FINITE SEMIGROUPS AND DECIDABILITY OF AMALGAMATION BASES FOR SEMIGROUPS*

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In this paper, we prove that the decision problem of whether or not a finite semigroup $S$ is an amalgamation base for all semigroups is decidable.

1 Introduction and preliminaries

In [3], M. Spair investigated problems of amalgams in the class of finite semigroups and showed that it is undecidable whether or not an amalgam of finite semigroups is embedded in a semigroup or a finite semigroup. However it is not known whether or not the problem of decidability of amalgamation bases. In this paper we prove decidability of whether or not a finite semigroup is an amalgamation base for all semigroups.

Let $S$ be a semigroup. Let $M$ be a nonempty set with a unitary and associative operation of $S: S^1 \times M \to M((s, w) \mapsto sw)$, where $S^1$ is the monoid obtained from $S$ by adjoining a new identity $1$. Then $M$ is called a left $S$-set. Dually, right $S$-set is defined. If $M$ is a left $S$-set and a right $S$-set satisfying that $(sm)t = s(mt)$ for all $s, t \in S$ and $m \in M$, then $M$ is called an $S$-biset.

A relation $\rho$ of a left $S$-set $M$ [resp. right $S$-set] is called $S$-congruence if $(m, m') \in \rho$ and $s \in S$ implies $(sm, sm') \in \rho$ [resp. $(ms, sm's) \in \rho$]. Let $M, N$ be a right $S$-sets [left $S$-set]. Then a map $\phi: M \to N$ is called an $S$-map if $\phi(sm) = s\phi(m)$ for any $m \in M$ and $s \in S$ [resp. $\phi(sm) = s\phi(m)$] for any $m \in M$ and $s \in S$.

Result 1 ([5, Proposition 1.5]). Let $S$ be a semigroup and $A, B, C, D \in S$-Ens [Ens-$S$, $S$-Ens-$S$] such that $A \supset C$, $B \supset D$. Let $\alpha$ be a bijective $S$-map [ $(S,S)$-map] : $C \to D$. Then there exist $W \in S$-Ens [Ens-$S$, $S$-Ens-$S$] and injective $S$-maps [$(S,S)$-maps] $\beta : A \to W$, $\lambda : B \to W$ such that $\alpha \lambda = \beta$ on $C$, $W = A\beta \cup B\lambda$, $A\beta \cap B\lambda = C\beta$.

*This is an abstract and the paper will appear elsewhere.
In this case, we say that the left $S$-set [-biset] $W$ is the left $S$-set gluing $C$ and $D$ by $\xi$ and write $W = C \#_{\xi} D$, where $\xi = \beta^{-1} \alpha : A \to B$.

If $A$, $B$ are generated by $x$ and $y$ respectively and $\xi(x) = y$, then we write $C \neq_{x=y} D$ instead of $C \neq_{\xi} D$.

Let $\mathrm{Y}$ a left $S$-set and $s, t: y: \in S$ with $s_{i}y_{i} := t_{i+1}y_{i+1}$ for all $1 \leq i \leq n-1$. Then for any $1 \leq i \leq n-1$, we define left congruences $\rho(s_{i})$, $\rho(t_{1})$ on $S^{1}$ as follows:

$$\rho(s_{i}) = t_{i+1}y_{i+1} = \ldots = t_{n-1}y_{n} = \rho(t_{1})$$

Then we have the set $A'$ of equations $s_{i}y_{i} = t_{i+1}y_{i+1}$ in $X$ $(1 \leq i \leq n-1)$.

We call $Y$ relatively free the relatively free left $S$-set associated to $Y$ with respect to $A$.

2 The decision problem of amalgamation bases for all semigroups

A semigroup $S$ is called an amalgamation base in the class of all semigroups (simply called a semigroup amalgamation base) if for any semigroups $T_{i}$ $(i \in I)$ containing $S$ as a subsemigroup the semigroup amalgam $[T_{i} (i \in I) ; S]$ is embedded into a semigroup.

This is a characterization of semigroup amalgamation bases.

Result 2 [4, Theorem 2.2]. A semigroup $S$ is a semigroup amalgamation base if and only if for each $X \in \text{Ens-S}$, $Y \in \text{S-Ens}$ and $N \in \text{S-Ens-S}$ with $N \supset S^{1}$, the map $: X \otimes Y \to X \otimes N \otimes Y (x \otimes y \to x \otimes 1 \otimes y)$ is injective.

We recall Bulman-Flemming and MxDowell's characterization of equality in tenser product.

Result 3 [1, Lemma 1.2]. Let $X \in \text{Ens-S}$, $Y \in \text{S-Ens}$. Then $x \otimes y = x' \otimes y'$ in $X \otimes Y$ if and only if there exist $s_{1}, \ldots, s_{n}, t_{1}, \ldots, t_{n} \in S^{1}, x_{1}, \ldots, x_{n} \in X$ and $y_{2}, \ldots, y_{n} \in Y$
such that

\[
\begin{align*}
    x &= x_1 s_1, & s_1 y &= t_1 y_2 \\
    x_1 t_1 &= x_2 s_2, & s_2 y_2 &= t_2 y_3 \\
    & \vdots & \vdots \\
    x_{n-1} t_{n-1} &= x_n s_n, & s_n y_n &= t_n y' \\
    x_n t_n &= x'
\end{align*}
\]

(1)

Then we call the system of equations (1) a scheme of length \( n \) over \( X \) and \( Y \) joining \((x, y)\) to \((x', y')\).

**The main theorem.** The decision problem whether or not a finite semigroup is an amalgamation base for all semigroups is decidable.

### 2.1 Schemes and Automata

We know that a semigroup \( S \) is an amalgamation base for all semigroups if and only if so is the semigroup \( S^1 \) obtained from \( S \) by adjoining an identity element. So we assume that \( S \) is a monoid.

In this section we construct an automata associated to an equation on tensor product of a certain right \( S \)-set and certain a left \( S \)-set in order to complete the proof of Theorem 1.

Let \( S \) be a finite monoid.

I. Let \( RC(S) \) be the set of all right congruences of \( S \).

Let \( \{ (\xi, a) \mid \xi \in RC(S), a \in S \} \) be the sets of initial vertices and terminal vertices and \( \{ (\xi, a, b) \mid \xi \in RC(S), a, b \in S \} \) be the sets of vertices.

Edges are of the form \((\xi, a) \xrightarrow{\theta} (\xi', a', b')\), where \( \theta \) is an \( S \)-isomorphism \( : (\xi a)S \rightarrow (\xi' a')S \) with \( \theta(\xi a) = \xi'a' \) or of the form \((\xi, a, b) \xrightarrow{\theta} (\xi', a', b')\), where \( \theta \) is an \( S \)-isomorphism \( : (\xi b)S \rightarrow (\xi' a')S \) with \( \theta(\xi b) = \xi'a' \).

II. Let \( LC(S) \) be the set of all left congruences of \( S \).

Let \( \{ (\phi, u) \mid \phi \in LC(S), u \in S \} \) be the sets of initial vertices and terminal vertices and \( \{ (\phi, u, v) \mid \phi \in LC(S), u, v \in S \} \) be the sets of vertices.

Edges are of the form \((\phi, u) \xrightarrow{\theta} (\phi', u', v')\), where \( \theta \) is an \( S \)-isomorphism \( : (\phi u)S \rightarrow (\phi' u')S \) with \( \theta(\phi u) = \phi'u' \) or of the form \((\phi, u, v) \xrightarrow{\theta} (\phi', u', v')\), where \( \theta \) is an \( S \)-isomorphism \( : S(\phi v) \rightarrow S(\phi' u') \) with \( \theta(\phi v) = \phi'u' \).

III. \( \{ (\xi, a, b, m, \phi, u, v) \mid \xi \in RC(S), a, b \in S, m \in E(sS_S), \phi \in LC(S), u, v \in S \} \)

To obtain all schemes joining \((\xi_0,*,1,\rho_0,*,u)\) to \((\xi'_0,1,*,1,\phi'_0 u',*)\) over right \( S \)-sets, the bi-\( S \)-sets \( E(sS_S) \) and left \( S \)-sets, we make a non-deterministic automaton \( A(\xi_0,1,\phi_0,u, \)
$E(sS_s), \xi_0, a', \rho_0', u'$ as follows:

Vertices are of the form $(\xi, a, b, m, \phi, u, v)$ where $\xi \in RC(S), a, b, u, v \in S, m \in E(sS_s), \phi \in LC(S)$,

$(\xi_0, *, 1, 1, \phi_0, *, v_0)$ is the initial vertex, $(\xi_0', 1, *, 1, \phi_0', u_0, *)$ is the terminal vertex, where $\xi_0, \xi_0' \in RC(S), v \in S$ and $\phi_0, \phi_0' \in LC(S)$.

Edges are of the form

1. $(\xi, a, b, m, \phi, u, v) \xrightarrow{\theta} (\xi', a', b', m', \phi, u, v)$, where $\xi, \xi' \in RC(S), a, a', b, b', u, v \in S, m, m' \in E(sS_s), \phi, \phi' \in LC(S), \theta$ is an $S$-isomorphism : $(\xi b)S \rightarrow (\xi' a')S$ with $\theta(\xi b) = \xi' a'$ and there exists an element $z \in E(sS_s)$ with $m = bz$ and $m' = a'z$.

2. $(\xi, a, b, m, \rho, u, v) \xrightarrow{\theta} (\xi, a, b, m', \rho', u', v')$, where $\xi, \xi' \in RC(S), a, b, u, v', v' \in S, m, m' \in E(sS_s), \rho, \rho' \in LC(S), \theta$ is an $S$-isomorphism : $(\xi b)S \rightarrow (\xi' a')S$ with $\theta(\xi b) = \xi' a'$ and there exists an element $w \in E(sS_s)$ with $m = uw$ and $m' = uw'$.

3. $(\xi_0, *, 1, 1, \phi_0, *, v_0)$ \rightarrow $(\xi, a, b, m, \phi, u, v)$, with $\xi, \xi' \in RC(S), a, a', b, b', u, v \in S, m, m' \in E(sS_s), \phi, \phi' \in LC(S), \theta$ is an $S$-isomorphism : $(\xi b)S \rightarrow (\xi' a')S$ with $\theta(\xi b) = \xi' a'$, where and there exists an element $w \in E(sS_s)$ with $a = bw$. These edges are labelled by $\theta$.

4. $(\xi, a, b, m, \rho, u, v) \xrightarrow{\theta} (\xi, a, b, m', \rho, u', v)$, where $\xi, \xi' \in RC(S), a, b, u, v \in S, m, m' \in E(sS_s), \rho, \rho' \in LC(S), \theta$ is an $S$-isomorphism : $(\xi b)S \rightarrow (\xi' a')S$ with $\theta(\xi b) = \xi' a'$, where and there exists an element $z \in E(sS_s)$ with $m = bz, m' = a'z$. These edges are labelled by $\theta$.

5. Edges $(\xi, a, b, m, \rho, u, v) \xrightarrow{\theta} (\xi, a, b, m', \rho, u, v)$ are with no label, where $\xi, \xi' \in RC(S), a, b, u, v \in S, m \in E(sS_s)$ and there exist elements $z \in E(sS_s)$ with $b = aw$ and the edge is labelled by $\theta$.

6. $(\xi, a, b, m, \rho, u, v) \xrightarrow{\theta} (\xi, a, b, m', \rho, u', v)$, where $\xi, \xi' \in RC(S), a, b, u, v \in S, m, m' \in E(sS_s), \rho, \rho' \in LC(S), \theta$ is an $S$-isomorphism : $(\xi b)S \rightarrow (\xi' a')S$ with $\theta(\xi b) = \xi' a'$, where and there exists an element $z \in E(sS_s)$ with $m = bz, m' = a'z$. These edges are labelled by $\theta$.

We make an automaton $E(\xi_0, 1, \phi_0, v, \xi_0', 1, \rho_0', u')$ as follows:

Vertices are of the form $(\xi, a, b, \phi, u, v)$ where $\xi \in RC(S), a, b, u, v \in S, m \in E(sS_s), \phi \in LC(S)$,

$(\xi_0, *, 1, \phi_0, *, v)$ is the initial vertex, $(\xi_0', 1, *, 1, \phi_0', u_0, *)$ is the terminal vertex, where $\xi, \xi' \in RC(S), v \in S$ and $\phi_0, \phi_0' \in LC(S)$.

Edges are of the form:

1. $(\xi, a, b, \phi, u, v) \xrightarrow{\theta} (\xi', a', b', \phi, u, v)$, where $\xi, \xi' \in RC(S), a, a', b, b', u, v \in S, \phi, \phi' \in LC(S), \theta$ is an $S$-isomorphism : $(\xi b)S \rightarrow (\xi' a')S$ with $\theta(\xi b) = \xi' a'$.

2. $(\xi, a, b, \rho, u, v) \xrightarrow{\theta} (\xi, a, b, m', \rho', u', v')$, where $\xi, \xi' \in RC(S), a, b, u, u', v, v' \in S, m, m' \in$
$E(SS), \rho, \rho' \in LC(S)$, $\theta$ is an $S$-isomorphism : $S(\phi v) \to (\phi' u')S$ with $\theta(\phi v) = \phi' u'$.

(3) $(\xi_0, \alpha, \rho_0, \pi, v_0) \xrightarrow{\theta} (\xi, a, b, \alpha, \rho, \pi, v, 0)$, where $\xi, \pi \in RC(S), v, \phi, \phi' \in LC(S)$, $\theta$ is an $S$-isomorphism : $(\xi_0)S \to (\xi a)S$ with $\theta(\xi_01) = \xi a$ and these edges are labelled by $\theta$.

(4) $(\xi, a, b, \alpha, \rho_0, \pi, v_0) \xrightarrow{\theta} (\xi', a', b', \alpha, \rho_0, \pi, v_0)$, where $\xi, \pi \in RC(S), a, b, a', b' \in S$, $\theta$ is an $S$-isomorphism : $(\xi a)S \to (\xi' a')S$ with $\theta(\xi a) = \xi' a'$. These edges are labelled by $\theta$.

(5) Edges $(\xi, a, b, \rho_0, u_0, 0) \xrightarrow{\theta} (\xi', 1, a, 1, b, \rho_0, u_0, 0)$ are with no label, where $\xi, \xi' \in RC(S)$, $a, b, u \in S$ and the edge is labelled by $\theta$.

(6) $(\xi, a, b, \alpha, \rho_0, u_0, 0, \pi) \xrightarrow{\theta} (\xi', a', b', \alpha, \rho_0, u_0, 0)$, where $\xi, \xi' \in RC(S), a, b, a', b' \in S$, $\theta$ is an $S$-isomorphism : $(\xi b)S \to (\xi' b')S$ with $\theta(\xi b) = \xi' b'$. These edges are labelled by $\theta$.

Lemma 1. Let $S$ be a finite monoid. Let $E(SS)$ be the injective hull of the $S$-biset $SS$. Let $X$ be a right $S$-set with an element $x, x'$ and $Y$ be a left $S$-set with an element $y, y'$. Let $\xi$ [resp. $\xi'$] be right congruences on $S$ such that there exists an isomorphism $\phi$ [resp. $\phi'$] of $S/\xi$ to $xS$ with $\theta(11) = x$ [resp. $S/\xi'$ to $x'S$ with $\theta'(11) = x'$]. Let $\rho$ [resp. $\rho'$] be left congruences on $S$ such that there exists an isomorphism $\gamma$ [resp. $\gamma'$] of $S/\rho$ to $xS$ with $\gamma(1\rho) = y$ [resp. $S/\rho'$ to $x'S$ with $\gamma'(1\rho') = y'$].

Then $x \otimes 1 \otimes y' = x' \otimes 1 \otimes y'$ in $X \otimes E(SS) \otimes Y$ if and only if there exists an element $u, v \in S$ such that there exists a successful pass from the initial vertix $(\xi, 1, 1, \rho, *, v)$ to the terminal vertix $(\xi', 1, 1, \rho', u)$ on the automaton $A(\xi, \xi', E(SS), \rho, \rho', v, u)$.

Lemma 2. Let $S$ be a finite semigroup. Let $X$ be a right $S$-set with an element $x, x'$ and $Y$ be a left $S$-set with an element $y, y'$. Let $\xi$ [resp. $\xi'$] be right congruences on $S$ such that there exists an isomorphism $\phi$ [resp. $\phi'$] of $S/\xi$ to $xS$ with $\theta(11) = x$ [resp. $S/\xi'$ to $x'S$ with $\theta'(11) = x'$. Let $\rho$ [resp. $\rho'$] be left congruences on $S$ such that there exists an isomorphism $\gamma$ [resp. $\gamma'$] of $S/\rho$ to $xS$ with $\gamma(1\rho) = y$ [resp. $S/\rho'$ to $x'S$ with $\gamma'(1\rho') = y'$]. Let $X$ be the right $S$-set associated with $\theta_1 \theta_2 \cdots \theta_n$ and $Y$ the left $S$-set associated with $\gamma_0 \gamma_1 \gamma_2 \cdots \gamma_m$. 
Then $x \otimes y' = x' \otimes y'$ in $X \otimes Y$ if and only if there exists a successful pass with the label $(\theta_0 \theta_1 \theta_2 \cdots \theta_n, \gamma_0 \gamma_1 \gamma_2 \cdots \gamma_m)$ from the initial vertex $(\xi, 1, 1, \rho, v)$ to the terminal vertex $(\xi', 1, 1, \rho', u)$ on the automaton $B(\xi, \xi', \rho, \rho', v, u)$.

Finally, by using Lemma 1 and Lemma 2, we can prove the main theorem.

References


