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# A Note on the Growth of Neighborhoods of Cellular Automata

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**Abstract**

In this paper we investigate the growth of neighborhoods of cellular automata following such of finitely generated groups particularly concerning the Garden of Eden theorem.

## 1 Introduction

A cellular automaton (CA for short) is a uniformly structured information processing system defined on a regular discrete space $S$, which is typically presented by a Cayley graph of a finitely generated group. The same finite automaton (cell) is placed at every point of the space. Every cell simultaneously changes its state following the local function defined on the neighboring cells. The neighborhood $N$ is also spatially uniform. Most studies on CA assume the standard neighborhoods after John von Neumann and E. F. Moore.

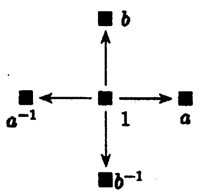


Figure 1: The von Neumann neighborhood $N_V$

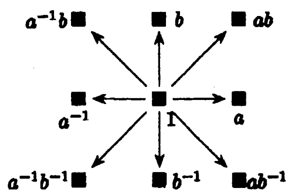


Figure 2: The Moore neighborhood $N_M$

Changing the view point, however, we posed an algebraic theory of neighborhoods of CA for clarifying the significance of the neighborhood itself, where the neighborhood $N$ can be an arbitrary finite subset of $S$, see Nishio, Margenstern and von Haeseler (2004, 2005) [11, 13], where the main topics is a question if an arbitrary neighborhood $N$ *fills* or generates $S$. Evidently the von Neumann and the Moore neighborhoods fill $Z^2$. A typical nonstandard neighborhood a *3-horse* $N_{3H} = \{a^2b, a^{-2}b, ab^{-2}\}$ was proved to fill $\mathbb{Z}^2$ [12]. Note that $N_{3H}$ dose not contain the identity 1 of the group (origin of the space).

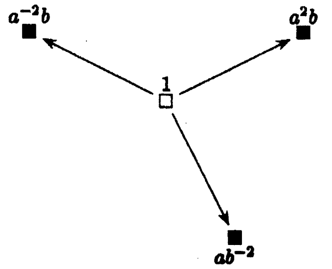


Figure 3: A 3-horse $N_{3H}$.

Now in this paper, assuming that the neighborhood fills the space, we study the *growth of neighborhoods* of cellular automata following the growth of finitely generated groups obtained my many authors with respect to the Garden of Eden theorem (GOE theorem for short), see Machi and Mignosi (1993) [6] and others. We discuss the growth of neighborhoods of CA particularly on the $n$-dimensional Euclidean space and the hyperbolic plane. For example we will see that the GOE theorem holds for a CA with the neighborhood $N_{3H}$ in $Z^2$ since its growth is polynomial, but not for a CA with a similar neighborhood in the pentagrid $\{5,4\}$ since its growth is exponential.

# 2 Preliminaries

## 2.1 Cellular Automaton CA

A CA is defined by a 4-tuple $(\Gamma(S), N, Q, f)$.

- *Cellular space* $\Gamma(S)$ is a Cayley graph of a finitely generated group $S = \langle G|R\rangle$ with generators $G$ and relators $R$. If $G = \{g_1, g_2, ..., g_r\}$, every element of $S$ is presented by a word $x \in (G \cup G^{-1})^*$, where $G^{-1} = \{g^{-1}|\ g \cdot g^{-1} = 1,\ g \in G\}$. The set $R$ of relators is written as

$$R = \{w_i = w_i' \mid w_i, w_i' \in (G \cup G^{-1})^*, i = 1, ..., n\}. \tag{1}$$

  For $x, y \in \Gamma(S)$, if $y = xg$, where $g \in G \cup G^{-1}$, then an edge labelled by $g$ is drawn from vertex $x$ to vertex $y$. In the sequel $\Gamma(S)$ and $S$ are not distinguished.

- *Neighborhood* $N = \{n_1, n_2, ..., n_s\}$ is a finite subset of $S$. For any cell $x \in S$, the information of cell $xn_i$ reaches $x$ in a unit of time. The set of all neighborhoods is denoted by $\mathcal{N}^S$. If $N \subset N'$, where $N$ and $N' \in \mathcal{N}^S$, $N$ is called a *subneiborhood* of $N'$. When $S$ is understood, the set of neighborhoods is written without superfix $S$. The cardinality $\#(N)$ is called the neighborhood size of CA. The set of the neighborhoods of size $s$ is denoted by $\mathcal{N}_s$.

- *Set of cell states* $Q = GF(q)$ where $q = p^n$ with prime $p$ and positive integer $n$. $Q = \mathbb{Z}/m\mathbb{Z}$ is also considered.

- *Local map* $f : Q^N \to Q$, where an element of $Q^N$ is called a local configuration.

- *Global map* $F : C \to C$, where an element of $C = Q^S$ is called a global configuration. $F$ is uniquely defined by $f$ and $N$ as follows.

$$F(c)(x) = f(c(xn_1), c(xn_2), \cdots, c(xn_s)), \tag{2}$$

where $c(x)$ is the state of cell $x \in S$ for any $c \in C$.

When starting with a configuration $c$, the behavior (trajectory) of CA is given by

$$F^{t+1}(c) = F(F^t(c)) \text{ for any } t \geq 0, \text{ where } F^0(c) = c. \tag{3}$$

## 2.2 Neighborhood and Neighbors

Given a *neighborhood* $N = \{n_1, n_2, ..., n_s\} \subset S$ for a cellular space $S = \langle G \mid R \rangle$, we recursively define the *neighbors* of CA. Let $p \in S$.

(1) The *1-neighbors* of $p$, denoted as $pN^1$, is the set

$$pN^1 = \{pn_1, pn_2, ..., pn_s\}. \tag{4}$$

(2) The *$m$-neighbors* of $p$, denoted as $pN^m$, are given as

$$pN^m = pN^{m-1} \cdot N, \; m \geq 1, \tag{5}$$

where $pN^0 = \{p\}$. Note that the computation of $pn_i$ has to comply with the relation $R$. We may say that the information contained in the cells of $pN^m$ reaches the cell $p$ after $m$ time steps.

(3) *$\infty$-neighbors* of $p$, denoted as $pN^\infty$, is defined by

$$pN^\infty = \bigcup_{m=0}^{\infty} pN^m. \tag{6}$$

Without loss of generality, we can concentrate on the $m$-neighbors of the identity element 1 of $S$, which is called *$m$-neighbors of CA* and denoted by $N^m$. Then

(4) $\infty$-neighbors of 1, denoted as $N^\infty$ and called the *neighbors of CA*, is given by

$$N^\infty = \bigcup_{m=0}^{\infty} N^m. \tag{7}$$

In order to discuss the growth of neighbors later, we define here the $m$-ball of radius $m$, denoted by $\overline{N^m}$, as

$$\overline{N^m} = \bigcup_{k=0}^{m} N^k. \tag{8}$$

Obviously, if $1 \in N$, then $\overline{N^m} = N^m$. In Sections 3 and 4, we will discuss the difference between the $m$-ball and the ball of radius $n$ in the group theory, $B_n = \{w \mid |w| \leq n, w \in S\}$. If

$N = G \cup G^{-1}$, which is the case of $N_V$, then $\overline{N^m} = B_m$.

The *intrinsic m-neighbors* $[N^m] = N^m \setminus N^{m-1}$ are considered as the cells whose information can reach the origin in exactly $m$ steps. Obviously, $N^\infty = \bigcup_{m=0}^{\infty}[N^m]$.

Now we have an algebraic result, which is proved by the fact that the procedure to generate a subsemigroup is the same as the above mentioned recursive definition of $N^\infty$.

**Proposition 1**

$$N^\infty = \langle N \mid R \rangle_{sg}, \tag{9}$$

*where $\langle N \mid R \rangle_{sg}$ means the semigroup obtained by concatenating the words from $N$ complying with $R$.*

In general, if $\langle N \mid R \rangle_{sg} = S' \subseteq S$, $N$ is said to $sg$-generates $S'$.

We also have the following easily proved proposition.

**Proposition 2**

$$\langle N \mid R \rangle_g = \langle N \cup N^{-1} \mid R \rangle_{sg}, \tag{10}$$

*where $\langle N \mid R \rangle_g$ is the smallest subgroup of $S$ which contains $N$.*

If $N = G$, then we have the following lemma as a corollary to Proposition 2.

**Lemma 1**

$$\langle g_1, g_2, ..., g_r | R \rangle_g = \langle g_1, g_2, ..., g_r, g_1^{-1}, g_2^{-1}, ..., g_r^{-1} | R \rangle_{sg}. \tag{11}$$

Example: $\mathbb{Z}^2 = \langle a, b | \; ab = ba \rangle_g = \langle a, b, a^{-1}, b^{-1} | \; ab = ba \rangle_{sg}$

# 3 Growth of groups

The *growth function* $\gamma_S$ of a finitely generated discrete group $S = \langle G | R \rangle$ is defined by means of the cardinality of the ball of radius $n$. That is

$$\gamma_S(n) = \# B_n = \#\{w \mid |w| \leq n, \; w \in S\}. \tag{12}$$

For a free group $F = \langle a, b | \emptyset \rangle$, $\gamma_F = 2^n$. For 2-dimensional Euclidean space $S = \mathbb{Z}^2 = \langle a, b | ab = ba \rangle$, $\gamma_S = 2n^2 + 2n + 1$.

## 3.1 Growth rate of groups

For most theory concerning the growth of groups, its asymptotic behavior called the *growth rate* is of interest. Though there are several definitions of the growth rate, they are all equivalent. The following one is due to Babai (1997) [1].

**Definition 1** *Two monotone non-decreasing functions $f_1, f_2 : \mathbb{N} \to \mathbb{N}$ are said to be equivalent ($f_1 \sim f_2$), if there exist constants*
$c_1, c_2, C_1, C_2, n_0 > 0$ *such that for all $n \geq n_0$,*

$$C_1 f_1(c_1 n) \leq f_2(n) \leq C_2 f_1(c_2 n). \tag{13}$$

The relation $\sim$ is evidently an equivalence relation. The equivalence class of $f$ is denoted by $[f]$. Let $[f_1]$ and $[f_2]$ be the equivalence classes to which $f_1$ and $f_2$ belong, respectively and define an order $[f_1] \precsim [f_2]$ if $Cf_1(cn) \leq f_2(n)$ for constants $C, c, n_0 \geq 0$ and for all $n \geq n_0$.

**Example 1** $[n^2] \precnsim [n^3]$, $[a^n] \precnsim [n^b]$, *and* $a^n \sim b^n$ for any positive integers $n,\ a,\ b \geq 1$.

The *growth rate* $[\gamma_S]$ of a group $S$ is an equivalence class to which $\gamma_S$ belongs. For $\mathbb{Z}^2$, $[\gamma_S] \sim n^2$. The growth rate of groups is simply called the *growth of groups*.

## 3.2 Past results on the growth of groups

1) The growth of a group is independent from the generators, Milnor(1968)[8].

**Lemma 2** *(Lemma 1 of [8]) Let $G_1 = \{g_1, ..., g_p\}$ and $G_2 = \{h_1, ..., h_q\}$ be two different sets of generators and let $\gamma_1(n)$ and $\gamma_2(n)$ be the corresponding growth functions. Then, there exist positive constants $k_1$ and $k_2$ so that*

$$\gamma_2(n) \leq \gamma_1(k_1 n)$$

*and*

$$\gamma_1(n) \leq \gamma_2(k_2 n)$$

*for all $n$.*

By Lemma 2 we have $[\gamma_1] \sim [\gamma_2]$ in Babai's sense.

2) There are three classes of the growth of groups; *polynomial $\precnsim$ subexponential $\precnsim$ exponential.* Grigorchuk (1983) gives a group $S$ which has subexponential growth [3][4];
$2^{n^{1/2-\epsilon}} \precsim [\gamma_S] \precsim 2^{n^\alpha}$ with $\epsilon > 0$ and $\alpha = \log_{32} 31 < 1$.

3) A nilpotent group $G$ has polynomial growth [14][2];
$[\gamma_G] \sim n^d$, where $d = \sum_k k\mathrm{rank}(G_k/G_{k+1})$ and $\{G_k\}$ is the lower central series of G.

# 4 Growth of neighborhoods

The *growth function* $\delta_{(N,S)}$ of neighborhood $N$ in $S = \langle G|R\rangle$ is defined by

$$\delta_{(N,S)}(m) = \#\{w \mid w \in \overline{N^m}\}, \tag{14}$$

where $\overline{N^m}$ is the $m$-ball of radius $m$ defined by Equation (8). Note that $\overline{N^m}$ is generally different from $B_m$. For example, in case of a 3-horse $N_{3H}$, the word $a$ of length 1 is included by $N^3$ (and $\overline{N^3}$) but not by $N^1$. Moreover the identity 1 of length 0 first appears in $N^{12}$ [12].

The *growth rate* $[\delta_{(N,S)}]$ of a neighborhood $N \subseteq S$ is similarly defined to be an equivalence class to which $\delta_{(N,S)}$ belongs.

We discuss here the difference between the growth functions/rates of the neighborhood $N$ and of the group $S$ itself. The problem is not trivial even though we have Lemma 2 by Milnor. First, it is seen that if $N = G \cup G^{-1}$ which is the case of $N_V$, then $\delta_{(N,S)} = \gamma_S$.

**Example 2** *For the von Neumann and the Moore neighborhoods in $S = \mathbb{Z}^2$, we have $\delta_{(N_V,S)}(m) = 2m^2 + 2m + 1 = \gamma_S(m)$ and $\delta_{(N_M,S)}(m) = 4m^2 + 4m + 1 > \gamma_S(m)$, respectively. Both neighborhoods have the same growth rate $m^2$, which is equal to the growth rate of $S = \mathbb{Z}^2$.*

*We show below a numerical computation of the growth function of $N_{3H}$ compared with the von Neumann neighborhood $N_V$. The 3-horse seems to grow faster than the von Neumann, but both growth rates are equal to $n^2$.*

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\delta_{N_V}(n)$ | 5 | 13 | 25 | 41 | 61 | 81 | 113 | 145 | 181 | 221 |
| $\delta_{N_{3H}}(n)$ | 3 | 9 | 18 | 35 | 62 | 100 | 147 | 208 | 277 | 353 |

## 4.1 Basic properties of the growth of neighborhoods

First we notice some basic properties of the growth of neighborhoods.

**Lemma 3** *If $N' \subseteq N$ then*

$$\delta_{(N',S)} \leq \delta_{(N,S)}$$

.

From this lemma and Proposition 2, we have

**Lemma 4** *For any $N \subseteq S$,*

$$\delta_{(N,S)} \leq \delta_{(N \cup N^{-1},S)} \text{ and } [\delta_{(N,S)}] \precsim [\gamma_N], \tag{15}$$

*where $\gamma_N$ is the growth function of the group $\langle N \mid R\rangle_g$.*

Then we have the following theorem.

**Theorem 1** *For a cellular space $S = \langle G|R\rangle_g$ and any neighborhood $N \subset S$,*

$$[\delta_{(N,S)}] \precsim [\gamma_S], \tag{16}$$

*where the equivalence holds if and only if $N$ fills $S$.*

*Proof*: By Lemmas 3 and 4 we have the theorem. ■

## 4.2 Growth of the hyperbolic horse

The CA on the hyperbolic space has been investigated by M.Margenstern and other authors, see [7] for his latest literature. Generally, the hyperbolic space does not allow a Cayley graph presentation of a group. However, the pentagrid $\{5,4\}$ can be treated by means of its dual hyperbolic grid $\{4,5\}$, which is seen to be a Cayley graph of the group $H_{\{4,5\}}$ as is shown below.

$H_{\{4,5\}} = \langle 1,2,3,4,5 \mid 12 = 21, 23 = 32, 34 = 43, 45 = 54, 51 = 15,\ i = i^{-1}, 1 \leq i \leq 5\rangle$, where $\{1,2,3,4,5\}$ is the symbol set of generators.

First, as for the growth of the group $H_{\{4,5\}}$, the following proposition holds.

**Proposition 3** *The growth rate of the pentagrid* $\{4,5\}$ *is exponential.*

Next, we investigate the horse power problem on the hyperbolic plane: for a neighborhood $N$, decide if $N$ fills the space $S$ or not. If $N$ consists of $s$ elements, it is called an $s$-horse.

First we give the following Theorem 3.8 of [13], which has been rewritten in my formulation.

**Theorem 2 (Theorem 3.8 of [13])** $N_{5HH}$ *fills* $H_{\{4,5\}}$, *where* $N_{5HH} = \{n_1, n_2, n_3, n_4, n_5\}$,

$$\begin{aligned} n_1 &\triangleq 145 \cdot 412 \cdot 254 = 4 \\ n_2 &\triangleq 251 \cdot 523 \cdot 315 = 5 \\ n_3 &\triangleq 312 \cdot 134 \cdot 421 = 1 \\ n_4 &\triangleq 423 \cdot 245 \cdot 532 = 2 \\ n_5 &\triangleq 534 \cdot 351 \cdot 143 = 3, \end{aligned}$$

*where for instance* $145 \cdot 412 \cdot 254 = 4$ *means that a concatenation of words 145, 412 and 254 makes 4.*

*Proof*: The above 5 elements $n_1 = 4, ..., n_5 = 3$ constitute the generators of $H_{\{4,5\}}$. ■

Finally, contrary to a 3-horse $N_{3H}$ in 2-dimensional Euclidean space, we see that a 2-horse $N_{2HH}$ is enough to fill $H_{\{4,5\}}$ by Theorem 3.9 of [13]. Then by Theorem 1, we have

**Proposition 4** *The growth rate of* $N_{2HH}$ *is exponential;* $[\delta_{N_{2HH}}] \sim 2^n$.

# 5 Garden of Eden theorem

The *Garden of Eden (GOE) theorem* was originally proved for $\mathbb{Z}^2$ by E.Moore(1962) [9] and J.Myhill(1963)[10].

**Definition 2** *A finite configuration (pattern) is called a Garden of Eden (GOE), if it is not in the image of F (A GOE has not an ancestor). Two distinct patterns* $p_1$ *and* $p_2$ *are called mutually erasable if two configurations* $c_1, c_2$*, which contain* $p_1$ *and* $p_2$*, respectively and coincide outside of the supports of* $p_1$ *and* $p_2$*, are mapped to the same configuration.*

**Theorem 3 (Moore)** *If there are mutually erasable patterns, then there are GOE patterns.*

**Theorem 4 (Myhill)** *If there are GOE patterns, then there are mutually erasable patterns.*

If there is no GOE patterns then $F$ is *surjective* and if there is no mutually erasable patterns then $F$ is *injective* when it is restricted to the finite configurations. Therefore these theorems together claim the following theorem, which is called the *GOE theorem* today.

**Theorem 5 (GOE theorem)** *F is surjective if and only if F is injective when it is restricted to the finite configurations.*

**Idea of Moore's proof of Theorem 3**: Let $\#(c(N^m))$ be the cardinality of different patterns contained by cells in $m$-neighbors $N^m$. For $S = \mathbb{Z}^2$ and Moore neighborhood, if there are mutually erasable patterns, then $\#(c(N^{m-1}))$ becomes greater than $\#(c(N^m))$ when $m$ becomes large enough, which implies the existence of GOE patterns. This proof is based on the fact that the growth of neighbors is not too fast (polynomial).

After the seminal papers by Moore and Myhill, group theorists have revealed that the GOE theorem holds for groups of polynomial and subexponential growth, but does not for exponential growth, see Machi&Mignosi(1993) [6] and Gromov(1999) [5]. The group theorists usually discuss the GOE theorem assuming the generators of the group as the neighborhood. This fact is one of the reasons why we are interested in the growth of neighborhoods.

### 5.1 GOE theorem for general neighborhoods

We discuss here the problem if the GOE theorem holds for CAs having the neighborhood which is not necessarily the generator set of the group.

**Theorem 6** *The GOE theorem holds for a CA which has a neighborhood of polynomial growth.*

*Proof*: The Moore's proof shown above generally applies to such a CA. ■

We conjecture that the GOE theorem does not hold for CAs having the neighborhoods of exponential growth.

## 6 Concluding remarks

In this paper, we have defined and analyzed the growth of neighborhoods of CA. A problem for future research is to consider other *growth-specific* properties than the GOE theorem. Many thanks are due to Maurice Margenstern and Thomas Worsch.

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