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On Numerical Semigroups of Genus 9

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§1. Introduction.
Let $\mathbb{N}_0$ be the additive semigroup of non-negative integers. A subsemigroup $H$ of $\mathbb{N}_0$ is called a numerical semigroup if the complement $\mathbb{N}_0 \setminus H$ of $H$ in $\mathbb{N}_0$ is a finite set. The cardinality $g(H)$ of the set $\mathbb{N}_0 \setminus H$ is called the genus of $H$. In this paper we are interested in numerical semigroups of genus 9. For a non-singular irreducible curve $C$ over an algebraically closed field $k$ of characteristic 0 (which is called a curve in this paper) and its point $P$ we set

$$H(P) = \{ n \in \mathbb{N}_0 | \exists \text{ a rational function } f \text{ on } C \text{ with } (f)_{\infty} = nP \}.$$ 

A numerical semigroup is Weierstrass if there exists a curve $C$ with its point $P$ such that $H(P) = H$. We have the following results:

**Fact 1.** Every numerical semigroup of genus $g \leq 8$ is Weierstrass. (See Lax [10], Komeda [4] and Komeda-Ohbuchi [8] for the case $g = 4$, $5 \leq g \leq 7$ and $g = 8$ respectively.)

We note that for any $g \geq 16$ there exists a non-Weierstrass numerical semigroup of genus $g$ (see Buchweitz [1].) A numerical semigroup $H$ is primitive if the largest positive integer not in $H$ is less than twice the least positive integer in $H$. Then we know the following fact:

**Fact 2.** Every primitive numerical semigroup of genus 9 is Weierstrass. (See Komeda [6].)

We want to study non-primitive numerical semigroups of genus 9.

An $n$-semigroup, i.e., a numerical semigroup in which the least positive integer is $n$. When $n$ is lower, we have the following result:

**Fact 3.** For $1 \leq n \leq 5$ every $n$-semigroup is Weierstrass. (See Macalachlan [11], Komeda [2] and [3] for the case $n = 3$, $n = 4$ and $n = 5$ respectively.)

Moreover, we have the following facts for two kinds of numerical semigroups:

**Fact 4.** Every $g$-semigroup of genus $g$ is Weierstrass. (See Pinkham [12].)
Fact 5. There is a unique non-primitive \((g-1)\)-semigroup of genus \(g\), which is Weierstrass. (See Komeda [5].)

Therefore, we are interested in non-primitive \(n\)-semigroups of genus 9 for \(n = 6, 7\).


**Definition 1.** A numerical semigroup \(H\) with \(#\mathcal{M}(H) = m\) is said to be of toric type if there are a positive integer \(l\), monomials \(g_j\)'s \((j = 1, \ldots, l + m - 1)\) in \(k[X_1, \ldots, X_m]\) and a saturated subsemigroup \(S\) of \(\mathbb{Z}^l\) generated by \(b_1, \ldots, b_{l+m-1}\) which generates \(\mathbb{Z}^l\) as a group such that

\[
\begin{array}{ccc}
\text{Spec } k[H] & \hookrightarrow & \text{Spec } k[X_1, \ldots, X_m] \\
\downarrow & & \downarrow \\
\text{Spec } k[S] & \hookrightarrow & \text{Spec } k[Y_1, \ldots, Y_{l+m-1}]
\end{array}
\]

where the horizontal maps are the embeddings through the generators and the right vertical map is induced by the \(k\)-algebra morphism from \(k[Y_1, \ldots, Y_{l+m-1}]\) to \(k[X_1, \ldots, X_m]\) sending \(Y_j\) to \(g_j\).

**Definition 2.** A 2m-semigroup \(H\) is of double covering type if there is a double covering \(\pi : C \rightarrow C_0\) of curves with ramification point \(P\) such that \(H(P) = H\).

We can show the following:

**Theorem 1.** Every non-primitive 6-semigroup of genus 9 is either of toric type or double covering type, hence Weierstrass. (See Komeda [9].)


We know that every non-primitive 7-semigroup of genus 9 is generated by 5 or 6 elements. We list up all non-primitive 7-semigroups of genus 9.

**Remark 2.** A non-primitive 7-semigroup of genus 9 generated by 5 elements is one of the following:

\(<7,9,10,11,13>, <7,9,10,11,12>, <7,9,10,12,13>, <7,8,11,12,13>\).

**Theorem 3.** Every non-primitive 7-semigroup of genus 9 generated by 5 elements is of toric type, hence Weierstrass. (See Komeda [7]).

**Remark 4.** A non-primitive 7-semigroup of genus 9 generated by 6 elements is one of the following:

\(<7,9,11,12,13,17>, <7,9,11,12,13,15>, <7,10,11,12,13,16>, <7,10,11,12,13,15>\).

First, we shall show that \(<7,9,11,12,13,17>\) is of toric type. We set \(a_1 = 7, a_2 = 9, a_3 = 11, a_4 = 12, a_5 = 13, a_6 = 17\). Then we have a generating system of
relations among $a_1, a_2, a_3, a_4, a_5$ and $a_6$ as follows:

$$3a_1 = a_2 + a_4, 2a_2 = a_1 + a_3, 2a_3 = a_2 + a_5, 2a_4 = a_1 + a_6, 2a_5 = a_2 + a_6,$$

$$2a_6 = a_2 + a_4 + a_5, a_1 + a_5 = a_2 + a_3, a_1 + a_6 = a_3 + a_5, 2a_1 + a_3 = a_4 + a_5,$$

$$a_3 + a_6 = a_1 + a_2 + a_4, a_5 + a_6 = a_1 + a_3 + a_4, 2a_1 + a_2 = a_3 + a_4,$$

$$2a_1 + a_4 = a_2 + a_6, a_4 + a_6 = a_1 + a_2 + a_5.$$

We set

$$b_i = e_i \in \mathbb{Z}^6, i = 1, \ldots, 6, b_7 = (1, 1, -1, 0, 0, 0), b_8 = (1, 0, -1, 0, 0, 0),$$

$$b_9 = (1, 0, 1, -1, 1, 0), b_{10} = (1, 0, 1, 1, 0, 1), b_{11} = (0, 0, 2, -1, 0, 0).$$

Let $S$ be the subsemigroup of $\mathbb{Z}^6$ generated by $b_1, \ldots, b_{11}$. Then $\text{Spec } k[S]$ is a 6-dimensional affine toric variety. We have a fiber product

$$\text{Spec } k[H] \hookrightarrow \text{Spec } k[X_1, \ldots, X_6] = \mathbb{A}^6$$

$$\downarrow \quad \varnothing$$

$$\text{Spec } k[S] \hookrightarrow \text{Spec } k[Y_1, \ldots, Y_{11}] = \mathbb{A}^{11}$$

where $\eta : k[Y_1, \ldots, Y_{11}] \hookrightarrow k[X_1, \ldots, X_6]$ is the $k$-algebra homomorphism sending $Y_i$ to $\xi_i$ for $1 \leq i \leq 11$ where

$$\xi_1 = X_1, \xi_2 = X_6, \xi_3 = X_3, \xi_4 = X_5, \xi_5 = X_1, \xi_6 = X_6,$$

$$\xi_7 = X_5, \xi_8 = X_2, \xi_9 = X_4, \xi_{10} = X_4, \xi_{11} = X_2.$$ Hence, the numerical semigroup $\langle 7, 9, 11, 12, 13, 17 \rangle$ is Weierstrass.

Second, we shall show that $\langle 7, 9, 11, 12, 13, 15 \rangle$ is of toric type. We set $a_1 = 7, a_2 = 9, a_3 = 11, a_4 = 12, a_5 = 13, a_6 = 15$. Then we have a generating system of relations among $a_1, a_2, a_3, a_4, a_5$ and $a_6$ as follows:

$$3a_1 = a_2 + a_4, 2a_2 = a_1 + a_3, 2a_3 = a_1 + a_6, 2a_4 = a_3 + a_5, 2a_5 = a_3 + a_6,$$

$$2a_6 = a_1 + a_3 + a_4, a_1 + a_5 = a_2 + a_3, a_1 + a_6 = a_2 + a_5,$$

$$2a_1 + a_3 = a_4 + a_5, a_5 + a_6 = 2a_1 + a_4, a_5 + a_6 = a_1 + a_2 + a_4,$$

$$2a_1 + a_2 = a_3 + a_4, 2a_1 + a_5 = a_4 + a_6, a_2 + a_6 = a_3 + a_5.$$

We set

$$b_i = e_i \in \mathbb{Z}^4, i = 1, \ldots, 4, b_5 = (1, 1, -1, 0), b_6 = (-1, 1, 1, 0),$$

$$b_7 = (-1, 0, 2, 0), b_8 = (2, 0, -1, 1), b_9 = (-1, 2, 0, -1).$$
Let $S$ be the subsemigroup of $\mathbb{Z}^4$ generated by $b_1, \ldots, b_9$. Then Spec $k[S]$ is a 4-dimensional affine toric variety. We have a fiber product

$$\begin{array}{ccc}
\text{Spec } k[H] & \leftarrow & \text{Spec } k[X_1, \ldots, X_6] = \mathbb{A}^6 \\
\downarrow & & \downarrow^{\eta} \\
\text{Spec } k[S] & \leftarrow & \text{Spec } k[Y_1, \ldots, Y_9] = \mathbb{A}^9
\end{array}$$

where $\eta : k[Y_1, \ldots, Y_9] \rightarrow k[X_1, \ldots, X_6]$ is the $k$-algebra homomorphism sending $Y_i$ to $\xi_i$ for $1 \leq i \leq 9$ where

$$\begin{align*}
\xi_1 &= X_1, \xi_2 = X_5, \xi_3 = X_2, \xi_4 = X_1, \xi_5 = X_6, \xi_6 = X_4, \\
\xi_7 &= X_3, \xi_8 = X_4, \xi_9 = X_4.
\end{align*}$$

Hence, the numerical semigroup $(7, 9, 11, 12, 13)$ is Weierstrass.

Third, we consider the semigroup $(7, 10, 11, 12, 13, 16)$. We set

$$a_1 = 7, a_2 = 10, a_3 = 11, a_4 = 12, a_5 = 13, a_6 = 16.$$ 

Then we have a generating system of relations among $a_1, a_2, a_3, a_4, a_5$ and $a_6$ as follows:

$$\begin{align*}
3a_1 &= a_2 + a_3, 2a_2 = a_1 + a_5, 2a_3 = a_2 + a_4, 2a_4 = a_3 + a_5, \\
2a_5 &= a_2 + a_6, 2a_6 = a_1 + a_4 + a_5, a_1 + a_6 = a_2 + a_5, a_1 + a_6 = a_3 + a_4, \\
2a_1 + a_2 &= a_3 + a_5, 2a_1 + a_3 = a_4 + a_5, 2a_1 + a_4 = a_2 + a_6, \\
2a_1 + a_5 &= a_3 + a_6, a_4 + a_6 = a_1 + a_2 + a_3, a_5 + a_6 = a_1 + a_2 + a_4.
\end{align*}$$

Let $S$ be the subsemigroup of $\mathbb{Z}^4$ generated by

$$b_i = e_i \in \mathbb{Z}^4, i = 1, \ldots, 4, b_5 = (2, -1, 0, 0), b_6 = (3, -2, 0, 0),$$

$$b_7 = (-1, 2, 1, 0), b_8 = (-2, 2, 1, 1), b_9 = (4, -3, -1, 0).$$

Then Spec $k[S]$ is a 4-dimensional non-normal variety such that we have a fiber product

$$\begin{array}{ccc}
\text{Spec } k[H] & \leftarrow & \text{Spec } k[X_1, \ldots, X_6] = \mathbb{A}^6 \\
\downarrow & & \downarrow^{\eta} \\
\text{Spec } k[S] & \leftarrow & \text{Spec } k[Y_1, \ldots, Y_9] = \mathbb{A}^9
\end{array}$$

Here $\eta : k[Y_1, \ldots, Y_9] \rightarrow k[X_1, \ldots, X_6]$ is the $k$-algebra homomorphism sending $Y_i$ to $\xi_i$ for $1 \leq i \leq 9$ where

$$\begin{align*}
\xi_1 &= X_2, \xi_2 = X_1, \xi_3 = X_1, \xi_4 = X_3, \xi_5 = X_5, \xi_6 = X_6, \\
\xi_7 &= X_3, \xi_8 = X_4, \xi_9 = X_4.
\end{align*}$$

Lastly we investigate the semigroup $(7, 10, 11, 12, 13, 15)$. We set

$$a_1 = 7, a_2 = 10, a_3 = 11, a_4 = 12, a_5 = 13, a_6 = 15.$$
Then we have a generating system of relations among $a_1, a_2, a_3, a_4, a_5$ and $a_6$ as follows:

$$3a_1 = a_2 + a_3, 2a_2 = a_1 + a_5, 2a_3 = a_1 + a_6, 2a_4 = 2a_1 + a_2,$$

$$2a_5 = 2a_1 + a_4, 2a_6 = a_1 + a_3 + a_4, a_1 + a_6 = a_2 + a_4, a_2 + a_6 = a_4 + a_5, 2a_1 + a_4 = a_3 + a_6,$$

$$2a_1 + a_5 = a_4 + a_6, a_2 + a_6 = a_3 + a_4, a_5 + a_6 = a_1 + a_2 + a_3.$$

Let $S$ be the subsemigroup of $\mathbb{Z}^3$ generated by

$$b_i = e_i \in \mathbb{Z}^3, i = 1, \ldots, 3, b_4 = (1, 1, -1), b_5 = (1, -1, 1),$$

$$b_6 = (2, -2, 1), b_7 = (-2, 1, 1), b_8 = (-1, 3, -1).$$

Then $\text{Spec } k[S]$ is a 3-dimensional non-normal variety where we have a fiber product

$$\text{Spec } k[H] \rightarrow \text{Spec } k[X_1, \ldots, X_6] = \mathbb{A}^6$$

$$\downarrow \quad \quad \square$$

$$\downarrow^{\eta}$$

$$\text{Spec } k[S] \rightarrow \text{Spec } k[Y_1, \ldots, Y_8] = \mathbb{A}^8$$

Here $\eta : k[Y_1, \ldots, Y_8] \rightarrow k[X_1, \ldots, X_6]$ is the $k$-algebra homomorphism sending $Y_i$ to $\xi_i$ for $1 \leq i \leq 8$ where

$$\xi_1 = X_2, \xi_2 = X_4, \xi_3 = X_6, \xi_4 = X_1, \xi_5 = X_5, \xi_6 = X_3, \xi_7 = X_1, \xi_8 = X_3.$$

References


