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On dense subsets of boundaries of Coxeter groups

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The purpose of this note is to introduce some results of recent papers [7], [8] and [9] about dense subsets of boundaries of Coxeter groups.

A Coxeter group is a group $W$ having a presentation

$\langle S \mid (st)^{m(s,t)} = 1 \text{ for } s,t \in S \rangle$,

where $S$ is a finite set and $m : S \times S \to \mathbb{N} \cup \{\infty\}$ is a function satisfying the following conditions:

(i) $m(s,t) = m(t,s)$ for any $s,t \in S$,
(ii) $m(s,s) = 1$ for any $s \in S$, and
(iii) $m(s,t) \geq 2$ for any $s,t \in S$ such that $s \neq t$.

The pair $(W, S)$ is called a Coxeter system. Let $(W, S)$ be a Coxeter system. For a subset $T \subset S$, $W_T$ is defined as the subgroup of $W$ generated by $T$, and called a parabolic subgroup. A subset $T \subset S$ is called a spherical subset of $S$, if the parabolic subgroup $W_T$ is finite. For each $w \in W$, we define $S(w) = \{s \in S \mid \ell(ws) < \ell(w)\}$, where $\ell(w)$ is the minimum length of word in $S$ which represents $w$. For a subset $T \subset S$, we also define $W^T = \{w \in W \mid S(w) = T\}$.

Let $(W, S)$ be a Coxeter system and let $S'$ be the family of spherical subsets of $S$. We denote $WS'$ as the set of all cosets of the form $wW_T$, with $w \in W$ and $T \in S'$. The sets $S'$ and $WS'$ are partially ordered by inclusion. Contractible simplicial complexes $K(W, S)$ and $\Sigma(W, S)$ are defined as the geometric realizations of the partially ordered sets $S'$ and $WS'$, respectively ([4]). The natural embedding $S' \to WS'$ defined by $T \mapsto W_T$ induces an embedding $K(W, S) \to \Sigma(W, S)$ which we regard as an inclusion. The group $W$ acts on $\Sigma(W, S)$ via simplicial automorphism. Then $\Sigma(W, S) = WK(W, S)$.
and $\Sigma(W,S)/W \cong K(W,S)$ ([4]). For each $w \in W$, $wK(W,S)$ is called a chamber of $\Sigma(W,S)$. If $W$ is infinite, then $\Sigma(W,S)$ is noncompact. In [10], G. Moussong proved that a natural metric on $\Sigma(W,S)$ satisfies the CAT(0) condition. Hence, if $W$ is infinite, $\Sigma(W,S)$ can be compactified by adding its ideal boundary $\partial\Sigma(W,S)$ ([4], [3]). This boundary $\partial\Sigma(W,S)$ is called the boundary of $(W,S)$. We note that the natural action of $W$ on $\Sigma(W,S)$ is properly discontinuous and cocompact ([4]), and this action induces an action of $W$ on $\partial\Sigma(W,S)$.

A subset $A$ of a space $X$ is said to be dense in $X$, if $\overline{A} = X$. A subset $A$ of a metric space $X$ is said to be quasi-dense, if there exists $N > 0$ such that each point of $X$ is $N$-close to some point of $A$.

Let $(W,S)$ be a Coxeter system. Then $W$ has the word metric $d_\ell$ defined by $d_\ell(w,w') = \ell(w^{-1}w')$ for each $w, w' \in W$.

In [7], the following theorems were proved.

**Theorem 1.** Let $(W,S)$ be a Coxeter system. Suppose that $W^{\{s_0\}}$ is quasi-dense in $W$ with respect to the word metric $d_\ell$ and $o(s_0t_0) = \infty$ for some $s_0, t_0 \in S$, where $o(s_0t_0)$ is the order of $s_0t_0$ in $W$. Then there exists $\alpha \in \partial\Sigma(W,S)$ such that the orbit $W\alpha$ is dense in $\partial\Sigma(W,S)$.

Suppose that a group $\Gamma$ acts properly and cocompactly by isometries on a CAT(0) space $X$. Every element $\gamma \in \Gamma$ such that $o(\gamma) = \infty$ is a hyperbolic transformation of $X$, i.e., there exists a geodesic axis $c : \mathbb{R} \to X$ and a real number $a > 0$ such that $\gamma \cdot c(t) = c(t + a)$ for each $t \in \mathbb{R}$ ([3]). Then, for all $x \in X$, the sequence $\{\gamma^t x\}$ converges to $c(\infty)$ in $X \cup \partial X$. We denote $\gamma^\infty = c(\infty)$.

**Theorem 2.** Let $(W,S)$ be a Coxeter system. If the set

$$\bigcup \{W^{\{s\}} \mid s \in S \text{ such that } o(st) = \infty \text{ for some } t \in S\}$$

is quasi-dense in $W$, then $\{w^\infty \mid w \in W \text{ such that } o(w) = \infty\}$ is dense in $\partial\Sigma(W,S)$.

**Remark.** For a negatively curved group $G$ and the boundary $\partial G$ of $G$,

1. we can show that $G\alpha$ is dense in $\partial G$ for each $\alpha \in \partial G$ by an easy argument, and
(2) it is known that \( \{ g^\infty \mid g \in G \text{ such that } o(g) = \infty \} \) is dense in \( \partial G \) ([2]).

As an application of Theorems 1 and 2, we obtained the following theorem in [7].

**Theorem 3.** Let \((W, S)\) be a Coxeter system. Suppose that there exist a maximal spherical subset \( T \) of \( S \) and an element \( s_0 \in S \) such that \( o(s_0t) \geq 3 \) for each \( t \in T \) and \( o(s_0t_0) = \infty \) for some \( t_0 \in T \). Then

1. \( W\alpha \) is dense in \( \partial\Sigma(W, S) \) for some \( \alpha \in \partial\Sigma(W, S) \), and
2. \( \{ w^\infty \mid w \in W \text{ such that } o(w) = \infty \} \) is dense in \( \partial\Sigma(W, S) \).

**Example.** The Coxeter system defined by the diagram in Figure 1 is not hyperbolic in Gromov sense, since it contains a copy of \( \mathbb{Z}^2 \), and it satisfies the condition of Theorem 3.

\[
\begin{array}{c}
\text{s}_0 \\
3 \\
3 \\
3 \\
T \\
t_0
\end{array}
\]

**FIGURE 1**

Suppose that a group \( G \) acts on a compact metric space \( X \) by homeomorphisms. Then \( X \) is said to be minimal, if every orbit \( Gx \) is dense in \( X \).

For a negatively curved group \( G \) and the boundary \( \partial G \) of \( G \), \( G\alpha \) is dense in \( \partial G \) for each \( \alpha \in \partial G \), that is, \( \partial G \) is minimal.

We note that Coxeter groups are non-positive curved groups and not negatively curved groups in general. There exist examples of Coxeter systems whose boundaries are not minimal as follows.

**Example.** Let \( S = \{ s, t, u \} \) and let

\[
W = \langle S \mid s^2 = t^2 = u^2 = (st)^3 = (tu)^3 = (us)^3 = 1 \rangle.
\]

Then \((W, S)\) is a Coxeter system and \( \Sigma(W, S) \) is the flat Euclidean plane. For any \( \alpha \in \partial\Sigma(W, S) \), \( W\alpha \) is a finite-points set and not dense in \( \partial\Sigma(W, S) \) which is a circle. This example implies that we can not omit the assumption "\( m(s_0, t_0) = \infty \)" in Theorem 3.
Example. Let $S = \{s_1, s_2, s_3, s_4\}$ and let

\[ W = \langle S \mid s_1^2 = s_2^2 = s_3^2 = s_4^2 = (s_1s_2)^2 = (s_2s_3)^2 = (s_3s_4)^2 = (s_4s_1)^2 = 1 \rangle. \]

Then $(W, S)$ is a Coxeter system and $\Sigma(W, S)$ is the Euclidean plane. For any $\alpha \in \partial \Sigma(W, S)$, $W\alpha$ is a finite-points set and not dense in $\partial \Sigma(W, S)$ which is a circle. Here we note that $\{s_1, s_2\}$ is a maximal spherical subset of $S$, $m(s_1, s_3) = \infty$ and $m(s_2, s_3) = 2$. This example implies that we can not omit the assumption "$m(s_0, t) \geq 3$" in Theorem 3.

As an extension of Theorem 3, we have obtained the following theorem in [9].

**Theorem 4.** Let $(W, S)$ be a Coxeter system which satisfies the condition in Theorem 3. Then every orbit $W\alpha$ is dense in $\partial \Sigma(W, S)$, that is, $\partial \Sigma(W, S)$ is minimal.

Here the following problems are open.

**Problem.** Does there exist a Coxeter system $(W, S)$ such that some orbit $W\alpha$ is dense in $\partial \Sigma(W, S)$ and $\partial \Sigma(W, S)$ is not minimal?

**Problem.** Suppose that a group $G$ acts geometrically on two CAT(0) spaces $X$ and $X'$. Is it the case that $\partial X$ is minimal if and only if $\partial X'$ is minimal?

**Problem (Ruane).** Suppose that a group $G$ acts geometrically on a CAT(0) space $X$. Is it always the case that the set $\{g^\infty \mid g \in G, o(g) = \infty\}$ is dense in $\partial X$?

In [8], we also have obtained the following theorem.

**Theorem 5.** Let $(W, S)$ be a Coxeter system and let $T$ be a subset of $S$ such that $W_T$ is infinite. If the set

\[ \bigcup \{ W^{(s)} \mid s \in S \text{ such that } o(ss_0) = \infty \text{ and } s_0t \neq ts_0 \text{ for some } s_0 \in S \setminus T \text{ and } t \in \tilde{T} \} \]

is quasi-dense in $W$ with respect to the word metric, then $W\partial \Sigma(W_T, T)$ is dense in $\partial \Sigma(W, S)$, where $W_T$ is the essential parabolic subgroup of $(W_T, T)$. 
If $W$ is a hyperbolic Coxeter group, then $W\partial\Sigma(W_T, T)$ is dense in $\partial\Sigma(W, S)$ for any $T \subset S$ such that $W_T$ is infinite.

As an application of Theorem 5, we have obtained the following corollary in [8].

**Corollary 6.** Let $(W, S)$ be a Coxeter system and let $T$ be a subset of $S$ such that $W_T$ is infinite. Suppose that there exist a maximal spherical subset $U$ of $S$ and an element $s \in S$ such that $o(su) \geq 3$ for every $u \in U$ and $o(su_0) = \infty$ for some $u_0 \in U$. If

1. $s \not\in T$ and $u_0 \in \tilde{T}$, or
2. $u_0 \not\in T$ and $s \in \tilde{T}$,

then $W\partial\Sigma(W_T, T)$ is dense in $\partial\Sigma(W, S)$.

Concerning $W$-invariantness of $\partial\Sigma(W_T, T)$, the following theorem is known.

**Theorem 7** ([6]).

1. Let $(W, S)$ be a Coxeter system and $T \subset S$. Then $\partial\Sigma(W_T, T)$ is $W$-invariant if and only if $W = W_T \times W_{S \setminus \tilde{T}}$.
2. Let $(W, S)$ be an irreducible Coxeter system and let $T$ be a proper subset of $S$ such that $W_T$ is infinite. Then $\partial\Sigma(W_T, T)$ is not $W$-invariant.

Here the following problem is open.

**Problem.** Let $(W, S)$ be a Coxeter system and let $T$ be a subset of $S$ such that $W_T$ is infinite. Is it the case that if $\partial\Sigma(W_T, T)$ is not $W$-invariant then $W\partial\Sigma(W_T, T)$ is dense in $\partial\Sigma(W, S)$? Particularly, is it the case that if $(W, S)$ is an irreducible Coxeter system then $W\partial\Sigma(W_T, T)$ is dense in $\partial\Sigma(W, S)$ for any subset $T$ of $S$ such that $W_T$ is infinite?

**References**


appear.


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