Title
RELATIVE E-RINGS (Algorithmic problems in algebra, languages and computation systems)

Author(s)
Hirano, Yasuyuki

Citation
数理解析研究所講究録 (2006), 1503: 57-59

Issue Date
2006-07

URL
http://hdl.handle.net/2433/58477

Type
Departmental Bulletin Paper

Textversion
publisher

Kyoto University
RELATIVE E-RINGS

Faculty of Science, Okayama University

1. INTRODUCTION

In [4, Problem 45], L. Fuchs posed the following problem:
Which rings $R$ satisfy $R \cong \text{End}(R^+)$? The author presents
particular, he studied commutative rings $R$ satisfying $R \cong \text{End}(R^+)$
and he called such rings E-rings. For E-rings, see the book "Additive
Groups of Rings I ([3])" by S. Feigelstock. Recently, in [2] R. Göbel,
S. Shelah and L. Strüngmann constructed noncommutative rings $R$ sat-
sifying $R \cong \text{End}(R^+)$. 

2. RELATIVE E-RINGS

Let $R$ be a ring with identity. By $R^+$ we denote the additive group
of the ring $R$. For an element $a \in R$, we have the mapping $a_l : R \to R$
deﬁned by $x \to ax$. $a_l$ is called the left multiplication induced by $a$.
Similarly we have the right multiplication induced by $a$. Obviously the
sets $\{a_l \mid a \in R\}$ and $\{a_r \mid a \in R\}$ form rings. We denote these rings
by $R_l$ and $R_r$, respectively.

Definition 2.1. A ring $R$ is called an E-ring if $R_l = \text{End}(R^+)$. 

The detailed version of this paper has been submitted for publication elsewhere.
This notion is generalized as follows.

**Definition 2.2.** Let $S$ be a ring and let $R$ be a ring such that $R$ is a right $S$-module. A ring $R$ is called a left $E$-ring relative to $S$ if $R_l = \text{End}_S(R_S)$.

Let $\mathbb{Z}$ denote the ring of rational integers. Then a left $E$-ring relative $\mathbb{Z}$ is nothing else but an $E$-ring. Let $S$ be a ring and let $R$ be a ring such that $R$ is a right $S$-module. Then $\text{End}_S(R_S)$ always contains $R_l$. Hence we can say that left $E$-rings relative to $S$ are those rings $R$ such that $\text{End}_S(R_S)$ is small as possible.

From the definition of a relative $E$-ring, the following is obvious.

**Proposition 2.3.** Let $S$ be a ring and let $R$ be a ring such that $R$ is a right $S$-module. If $R$ is a left $E$-ring relative to $S$ and if $f \in \text{End}(R_S)$, then $f(R)$ is a principal right ideal of $R$.

Also we can easily see the following:

**Proposition 2.4.** Let $S$ be a ring and let $R$ be a ring such that $R$ is a right $S$-module.

1. The ring $R$ is a left $E$-ring relative to $S$.

2. Every element of $R_r$ commutes with any element of $\text{End}_S(R_S)$.

As a corollary, we have the following characterizations of an $E$-algebra relative to a commutative ring.

**Corollary 2.5.** Let $S$ be a commutative ring and $R$ be an $S$-algebra. Then the following are equivalent:

1. $R$ is an $E$-ring relative to $S$.

2. $R_r = \text{End}_S(R_S)$.

3. $R$ is a commutative ring and $R \cong \text{End}_S(R_S)$.

4. $\text{End}_S(R_S)$ is a commutative ring.
Example 2.6. Let $R$ be a commutative ring and let $S = R[x, y]$. Consider the ring $A = S/(x) \oplus S/(y)$. Then $A$ is a $S$-algebra, but $A$ is not a cyclic $S$-module. Clearly $End_S(A) \cong A$ and so $End_S(A)$ is commutative. Therefore $A$ is a relative E-algebra over $S$.

Example 2.7. Let $R$ be a commutative ring and let $S$ be a multiplicatively closed subset of $R$. Then $S^{-1}R$ is a relative $E$-algebra over $S$.

Let $R$ be a commutative ring and let $\{I_n\}_{n \geq 0}$ be a family of ideals of $R$ satisfying the condition that $I_n \subseteq I_m$ whenever $n \geq m$. We can then define a topology on the set $R$ with an open basis $\{a + I_n \mid a \in R, n \geq 0\}$. This topology is called the linear topology defined by a family of ideals $\{I_n\}_{n \geq 0}$. Then we can construct the completion $\hat{R}$ of $R$. It is well-known that

$$\hat{R} \cong \lim_{\longleftarrow n} R/I_n.$$ 

Example 2.8. Let $R$ be a commutative ring and consider the linear topology defined by a family of ideals $\{I_n\}_{n \geq 0}$. Then the completion $\hat{R}$ of $R$ is a relative $E$-algebra over $R$.

REFERENCES