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Examples of Peirce decomposition of Generalized Jordan triple systems of second order ——Balanced classical cases—— [1]

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Abstract In this paper, we consider examples of the Peirce decomposition of simple balanced generalized Jordan triple systems of second order associated with Lie algebras. By means of choice of a tripotent element for these triple systems, we can realize the decomposition without using the root systems of Lie algebras.

Introduction

One of the main object of study in this article is to provide examples of a Peirce decomposition of simple balanced generalized Jordan triple systems of second order.

It is known that the all simple Lie algebras L have a decomposition of 5- graded Lie algbras as follows;

$$L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2,$$

starting with a triple system,which has a triple product's structure into the subspace component L_1 of L. And if $dim\ L_{-2} = dim\ L_2 = 1$,it is said to be a balanced triple system for L_1, furthermore,a property of 5-grading of Lie algebras is reduced from that property of triple systems equipped with 2nd order (to see ,[K1]-[K5]). This is one of simple reasons for us to consider about the triple systems.

General speaking for our mathematical field (that is, nonassociative algebras), it seems that nonassociative algebras are rich in algebraic structures and mathematical physics. They provide an important common ground for various branches of mathematics, not only for pure algebra and differential geometry, but also for representation theory and algebraic geometry. That is, the concept of nonassociative algebras which contain Jordan algebras (superalgebras) and Lie algebras (superalgebras) plays an important role in many mathematical and physical subjects (for example,[J.1],[K.8],[K-O.3],[N], [O],[Sch],[Z-S-S-S] etc.). We have determined that the construction and characterization of these algebras can be expressed in terms of the notion of triple systems ([K-K],[K.4],[K-5],[O-K.1]), in particular, by using the standard embedding method ([Li],[M],[K.6],[K-O.1],[O-K.2])).

Describing our recent results in brief, we find the following:

* For the construction of simple Lie algebras, the generalized Jordan triple system of second order (that is, the $(-1,1)$-Freudenthal-Kantor triple system) is a useful concept([Kan],[K.1],[K.2],[K.3],[K.4],[K.5],[K6]).
* For the construction of simple Lie superalgebras, the $(-1,-1)$-Freudenthal-Kantor triple system is a useful concept([K-O.1],[K-O.2],[E-K-O.1],[E-K-O.2],[K-O.4]).
* For the construction of Jordan superalgebras, the δ-Jordan-Lie triple system is a useful concept ([O-K.1],[K-O.5],[K-O.6]).
* For the characterization and representation of mathematical physics, the triple system is useful concept, in particular, Yang-Baxter equations, generalized Zorn vector matrix,etc, ([O],[O-K.2],[K-O.3],[K-O.7]).

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[1] This is an announcement and the details will be published elsewhere.

Our purpose is to propose a unified structural theory for triple systems. In previous work ([K-K]), we have studied the Peirce decomposition of the generalized Jordan triple system U of second order by employing a tripotent element e of U, (tripotent element means $\{eee\} = e$). The Peirce decomposition of U is described as follows:

$$U = U_{00} \oplus U_{\frac{1}{2}\frac{1}{2}} \oplus U_{11} \oplus U_{\frac{3}{2}\frac{3}{2}} \oplus U_{-\frac{1}{2}0} \oplus U_{01} \oplus U_{\frac{1}{2}2} \oplus U_{13},$$

where $L(a) = \{eea\} = \lambda a$, and $R(a) = \{aee\} = \mu a$ if $a \in U_{\lambda\mu}$.

In particular, if the tripotent element is the left unit (left unit element e means $eex = x, \forall x \in U$), then we have

$$U = U_{11}^{+} \oplus U_{11}^{-} \oplus U_{13}^{+} \oplus U_{13}^{-},$$

where $Q(x) = \pm x$ if $x \in U_{11}^{\pm}$, and $Q(x) = \pm 3x$ if $x \in U_{13}^{\pm}$.

On the other hand, for the Peirce decomposition of a Jordan triple system U, it is well known that

$$U = U_{00} \oplus U_{\frac{1}{2}\frac{1}{2}} \oplus U_{11}, \text{ (only } 3-\text{component's decomposition).}$$

In the present article, we shall investigate examples of the Peirce decomposition of simple balanced generalized Jordan triple systems of second order. And only consider classical types cases, for exceptional cases, we deal with it in other paper ([K.7]).

We are concerned with triple systems which have finite dimensionality over a field Φ of characteristic $\neq 2$ *or* 3, unless otherwise specified.

§ 1. Definitions and Preamble

In order to render this paper as self-contained as possible, we first recall the definition of a generalized Jordan triple system of second order (hereafter, referred to as GJTS of 2nd order), and the construction of Lie algebras associated with GJTS of 2nd order.

A vector space V over a field Φ, endowed with a trilinear operation $V \times V \times V \to V$, $(x, y, z) \longmapsto \{xyz\}$, is said to be a GJTS of 2nd order if the following two conditions are satisfied:

$$\begin{aligned} &(J1) \quad \{ab\{xyz\}\} = \{\{abx\}yz\} - \{x\{bay\}z\} + \{xy\{abz\}\}, (GJTS) \\ &(K1) \quad K(K(a,b)x,y) - L(y,x)K(a,b) - K(a,b)L(x,y) = 0, \text{(2nd order)} \end{aligned}$$

where $L(a,b)c = \{abc\}$ and $K(a,b)c = \{acb\} - \{bca\}$.
Furthermore if the GJTS of 2nd order satisfies

$$dim_{\Phi}\{K(a,b)\}_{span} = 1,$$

then it is said to be balanced.

On the other hand, we can generalize the concept of GJTS of 2nd order as follows (see [K.1],[K.2],[K.5],[K-O.1] and the references therein).

For $\varepsilon = \pm 1$ and $\delta = \pm 1$, if the triple product satisfies

$$\begin{aligned} &(ab(xyz)) = ((abx)yz) + \varepsilon(x(bay)z) + (xy(abz)), \\ &K(K(a,b)c,d) - L(d,c)K(a,b) + \varepsilon K(a,b)L(c,d) = 0, \end{aligned}$$

where $L(x,y)z = (xyz)$ and $K(a,b)c = (acb) - \delta(bca)$, then it is said to be a (ε,δ)-Freudenthal-Kantor triple system (hereafter abbreviated as (ε,δ)-F-K.t.s).

The triple products are generally denoted by $\{xyz\}$, (xyz), $[xyz]$, and $< xyz >$, as is our convention.

Remark. We note that the concept of GJTS of 2nd order coincides with that of $(-1,1)$-F-K.t.s. Thus we can construct the simple Lie algebras or superalgebras by means of the standard embedding method ([Kan.1],[K.1]-[K.5],[E-K-O], [K-O.1],[K-O.2],[K-O.4]).

Proposition 1.1 ([K.2],[K-O.1]). Let $U(\varepsilon,\delta)$ be a (ε,δ)-F-K.t.s. If J is an endomorphism of $U(\varepsilon,\delta)$ such that $J < xyz > = < JxJyJz >$ and $J^2 = -\varepsilon\delta Id$, then $(U(\varepsilon,\delta),[xyz])$ is a Lie triple system (the case of $\delta = 1$) or an anti-Lie triple system (the case of $\delta = -1$) with respect to the product

$$[xyz] := < xJyz > -\delta < yJxz > +\delta < xJzy > - < yJzx > .$$

Corollary Let $U(\varepsilon,\delta)$ be a (ε,δ) -F-K.t.s. Then the vector space $T(\varepsilon,\delta) = U(\varepsilon,\delta) \oplus U(\varepsilon,\delta)$ becomes a Lie triple system (the case of δ = 1) or an anti-Lie triple system (the case of δ = -1) with respect to the triple product defined by

$$[\begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} \begin{pmatrix} e \\ f \end{pmatrix}] = \begin{pmatrix} L(a,d) - \delta L(c,b) & \delta K(a,c) \\ -\varepsilon K(b,d) & \varepsilon(L(d,a) - \delta L(b,c)) \end{pmatrix} \begin{pmatrix} e \\ f \end{pmatrix}.$$

Thus we can obtain the standard embedding Lie algebra (the case of $\delta = 1$) or Lie superalgebra (the case of $\delta = -1$), $L(\varepsilon,\delta) = D(T(\varepsilon,\delta),T(\varepsilon,\delta)) \oplus T(\varepsilon,\delta)$, associated with $T(\varepsilon,\delta)$, where $D(T(\varepsilon,\delta),T(\varepsilon,\delta))$ is the set of inner derivations of $T(\varepsilon,\delta)$. That is, these vector spaces $D(T(\varepsilon,\delta),T(\varepsilon,\delta))$ and $T(\varepsilon,\delta)$ mean

$$D(T(\varepsilon,\delta),T(\varepsilon,\delta)) := \{\begin{pmatrix} L(a,b) & K(c,d) \\ K(e,f) & \varepsilon L(b,a) \end{pmatrix}\}_{span}, \text{ and}$$

$$T(\varepsilon,\delta) := \{\begin{pmatrix} x \\ y \end{pmatrix} | x,y \in U(\varepsilon,\delta)\}_{span}$$

Remark. We note that $L(\varepsilon,\delta) := L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_{-1} \oplus L_{-2}$ is the five graded Lie algebra or Lie superalgebra, such that $L_{-1} = U(\varepsilon,\delta), D(T(\varepsilon,\delta),T(\varepsilon,\delta)) = L_{-2} \oplus L_0 \oplus L_{-2}$ with $[L_i, L_j] \subseteq L_{i+j}$.

By straightforward calculations for the correspondence of the (1,1) balanced F.K.t.s with the (-1,1) balanced F.K.t.s, we obtain the following.

Proposition 1.2. Let $(U, < xyz >)$ be a $(1,1)$ F-K.t.s. If there is an endomorphism J of U such that $J < xyz > = < JxJyJz >$ and $J^2 = -Id$, then $(U,\{xyz\})$ is a GJTS of 2nd order with respect to the product defined by $\{xyz\} := < xJyz >$.

In [K-4], we obtained all simple $(1,1)$-balanced F-K.t.s over the complex number field. Thus, these results (by the special case of above Proposition 1.2) give us a list of the simple balanced GJTSs of 2nd order.

In the next section, we will discuss the explicit forms of this list and investigate examples of the Peirce decomposition by providing a tripotent element of the simple balanced GJTSs of 2nd order.

§ 2. Main results (Classical types)

On the basis of the results presented in section 1 and [K-4], in order to make this section as comprehensive as possible, we first summarize the classical types of simple balanced GJTSs of 2nd order as follows:

A_n**-type**: Let $M_A(n)$ be a set of the matrix $\{\begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} | x, y \in Mat(1, n; \mathbf{C})\}$. For $M_A(n)$, we can define a triple product by

$$\{xyz\} = x \circ (PJy \circ z) + z \circ (PJy \circ x) - PJy \circ (x \circ z),$$

where

$$x \circ y = \begin{pmatrix} 0 & x_1 \\ x_2 & 0 \end{pmatrix} \circ \begin{pmatrix} 0 & y_1 \\ y_2 & 0 \end{pmatrix} = \begin{pmatrix} B(x_1, y_2) & 0 \\ 0 & B(y_1, x_2) \end{pmatrix},$$

$B(x, y) = xy^T$ (y^T is the transpose matrix of y), and furthermore

$$P : \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} \to \begin{pmatrix} 0 & x \\ -y & 0 \end{pmatrix} \text{ and } J : \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} \to \begin{pmatrix} 0 & y \\ -x & 0 \end{pmatrix}.$$

That is, if we set $a = B(z_1, y_1)x_1 + B(x_1, y_1)z_1 - B(z_1, x_2)y_2$, and $b = B(y_2, z_2)x_2 + B(y_2, x_2)z_2 - B(x_1, z_2)y_1$, then by straightforward calculations,

$$\{xyz\} = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}.$$

C_n**-type**: We identify the vector space $\{x | x \in Mat(1, 2n; \mathbf{C})\}$ with

$$M_c(n) = \{\begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix} | x \in Mat(1, 2n; \mathbf{C})\}.$$

For $M_c(n)$, we can define a triple product by

$$\{xyz\} = \frac{1}{2}\{< Jy|x > z + < Jy|z > x + < x|z > Jy\},$$

where J is an endomorphism of $M_c(n)$ such that $J^2 = -Id$ and $< x|y >$ is an anti-symmetric bilinear form satisfying the relation $< Jx|y > = < Jy|x > = - < x|Jy >$.

Remark. For the C_n-type of simple balanced GJTS of 2nd order, there exist an endomorphism and a bilinear form such that

$$\begin{aligned} J: \quad & (x_1, \cdots, x_n, x_{n+1}, \cdots, x_{2n}) \to (-x_{n+1}, \cdots, -x_{2n}, x_1, \cdots, x_n) \\ \text{and} \quad & < x|y > = x_1 y_{n+1} + \cdots + x_n y_{2n} - x_{n+1} y_1 - \cdots - x_{2n} y_n, \end{aligned}$$

for $x = (x_1, \cdots x_n, x_{n+1}, \cdots, x_{2n})$ and $y = (y_1, \cdots, y_n, y_{n+1}, \cdots, y_{2n})$.

B_n, D_n **-types**: We identify the space $\{x | x \in Mat(2, p : \mathbf{C})\}$ with

$$M_{B,D}(p) = \{\begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix} | x \in Mat(2, p; \mathbf{C})\}.$$

For $M_{B,D}(p)$, we can define a triple product by

$$\{xyz\} = x \circ (PJy \circ z) + z \circ (PJy \circ x) - PJy \circ (x \circ z),$$

where

$$x \circ y = \begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix} \circ \begin{pmatrix} 0 & y \\ y & 0 \end{pmatrix} = \begin{pmatrix} (\sigma_0 \circ B(x, y))^T & 0 \\ 0 & B(y, x)\sigma_0 \end{pmatrix},$$

$B(x, y) = xy^T$ (2 by 2 matrix), $\sigma_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and $J = \sigma_0$.

That is,

$$\{xyz\} = \begin{pmatrix} 0 & -zy^Tx - xy^Tz + zx^Ty \\ -zy^Tx - xy^Tz + zx^Ty & 0 \end{pmatrix}.$$

Remark. The standard embedding Lie algebras, which are obtained from the types of the triple systems $A_{n-1}, B_n(p = 2n-3), C_n,$ and $D_n(p = 2n-4)$ correspond to the types of the classical simple Lie algebras, respectively ([K.4],[K.5]).

From now on, we will give examples of Peirce decomposition of balanced classical type's triple systems.

In the A_n-type balanced GJTS of 2nd order;

if we set $e = \begin{pmatrix} 0 & e_1 \\ e_1 & 0 \end{pmatrix}$, where e_1 is a $(1,0\cdots,0) : 1 \times n$ matrix, then by straightforward calculations, we obtain $\{eee\} = e$ and $\{eex\} = x, \forall x \in U$.

On the other hand, we have

$$R(x) = \{xee\} = x \text{ and } x = \begin{pmatrix} 0 & x_1 \\ x_2 & 0 \end{pmatrix}$$
$$<=> \begin{cases} B(e_1,e_1)x_1 + B(x_1,e_1)e_1 - B(e_1,x_2)e_1 = x_1 \\ B(e_1,e_1)x_2 + B(e_1,x_2)e_1 - B(x_1,e_1)e_1 = x_2 \end{cases}$$
$$<=> B(e_1,x_2) = B(x_1,e_1)$$
$$<=> \text{if } x_1 = (a_1,\cdots,a_n) \text{ and } x_2 = (b_1,\cdots,b_n), \text{then } a_1 = b_1.$$

Similarly, we have

$R(x) = \{xee\} = 3x$ and $x = \begin{pmatrix} 0 & x_1 \\ x_2 & 0 \end{pmatrix}$ $<=>$

if $x_1 = (a_1,\cdots,a_n)$ and $x_2 = (b_1,\cdots,b_n)$, then $a_1 = -b_1, a_i = b_i = 0\ (2 \leq i)$.

Furthermore, we have

$Q(x) = \{exe\} = x <=> a_2 = -b_2,\cdots,a_n = -b_n$,

$Q(x) = -x <=> a_1 = b_1 = 0,\ a_i = b_i(2 \leq i)$,

$Q(x) = 3x <=> a_1 = -b_1, a_i = b_i = 0(2 \leq i)$,

$Q(x) = -3x <=> x = 0$.

Hence, we obtain a Peirce decomposition with respect to the above tripotent e as follows.

$$x = \begin{pmatrix} 0 & x_1 \\ x_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & (a_1,\cdots,a_n) \\ (b_1,\cdots,b_n) & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & (\frac{a_1+b_1}{2}, \frac{a_2-b_2}{2}, \cdots, \frac{a_n-b_n}{2}) \\ (\frac{a_1+b_1}{2}, -\frac{a_2-b_2}{2}, \cdots, -\frac{a_n-b_n}{2}) & 0 \end{pmatrix}$$
$$+ \begin{pmatrix} 0 & (0, \frac{a_2+b_2}{2}, \cdots, \frac{a_n+b_n}{2}) \\ (0, \frac{a_2+b_2}{2}, \cdots, \frac{a_n+b_n}{2}) & 0 \end{pmatrix}$$
$$+ \begin{pmatrix} 0 & (\frac{a_1-b_1}{2}, 0\cdots,0) \\ (-\frac{a_1-b_1}{2}, 0\cdots,0) & 0 \end{pmatrix}$$
$$\in U_{11}^+ \oplus U_{11}^- \oplus U_{13}^+ = U.$$

In the B_n and D_n types of balanced GJTS U of 2nd order;

if we set $i = \sqrt{-1}$, and e is a $\begin{pmatrix} i\,0\cdots 0 \\ 0\,i\cdots 0 \end{pmatrix}, \cdots 2 \times p$ matrix, then by straightforward calculations, we obtain

$$\{eee\} = e \text{ and } \{eex\} = x, \forall x \in U.$$

On the other hand, we have

$$
\begin{aligned}
&R(x) = \{xee\} = x \\
&<=> \begin{pmatrix} 0 & -ee^Tx - xe^Te + ex^Te \\ -ee^Tx - xe^Te + ex^Te & 0 \end{pmatrix} = \begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix} \\
&<=> xe^Te = ex^Te \ <=> xe^T = ex^T, \text{ by } ee^T = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{aligned}
$$

Similarly, we have

$$R(x) = \{xee\} = 3x <=> \ xe^Te = -ex^Te <=> \ xe^T = -ex^T.$$

Furthermore, we obtain

$$Q(x) = \{exe\} = x <=> \{exe\} = \begin{pmatrix} 0 & -2ex^Te - x \\ -2ex^Te + x & 0 \end{pmatrix} = x$$

$$<=> x = -ex^Te \ <=> \ xe^T = -ex^T.$$

$$Q(x) = \{exe\} = -x <=> xe^T = 0$$

$$Q(x) = 3x <=> cx^Tc = -2x \ <=> \ 2xc^T = cx^T.$$

$$Q(x) = -3x <=> x = ex^Te <=> \ xe^T = -ex^T.$$

Hence, we obtain a Peirce decomposition with respect to the tripotent defined by using the above e,

$$x = \frac{x + ex^Te}{2} + \frac{x - ex^Te}{2} \in U_{13}^- \oplus U_{11}^+ = U.$$

In the C_n type balanced GJTS U of 2nd order;
if we set e as a $(i, 0\cdots0, 0\cdots0)\cdots 1 \times 2n$ matrix, then we obtain

$$\{eee\} = e \text{ and } < Je|e >= Id.$$

By straightforward calculations, we have

$$
\begin{aligned}
&\{eex\} = \tfrac{1}{2}(< Je|x > e + < e|x > Je + x), \\
&\{xee\} = \tfrac{1}{2}(x + < Je|x > e + < x|e > Je), \\
&\{exe\} =< Jx|e > e.
\end{aligned}
$$

On the other hand, by the relation $< Jx|y >= - < x|Jy >$, we have

$$\{exe\} =< Jx|e > e = - < x|Je > e =< Je|x > e.$$

Hence, we obtain

$$\{eex\} = x <=> x = (x_1, 0\cdots0) \text{ for } x = (x_1, x_2, \cdots x_{2n}),$$

$$\{eex\} = \frac{1}{2}x <=> x = (0, x_2, \cdots, x_n, 0, x_{n+2}, \cdots, x_{2n}) \text{ for } x = (x_1, \cdots, x_{2n}),$$

$$\{eex\} = 0 <=> x = (0\cdots0, x_{n+1}, 0\cdots0) \text{ for } x = (x_1, \cdots, x_{2n}),$$

$$\{ccx\} = \frac{3}{2}x <=> x = 0, \ \{ccx\} = -\frac{1}{2}x <=> x = 0, \{xcc\} = 3x <=> x = 0,$$

$$\{xee\} = \frac{1}{2}x <=> x = (0, x_2, \cdots, x_n, 0, x_{n+2}, \cdots, x_{2n}),$$

$$\{xee\} = x <=> x = (x_1, 0, \cdots, 0, x_{n+1}, 0 \cdots, 0).$$

Therefore, we obtain a Peirce decomposition with respect to the tripotent element e as follows:

$$U = U_{\frac{1}{2}\frac{1}{2}} \oplus U_{11} \oplus U_{01},$$

where

$$U_{\frac{1}{2}\frac{1}{2}} = \{(0, x_2, \cdots, x_n, 0, x_{n+2}, \cdots, x_{2n})\}_{span},$$
$$U_{11} = \{(x_1, 0 \cdots, 0)\}_{span}, \text{ and } U_{01} = \{(0 \cdots 0, x_{n+1}, 0 \cdots 0)\}_{span}.$$

These imply the relation:

$$L(x)(2L(x) - Id)(L(x) - Id) = 0, \quad for\ L(x) = \{eex\}.$$

From these results, we note that there are several Peirce decompositions by virtue of choice of tripotent elements.

Remark. For the balanced GJTSs of 2nd order of exceptional types G_2, F_4, E_6, E_7 and E_8 associated with exceptional simple Lie algebras, we will consider their Peirce decompositions in another paper ([K.7]).

Remark. For the balanced GJTSs of 2nd order, one study has been considered from a geometrical approach(see [Ber]), that is, he conducted the correspondence of quaternionic structures on symmetric spases with balanced Freudenthal-Kantor triple systems. Thus it seems that our decompositions is useful in the detail's characterization.

Remark. It seems that this field in nonassociative algebras is very important subject in mathematical phisics and differential geometry as well as a characterization and constraction of Lie algebras, Lie superalgebras and Yang-Baxter equations. Also, it seems that these triple systems will become useful tools and concept to characterize about infinite dimensional Lie algebras and superalgebras.

Appendix

We will give examples of other types as follows.

Example A([K.7]) For a balanced exceptional G_2 type, we have a decomposition;

$$U = U_{11}^{+} \oplus U_{11}^{-} \oplus U_{13}^{+}, where\ e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Example B For a quadratic triple system (i.e., $xxy = yxx = (x,x)y, (x,y) = (y,x)$) ,we have

$$U = U^{+} \oplus U^{-}$$

where,$U^{\pm} = \{x | Q(x) = \pm x\}$.

Example C For a GJTS of 2nd order defined by

$$U = Mat_{p,p}(C), e = E_p, \quad xyz = x^t\bar{y}z + z^t\bar{y}x - z^tx\bar{y},$$

we have,

$$U = U_{11}^{+} \oplus U_{11}^{-} \oplus U_{13}^{+} \oplus U_{13}^{-}.$$

Example D For a balanced (-1,-1) -Freudenthal-Kantor triple system, we have,

$$U = U_{11} \oplus U_{1,-1}.$$

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