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Kyoto University
Endomorphisms of a Module over a Local Ring\textsuperscript{1}

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The matrix $A$ of an endomorphism $\sigma$ of a module $M$ over a ring $R$ is completely determined by the choice of a basis $X$ for $M$, where $A$ is called the matrix of $\sigma$ relative to $X$.

Therefore, it will be natural to seek $X$ giving a simple $A$, which is our primitive motivation.

Now, let $R$ be a field. Then we have a good example of such $A$ expressed in a nice form for a suitable $X$. Indeed, we know the following fact (see Lang[7,p557,Theorem 2.1] or Herstein[4,p307,Theorem 6.7.1]):

\textbf{Theorem.} Let $R$ be a field. Then there are $m$ elements $\{x_1, x_2, \cdots, x_m\}$ in $M$ and $m$ polynomials $\{g_1(t), g_2(t), \cdots, g_m(t)\}$ in the polynomial ring $R[t]$ over $R$ in one indeterminate $t$ such that $A$ is a direct sum of $m$ companion matrices of $\{g_1(t), g_2(t), \cdots, g_m(t)\}$.

What can we say about this result, if $R$ is a local ring? Is it possible to get a concise form of $A$ as above? To analyze this problem is the purpose of this note.

So, let $R$ be a local ring with the identity $1$ and the unique maximal ideal $\mathfrak{m}$, $M$ a free module of rank $n$ over $R$, and $\text{End}_R M$ the endomorphism ring of $M$.

Then we have two canonical maps

$$\pi_R : R \to \overline{R} = R/\mathfrak{m} \quad \text{defined by} \quad a \mapsto \overline{a} = a + \mathfrak{m}$$

and

$$\pi_M : M \to \overline{M} = M/\mathfrak{m}M \quad \text{defined by} \quad x \mapsto \overline{x} = x + \mathfrak{m}M.$$
Since $\overline{R}$ is a field, $\overline{M}$ is a vector space over $\overline{R}$ by the scalar multiplication $\overline{a}\overline{x} = \overline{ax}$ for $a \in R$ and $x \in M$. Clearly the ring homomorphism $\pi_R$ is an $R$-module homomorphism if we define $\overline{ab} = \overline{ab}$ for $a, b \in R$. Also $\pi_M$ is an $R$-module homomorphism.

Further, for $x \in M$ and $\sigma \in \text{End}_R M$, if we define $\overline{\sigma x} = \overline{\sigma x}$, we obtain an endomorphism $\overline{\sigma}$ of $\overline{M}$, that is, $\overline{\sigma} \in \text{End}_{\overline{R}} \overline{M}$. Thus we have the third canonical map

$$\pi_E : \text{End}_R M \to \text{End}_{\overline{R}} \overline{M} \quad \text{by} \quad \sigma \mapsto \overline{\sigma},$$

which is a ring homomorphism.

An element $\rho \in \text{End}_R M$ is called a permutation if it is a permutation on some basis for $M$. Also $\delta \in \text{End}_R M$ is diagonal if the matrix of $\delta$ is diagonal relative to some basis for $M$.

Also we denote the ring of $r \times s$ matrices over $R$ by $M_{r,s}(R)$, and by $M_r(R)$ if $r = s$. Then, our results are as follows:

**Theorem A.** For any $\sigma \in \text{End}_R M$ there is a new basis $X$ and a permutation $\rho$ on $X$ such that the matrix of $\rho^{-1} \sigma$ relative to $X$ is expressed as

$$\begin{pmatrix}
I_{n-m} & O_{n-m,m} \\
B_{m,n-m} & D_m
\end{pmatrix},$$

where

(i) $m$ is the number of the invariant factors of $\overline{\sigma}$,

(ii) $I_{n-m} \in M_{n-m}(R)$ is the identity matrix,

(iii) $O_{n-m,m} \in M_{n-m,m}(R)$ is the zero matrix,

(iv) $D_m \in (d_{ij}) \in M_m(R)$ is a matrix with $d_{ij} \equiv 0 \mod m$ if $i \neq j$, i.e., diagonal modulo $m$,

and

(v) $B_{m,n-m} = (b_{ij}) \in M_{m,n-m}(R)$ is a matrix such that for any $i = 1, 2, \ldots, m$ we have

$$b_{ij} \equiv 0 \mod m$$

for $j \leq \Pi_{\lambda=1}^{i-1}(n_{\lambda} - 1)$ or $\Pi_{\mu=1}^{i}(n_{\mu} - 1) < j$. 


References


