t), \cdots, g_m(t)\}$.

What can we say about this result, if R is a local ring ? Is it possible to get a concise form of A as above ? To analize this problem is the purpose of this note.

So, let R be a local ring with the identity 1 and the unique maximal ideal $\mathfrak{m}$, M a free module of rank n over R, and $\mathrm{End}_R M$ the endomorphism ring of M.

Then we have two chanonical maps

$$\pi_R : R \to \overline{R} = R/\mathfrak{m} \qquad \text{defined by} \qquad a \mapsto \bar{a} = a + \mathfrak{m}$$

and

$$\pi_M : M \to \overline{M} = M/\mathfrak{m}M \qquad \text{defined by} \qquad x \mapsto \bar{x} = x + \mathfrak{m}M.$$

[1]This is an abstract and the details will be published elsewhere.

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Since $\overline{R}$ is a field, $\overline{M}$ is a vector space over $\overline{R}$ by the scalar multiplication $\bar{a}\bar{x} = \overline{ax}$ for $a \in R$ and $x \in M$. Clearly the ring homomorphism π_R is an R-module homomorphism if we define $a\bar{b} = \overline{ab}$ for $a, b \in R$. Also π_M is an R-module homomorphism.

Further, for $x \in M$ and $\sigma \in \mathrm{End}_R M$, if we define $\bar{\sigma}\bar{x} = \overline{\sigma x}$, we obtain an endomorphism $\bar{\sigma}$ of $\overline{M}$, that is, $\bar{\sigma} \in \mathrm{End}_{\overline{R}}\overline{M}$. Thus we have the third chanonical map

$$\pi_E : \mathrm{End}_R M \to \mathrm{End}_{\overline{R}}\overline{M} \qquad \text{by} \qquad \sigma \mapsto \bar{\sigma},$$

which is a ring homomorphism.

An element $\rho \in \mathrm{End}_R M$ is called a permutation if it is a permutation on some basis for M. Also $\delta \in \mathrm{End}_R M$ is diagonal if the matrix of δ is diagonal relative to some basis for M.

Also we denote the ring of $r \times s$ matrices over R by $M_{r,s}(R)$, and by $M_r(R)$ if $r = s$. Then, our results are as follows:

Theorem A. For any $\sigma \in \mathrm{End}_R M$ there is a new basis X and a permutation ρ on X such that the matrix of $\rho^{-1}\sigma$ relative to X is expressed as

$$\begin{pmatrix} I_{n-m} & O_{n-m,m} \\ B_{m,n-m} & D_m \end{pmatrix},$$

where

(i) m is the number of the invariant factors of $\bar{\sigma}$,

(ii) $I_{n-m} \in M_{n-m}(R)$ is the identity matrix,

(iii) $O_{n-m,m} \in M_{n-m,m}(R)$ is the zero matrix,

(iv) $D_m \in (d_{ij}) \in M_m(R)$ is a matrix with $d_{ij} \equiv 0 \bmod \mathfrak{m}$ if $i \neq j$, i.e., diagonal modulo $\mathfrak{m}$,

and

(v) $B_{m,n-m} = (b_{ij}) \in M_{m,n-m}(R)$ is a matrix such that for any $i = 1, 2, \cdots, m$ we have

$$b_{ij} \equiv 0 \bmod \mathfrak{m}$$

for $j \leq \Pi_{\lambda=1}^{i-1}(n_\lambda - 1)$ or $\Pi_{\mu=1}^{i}(n_\mu - 1) < j$.

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