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Combinatorial structure of group divisible designs and finite geometry

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Abstract Balanced incomplete block (BIB) design and group divisible (GD) designs are connected with finite geometry. In this paper, at first, we denote BIB design, GD designs and finite geometry. Next, the combinatorial structure of GD designs with $r = \lambda_1 + 1$ is discussed. Moreover the combinatorial structure of GD designs is discussed from another point of view of assuming local structure in each group. Finally, we give a conjecture about combinatorial structure of GD designs with local structure corresponding finite geometry in each group.

Keywords: Group divisible design; Balanced incomplete block design; Finite geometry; Projective geometry

1. Introduction

Let V be a finite set and $\mathcal{B}$ be a collection of subsets of the same size of V. A pair $(V, \mathcal{B})$ is called a *block design*, or simply a *design*. Elements of V and $\mathcal{B}$ are called *points* and *blocks*, respectively. Let $v = |V|$ and $b = |\mathcal{B}|$. In discussing the combinatorial problems on designs, we adopt the terminology "points" instead of treatments used usually. For a block design $(V, \mathcal{B})$, let $V = \{p_1, p_2, \cdots, p_v\}$ and $\mathcal{B} = \{B_1, B_2, \cdots, B_b\}$, and the $v \times b$ matrix $N = (n_{ij})$, called an *incidence matrix* of a block design $(V, \mathcal{B})$, is defined as $n_{ij} = 1$ when $p_i \in B_j$, and $n_{ij} = 0$ when $p_i \notin B_j$. The *complement* of a design with the incidence matrix N is the design with the incidence matrix $\bar{N}$ which is obtained by exchanging 0's and 1's in N.

Now a group divisible (GD) design is defined. Let $v = mn$ $(m, n \geq 2)$, b, r, k, λ_1, λ_2 be positive integers. A *GD design* with parameters $v = mn$, b, r, k, λ_1, λ_2 is a triplet $(V, \mathcal{B}, \mathcal{G})$, where V is a v-set of *points*, $\mathcal{B}$ is a collection of b k-subsets, called *blocks*, of V and $\mathcal{G} = \{G_1, \cdots, G_m\}$ is a partition of V into m groups of n points each such that any two distinct points in the same group occur together in exactly λ_1 blocks of $\mathcal{B}$, while those in different groups occur together in exactly λ_2 blocks of $\mathcal{B}$. Here, r is the number of blocks containing a given point. Note that r is a constant not depending on the point chosen. Among parameters of a GD design, it holds that

$$bk = vr, \tag{1.1}$$

$$\lambda_1(n-1) + \lambda_2 n(m-1) = r(k-1). \tag{1.2}$$

When λ_1 equals λ_2, a GD design is called a *balanced incomplete block* (*BIB*) *design* with parameters v, b, r, k, λ $(= \lambda_1 = \lambda_2)$, which satisfy (1.1) and

$$r(k-1) = \lambda(v-1). \tag{1.3}$$

When $v = b$, a design is said to be *symmetric*.

Let N be an incidence matrix of a GD design and N' be the transpose of N. In the analysis of the design of experiment, the eigenvalues of the matrix NN' play an important role. For the incidence matrix N of a GD design with parameters v, b, r, k, λ_1 and λ_2, the determinant of NN' is given by

$$|NN'| = rk(r-\lambda_1)^{m(n-1)}(rk - v\lambda_2)^{m-1}$$

and the eigenvalues of NN' are rk, $r-\lambda_1$, $rk - v\lambda_2$ with multiplicities 1, $m(n-1)$ and $m-1$, respectively (see, for example, Raghavarao [8, pp.127-128]).

Bose and Connor [3] classified GD designs into three types in terms of the eigenvalues of NN' as follows:

(1) *Singular* if $r - \lambda_1 = 0$,
(2) *Nonsingular* if $r - \lambda_1 > 0$
  (2a) *Semi-regular* if $rk - v\lambda_2 = 0$
  (2b) *Regular* if $rk - v\lambda_2 > 0$.

By considering the rank of NN', it follows that $v \le b$ holds in the case of a regular GD design similarly to the case of a BIB design, which is called *Fisher's inequality*. A design is said to be *symmetric*, if $v = b$. We refer the reader to [2] and [5] for relevant design-theoretic terminology.

2. Finite geometry

From the standpoint of this paper, geometry is a particular kind of incidence system. The basic relation is the incidence relation $P \in L$, read the point P is on the line L. A finite geometry is one that contains a finite number of points. Let $PG(\ell, q)$ be a projective geometry of dimension ℓ over the finite field $F_q = GF(q)$ with $q = p^r$ elements, where p is a prime.

If we take the points as objects and the lines as blocks, a finite projective plane is a symmetric block design with parameters $v = \ell^2 + \ell + 1$, $k = \ell + 1$, $\lambda = 1$. Conversely, a block design with these parameters is a finite projective plane.

Several methods of constructing GD designs are given by Bose et al. [4]. A geometrical method of constructing symmetric regular GD designs is given by Sprott [10]. When s is a prime or a prime power, there exists a regular symmetric GD design with parameters $v = b = s(s-1)(s^2+s+1)$, $m = s^2+s+1$, $n = s(s-1)$, $r = k = s^2$, $\lambda_1 = 0$, $\lambda_2 = 1$.

3. Group divisible designs without α-resolution class

For a singular GD design $r = \lambda_1$ holds, while in case of a nonsingular GD design $r > \lambda_1$ holds. It may be natural to investigate the case of $r = \lambda_1 + 1$, since it may have some interconnecting property (the next saturated case) between singular and nonsingular cases.

In this section, we will characterize the combinatorial structure of GD designs with $r = \lambda_1 + 1$, and that of GD designs without "α-resolution class" in each group. All the results in this section are due to [9], [7], and [1].

To state the results, we will give some basic notations. We denote the identity matrix of order s, an $s\times t$ matrix all of whose elements are unity and an $s\times t$ matrix all of whose elements are zero, by I_s, $J_{s\times t}$ and $O_{s\times t}$, respectively. In particular, let $J_s = J_{s\times s}$ and $O_s = O_{s\times s}$. Moreover, let $\mathbf{1}_n = J_{1\times n}$ and $\mathbf{0}_n = O_{1\times n}$. Hence the above $\bar{A} = \mathbf{1}'_v\mathbf{1}_b - A = J_{v\times b} - A$. Here $\mathbf{1}'_n$ means the transpose of $\mathbf{1}_n$. $A \otimes B$ denotes the kronecker product of matrices A and B.

A symmetric BIB design with parameters v, $k = (v-1)/2$, $\lambda = (v-3)/4$ is called a *Hadamard design*. For a tournament, i.e., a complete simple digraph, with the $v\times v$ adjacency matrix N, if N is the incidence matrix of a Hadamard design, then the tournament is called a *Hadamard tournament of order* v (see [5]). A simple undirected graph is called a *strongly regular graph* if for any two distinct vertices i and j, there are p^1_{11} or p^2_{11} vertices which are connected to both of vertices i and j, according as i and j are connected or not. We refer the reader to [6] and [11] for relevant graph-theoretic terminology.

3.1. Group divisible designs with $r = \lambda_1 + 1$

The combinatorial property of a GD design with $r = \lambda_1 + 1$ was first investigated by Shimata and Kageyama [9] who showed that a GD design with $r = \lambda_1 + 1$ must be symmetric and regular. Jimbo and Kageyama [7] completely characterized a GD design with $r = \lambda_1 + 1$ in terms of Hadamard tournaments and strongly regular graphs.

In fact, in a GD design with parameters $v = mn$ $(m, n \geq 2) = b$, $r = k = \lambda_1 + 1$, λ_2, by the result given in [9], the $v\times v$ incidence matrix N of the GD design is divided into m^2 $n\times n$ submatrices such as $N = (N_{ij})$, where $N_{11} = N_{22} = \cdots = N_{mm} = I_n$ or $J_n - I_n$, and $N_{ij} = J_n$ or O_n for $i \neq j$. The incidence matrix N is completely characterized in terms of Hadamard tournaments and strongly regular graphs from the viewpoint of the construction as follows.

Theorem 3.1 (Jimbo and Kageyama [7]). *Let N be the incidence matrix of a regular GD design with $r = \lambda_1 + 1$ or of its complement such that $N_{ii} = I_n$ for any i.*

(i) *When $n \geq 3$ and $\lambda_2 \equiv 2 \pmod{n}$, the incidence matrix of the design is given by $N = I_m \otimes I_n + (J_m - I_m) \otimes J_n$ for general m and n, which leads to a symmetric* *regular GD design with parameters* $v = b = mn$, $r = k = (m-1)n+1$, $\lambda_1 = (m-1)n$, $\lambda_2 = (m-2)n+2$.

(ii) *When* $n \geq 2$ *and* $\lambda_2 \equiv 1 \pmod n$ (*i.e.*, $v = b = mn$, $r = k = n(m-1)/2+1$, $\lambda_1 = n(m-1)/2$, $\lambda_2 = n(m-3)/4+1)C$ *the existence of the design is equivalent to the existence of a Hadamard tournament of order* $m \equiv 3 \pmod 4$.

(iii) *When* $n = 2$ *and* λ_2 *is even* (*i.e.*, $v = b = 2m$, $r = k = 2s+1$, $\lambda_1 = 2s$, $\lambda_2 = 2s^2/(m-1))C$ *the existence of the design is equivalent to the existence of a strongly regular graph with parameters* $v = m$, $k = s$, $p_{11}^1 = x$, $p_{11}^2 = x+1$, *where* $s^2 = (x+1)(m-1)$. *Hence* $\lambda_2 = 2(x+1)$.

Remark. A regular GD design exists only when the parameters satisfy the conditions (i), (ii) or (iii).

Theorem 3.1 reveals that the inner structure of GD designs with $r = \lambda_1 + 1$ is characterized in terms of Hadamard tournaments and strongly regular graphs. For Hadamard tournaments and strongly regular graphs, there are some available existence or non-existence results Hence, the existence or nonexistence problem of GD designs with $r = \lambda_1 + 1$ can be reduced to those of Hadamard tournaments and strongly regular graphs.

3.2. Definition of an α-resolution class

In this subsection, we define an (r, λ)-design and an α-resolution class, which will be utilized when we consider some substructure in each group of GD designs.

For positive integers v, r, λ, an (r, λ)-*design* with parameters v, r, λ is a pair $(V, \mathcal{B})$ where V is a v-set of points and $\mathcal{B}$ is a collection of subsets of V such that every point of V occurs in r blocks of $\mathcal{B}$, and that any two distinct points of V occur together in exactly λ blocks of $\mathcal{B}$. In particular, when every block has the same size ($= k$, say), an (r, λ)-design is exactly a BIB design.

For a subcollection $\mathcal{B}'$ ($\subset \mathcal{B}$), if every point of V occurs in exactly α blocks ($1 \leq \alpha \leq r$) in $\mathcal{B}'$, then $\mathcal{B}'$ is called an α-*resolution class* of $(V, \mathcal{B})$. An α-resolution class is said to be *trivial* when $\alpha = r$, and *nontrivial* when $1 \leq \alpha \leq r-1$. In this paper, an α-resolution class implies a nontrivial α-resolution class if it is not specified.

Here, we will give examples of α-resolution classes, in which one has nontrivial α-resolution class, while the other does not.

Example 3.1. The following design is a (3,1)-design with nontrivial 1-resolution classes.

$$S = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Example 3.2. The following design is a (3,1)-design with no nontrivial α-resolution class.

$$T = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

3.3. Combinatorial structure of these designs

Let N be the $v \times b$ incidence matrix of a GD design with parameters $v = mn$ $(m, n \geq 2)$, b, r $(< b)$, k, λ_1, λ_2. Any groups G_l $(l = 1, 2, \cdots, m)$ of the GD design have the $n \times b$ incidence matrices $B_l = (N_l^* : J : O)$ after appropriate permutations of columns, where N_l^* are the incidence matrices of (r_l^*, λ_l^*)-designs with parameters $v_l^* = n$, b_l^*, r_l^* $(< b_l^*)$, λ_l^* $(< r_l^*)$ and with block sizes less than n. In this paper, we suppose that all (r_l^*, λ_l^*)-designs with the incidence matrices N_l^* do not have any α-resolution classes, if not specified. We call such design a GD design without α-resolution classes in each group. Then the following two main theorems can be established.

Theorem 3.2 (Adachi, Jimbo and Kageyama [1]). *Suppose that a GD design without α-resolution classes in each group has parameters $v = mn$ $(m, n \geq 2)$, b, r $(< b)$, k, λ_1, λ_2. Then, the incidence matrix N of the GD design is, after an appropriate permutation of rows and columns, represented by*

$$N = \begin{pmatrix} N_1^* & O_{n\times b^*} & \cdots & O_{n\times b^*} \\ O_{n\times b^*} & N_2^* & \cdots & O_{n\times b^*} \\ \vdots & \vdots & \ddots & \vdots \\ O_{n\times b^*} & O_{n\times b^*} & \cdots & N_m^* \end{pmatrix} + D \otimes J_{n\times b^*}, \tag{3.1}$$

where all of N_l^* are the incidence matrices of BIB designs with the same parameters $v_l^* = n$, $b_l^* = b^*$, $r_l^* = r^*$, $k_l^* = k^*$, $\lambda_l^* = \lambda^*$, and $D = (d_{ij})$ is an $m \times m$ matrix with entries 0 or 1 and $d_{ii} = 0$ for all i.

Since N is the incidence matrix of a GD design, each row of D has the same number of 1's. Let s (≥ 1) be the number of 1's in each row of D. For convenience, we denote the first term of (3.1), by $\mathrm{diag}(N_1^*, N_2^*, \cdots, N_m^*)$.

Theorem 3.3 (Adachi, Jimbo and Kageyama [1]). *Let N be the incidence matrix (3.1) of a GD design without α-resolution classes in each group. Then the GD design is regular and N is characterized as follows:*

(i) *When $b^* \neq 2r^*$ and $\lambda_2 \equiv 0 \pmod{b^*}$, the incidence matrix of the GD design is given by $N = \mathrm{diag}(N_1^*, N_2^*, \cdots, N_m^*)$ for general m and n, that is, $D = O_m$, which leads to a GD design with parameters $v = mn$, $b = mb^* = mnr^*/k^*$, $r = r^*$, $k = k^*$, $\lambda_1 = r^*(k^* - 1)/(n-1)$, $\lambda_2 = 0$.*

(ii) *When $b^* \neq 2r^*$ and $\lambda_2 \equiv 2r^* \pmod{b^*}$, the incidence matrix of the GD design is given by $N = \mathrm{diag}(N_1^*, N_2^*, \cdots, N_m^*) + (J_m - I_m) \otimes J_{n\times b^*}$ for general m and n, that is, $D = J_m - I_m$, which leads to a GD design with parameters $v = mn$, $b = mb^* = mnr^*/k^*$, $r = r^*(mn - n + k^*)/k^*$, $k = k^* + (m-1)n$, $\lambda_1 = r^*\{(m-1)n(n-1) + k^*(k^*-1)\}/\{k^*(n-1)\}$, $\lambda_2 = r^*(mn - 2n + 2k^*)/k^*$.*

(iii) *When $\lambda_2 \equiv r^* \pmod{b^*}$, D is the adjacency matrix of a Hadamard tournament of order $m \equiv 3 \pmod 4$, which leads to a GD design with parameters $v = mn$, $b = mb^* = mnr^*/k^*$, $r = r^*(mn - n + 2k^*)/(2k^*)$, $k = k^* + (m-1)n/2$, $\lambda_1 = r^*\{(m-1)n(n-1)+2k^*(k^*-1)\}/\{2k^*(n-1)\}$, $\lambda_2 = r^*(mn-3n+4k^*)/(4k^*)$. In this case, the existence of the GD design is equivalent to that of a Hadamard tournament of order m.*

(iv) *When $b^* = 2r^*$ and $\lambda_2 \equiv 0 \pmod{b^*}$, D is the adjacency matrix of a strongly regular graph with parameters $\tilde{v} = m$, $\tilde{k} = s$, $p_{11}^1 = x$, $p_{11}^2 = x+1$, where $s^2 = (x+1)(m-1)$, which leads to a GD design with parameters $v = mn$, $b = mb^* = 2mr^*$, $r = (2s+1)r^*$, $k = n(2s+1)/2$, $\lambda_1 = r^*\{4(n-1)s+n-2\}/\{2(n-1)\}$, $\lambda_2 = 2(x+1)r^*$. In this case, the existence of the GD design is equivalent to that of a strongly regular graph.*

Corollary 3.1. *Theorem 3.1 is a special case of Theorem 3.3.*

By Theorem 3.3, we see that the structure of GD designs without α-resolution classes in each group is characterized in terms of Hadamard tournaments and strongly regular graphs.

Four examples are given to show the structure of the present GD design.

Example 3.3. For $m = 3$, the incidence matrix of a GD design with $r = \lambda_1 + 1 = 2n + 1$, $\lambda_2 = n + 2$ is given by

$$N = \begin{pmatrix} I_n & J_n & J_n \\ J_n & I_n & J_n \\ J_n & J_n & I_n \end{pmatrix}.$$

Example 3.4. For $m = 7$, the incidence matrix of a GD design with $r = \lambda_1 + 1 = 3n + 1$, $\lambda_2 = n + 1$ is given by

$$N = \begin{pmatrix} I_n & O_n & O_n & J_n & O_n & J_n & J_n \\ J_n & I_n & O_n & O_n & J_n & O_n & J_n \\ J_n & J_n & I_n & O_n & O_n & J_n & O_n \\ O_n & J_n & J_n & I_n & O_n & O_n & J_n \\ J_n & O_n & J_n & J_n & I_n & O_n & O_n \\ O_n & J_n & O_n & J_n & J_n & I_n & O_n \\ O_n & O_n & J_n & O_n & J_n & J_n & I_n \end{pmatrix},$$

which corresponds to a Hadamard tournament of order 7. It is well known that a Hadamard tournament of order 7 is unique up to isomorphic.

Example 3.5. For $m = 10$, the incidence matrix of a GD design with $r = \lambda_1 + 1$, $n = 2$, $\lambda_2 = 2$ is given by

$$N = \begin{pmatrix} I_2 & J_2 & O_2 & O_2 & J_2 & J_2 & O_2 & O_2 & O_2 & O_2 \\ J_2 & I_2 & J_2 & O_2 & O_2 & O_2 & J_2 & O_2 & O_2 & O_2 \\ O_2 & J_2 & I_2 & J_2 & O_2 & O_2 & O_2 & J_2 & O_2 & O_2 \\ O_2 & O_2 & J_2 & I_2 & J_2 & O_2 & O_2 & O_2 & J_2 & O_2 \\ J_2 & O_2 & O_2 & J_2 & I_2 & O_2 & O_2 & O_2 & O_2 & J_2 \\ J_2 & O_2 & O_2 & O_2 & O_2 & I_2 & O_2 & J_2 & J_2 & O_2 \\ O_2 & J_2 & O_2 & O_2 & O_2 & O_2 & I_2 & O_2 & J_2 & J_2 \\ O_2 & O_2 & J_2 & O_2 & O_2 & J_2 & O_2 & I_2 & O_2 & J_2 \\ O_2 & O_2 & O_2 & J_2 & O_2 & J_2 & J_2 & O_2 & I_2 & O_2 \\ O_2 & O_2 & O_2 & O_2 & J_2 & O_2 & J_2 & J_2 & O_2 & I_2 \end{pmatrix},$$

which corresponds to the Petersen graph, i.e., a strongly regular graph with $p^1_{11} = 0$ and $p^2_{11} = 1$. It is well known that the Petersen graph is unique up to isomorphic.

Example 3.6. The following N^* is the incidence matrix of a BIB design with parameters $v^* = b^* = 7, r^* = k^* = 3$, $\lambda^* = 1$, and also of a (3,1)-design without

α-resolution classes,

$$N^* = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

which, by utilizing Theorem 3.2 and 3.3, shows that an incidence matrix of a GD design with $r = \lambda_1 + 2$, $n = 7$ is given by

$$N = I_m \otimes N^* + D \otimes J_7,$$

where $D = O_m$ in the case of $\lambda_2 \equiv 0 \pmod 7$, $D = J_m - I_m$ in the case of $\lambda_2 \equiv 6$ (mod 7), or D is the adjacency matrix of a Hadamard tournament of order $m \equiv 3 \pmod 4$ in the case of $\lambda_2 \equiv 3 \pmod 7$.

4. Concluding remark

A GD design with $r = \lambda_1$ is singular, whose existence is equivalent to that of a BIB design [8, Theorem 8.5.1]. While, if $r > \lambda_1$, a GD design is said to be regular or semi-regular.

It is known that GD designs with $r = \lambda_1 + 1$ are symmetric and regular, and the combinatorial structure of these designs is characterized in terms of Hadamard tournaments and strongly regular graphs from the viewpoint of the construction (see [7] and [9]). As the next interesting cases we can consider two cases: a GD design with $r = \lambda_1 + 2$ and another GD design which is characterized in terms of Hadamard tournaments and strongly regular graphs.

We can easily show that there exists a symmetric GD design with $r = \lambda_1 + 2$. In fact, a symmetric BIB design with parameters $v = b = 7$, $r = k = 3$, $\lambda = 1$ can be generated by a finite projective geometry PG(2,2). We can obtain a symmetric GD design with $r = \lambda_1 + 2$ and $n = 7$ together with a Hadamard tournament as in Example 3.6.

Thus we state the following open problem:

Open problem. What is a condition for the group size n such that there exists a symmetric GD design with $r = \lambda_1 + 2$?

Moreover, if it is shown that there are no $(r, r-2)$-designs with 7 points except for $(2,0)$-, $(3,1)$-, $(4,2)$-, $(6,4)$-designs which can be embedded in a GD design with $r = \lambda_1 + 2$ and $n = 7$, then the following conjecture may be made:

Conjecture. A GD design with $r = \lambda_1 + 2$ and $n = 7$ is regular and its $v \times b$ incidence matrix N is, after an appropriate permutation of rows and columns, divided into m^2 submatrices $N = (N_{ij})$. Every diagonal submatrix N_{ii} is one of the following:

(i) $(I_7 : I_7)$ or its complement,

(ii) the incidence matrix of a BIB design with parameters $v = b = 7$, $r = k = 3$, $\lambda = 1$ or its complement.

In case of (ii), it can be characterized as in Theorem 3.3 because the BIB design with parameters $v = b = 7$, $r = k = 3$, $\lambda = 1$ does not have any α-resolution class. The BIB design with parameters $v = b = 7$, $r = k = 3$, $\lambda = 1$ can be generated by a finite projective geometry PG(2,2).

References

[1] T. Adachi, M. Jimbo and S. Kageyama, *Combinatorial structure of group divisible designs without α-resolution classes in each group*, *Discrete Math*, **265** (2003), Issues 1–3 , pp. 1–11.

[2] T. Beth, D. Jungnickel, H. Lenz, *Design Theory*, Cambridge University Press, Cambridge, 1986.

[3] R. C. Bose and W. S. Connor, *Combinatorial properties of group divisible incomplete block designs*, *Ann. Math. Statist.*, **23** (1952), pp. 367–383.

[4] R. C. Bose, S. S. Shrikhande and K. N. Bhattacharya, *On the construction of group divisible incomplete block design*, *Ann. Math. Statist.*, **24** (1953) pp. 167–195.

[5] M. Hall, Jr., *Combinatorial Theory*, John Wiley, New York, 1967.

[6] N. Ito, *Hadamard tournaments with transitive automorphism groups*, *European J. Combin.* **5** (1984), pp. 37-42.

[7] M. Jimbo and S. Kageyama, *Discrete structure of group divisible designs with $r = \lambda_1 + 1$*, *ICA Bulletin*, **32** (2001), pp. 29-36.

[8] D. Raghavarao, *Constructions and Combinatorial Problems in Design of Experiments*, John Wiley, New York, 1971.

[9] T. Shimata and S. Kageyama, *Symmetry of a group divisible design with $r = \lambda_1 + 1$*, *J. Statist. Plann. Inference*, **106** (2002), pp. 31–37.

[10] D. A. Sprott, *A series of symmetric group divisible designs*, *Ann. Math. Statist.*, **30** (1959) pp. 249–251.

[11] V. D. Tonchev, *Combinatorial Configurations: Designs, Codes, Graphs.* Longman Scientific & Technical, England, 1988.