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The Rayleigh-Bénard Problems

Consider a plane horizontal layer (see Fig. 1) of an incompressible viscous fluid heated from below. At the lower boundary: \( z = 0 \) the layer of fluid is maintained at temperature \( T + \delta T \) and the temperature of the upper boundary \( (z = h) \) is \( T \).

As well known, under the vanishing assumption in \( y \)-direction, the two-dimensional \((x,z)\) heat convection model can be described as the following Oberbeck-Boussinesq approximations [1]:

\[
\begin{align*}
\rho_{t} + u\rho_{x} + w\rho_{z} &= -p_{x}/\rho_{0} + \nu\Delta u, \\
w_{t} + uw_{x} + wu_{z} &= -(p_{z} + g\rho)/\rho_{0} + \nu\Delta w, \\
u_{x} + w_{z} &= 0, \\
\theta_{t} + u\theta_{x} + w\theta_{z} &= \kappa\Delta\theta.
\end{align*}
\]

(1)

Here, \( u, w \): velocity in \( x \) and \( z \), respectively, \( p \): pressure, \( \theta \): temperature, \( \rho \): fluid density, \( \rho_{0} \): density at temperature \( T + \delta T \), \( \nu \): kinematic viscosity, \( g \): gravitational acceleration, \( \kappa \): coefficient of thermal diffusivity, \( \alpha \): coefficient of thermal diffusivity, \( \alpha \): coefficient of thermal expansion, \( \Delta \): coefficient of thermal expansion. And \( \rho \) is assumed to be represented by \( \rho - \rho_{0} = -\rho_{0}\alpha(\theta - T - \delta T) \), where \( \alpha \) is the coefficient of thermal expansion.
The Oberbeck-Boussinesq equations (1) have the following stationary solution:

\[ u^* = 0, \quad w^* = 0, \quad \theta^* = T + \frac{\delta T}{h} z, \quad p^* = p_0 - g \rho_0 (z + \frac{\alpha \delta T}{2h} z^2), \]

where \( p_0 \) is a constant. By setting

\[ \hat{u} := u, \quad \hat{w} := w, \quad \hat{\theta} := \theta^* - \theta, \quad \hat{p} := p^* - p, \]

we obtain the transformed equations:

\[
\begin{cases}
\hat{u}_t + \hat{u} \hat{u}_x + \hat{w} \hat{u}_z = \frac{\hat{p}_x}{\rho_0} + \nu \Delta \hat{u}, \\
\hat{w}_t + \hat{u} \hat{w}_x + \hat{w} \hat{w}_z = \frac{\hat{p}_z}{\rho_0} - g \alpha \theta + \nu \Delta \hat{w}, \\
\hat{u}_x + \hat{w}_z = 0, \\
\hat{\theta}_t + \delta T \hat{w}/h + \hat{u} \hat{\theta}_x + \hat{w} \hat{\theta}_z = \kappa \Delta \hat{\theta}.
\end{cases}
\]

By further transforming to dimensionless variables:

\[ t \rightarrow \kappa t, \quad u \rightarrow \hat{u}/\kappa, \quad w \rightarrow \hat{w}/\kappa, \quad \theta \rightarrow \hat{\theta}/\delta T, \quad p \rightarrow \hat{p}/(\rho_0 \kappa^2) \]

of (2), we have the dimensionless equations:

\[
\begin{cases}
u_u + \nu w_u + \nu w_z = \frac{p_x}{\rho_0} + \nu \Delta u, \\
u_t + \nu w_t + \nu w_z = \frac{p_z}{\rho_0} - \nu \Delta w, \\
u_x + \nu w_z = 0, \\
u \theta_t + \nu w + \nu \theta_x + \nu \theta_z = \Delta \theta.
\end{cases}
\]

Here \( \mathcal{R} := (\delta T \alpha g)/(\kappa \nu h) \) is the Rayleigh number and \( \mathcal{P} := \nu/\kappa \) is the Prandtl number.

## 2 Fixed-point formulation of problem

We describe the problem concerned as a fixed-point equation of a compact map on the appropriate function space. Since we only consider the steady-state solutions, \( u_t, \ w_t \) and \( \theta_t \) vanish in (3). And also assume that all fluid motion is confined to the rectangular region \( \Omega := \{0 < x < 2\pi/a, \ 0 < z < \pi \} \) for a given wave number \( a > 0 \).

Let us impose periodic boundary condition (period \( 2\pi/a \)) in the horizontal direction, stress-free boundary conditions \( (u_z = w = 0) \) for the velocity field and Dirichlet boundary conditions \( (\theta = 0) \) for the temperature field on the surfaces \( z = 0, \pi \), respectively. Furthermore, we assume the following evenness and oddness conditions:

\[ u(x, z) = -u(-x, z), \quad w(x, z) = w(-x, z), \quad \theta(x, z) = \theta(-x, z). \]

We use the stream function \( \Psi \) satisfying \( u = -\Psi_z, \ w = \Psi_x \) so that \( u_x + w_z = 0 \). By some simple calculations in (3) with setting \( \Theta := \sqrt{\mathcal{P} \mathcal{R}} \theta \), we obtain

\[
\begin{cases}
\mathcal{P} \Delta^2 \Psi = \sqrt{\mathcal{P} \mathcal{R}} \Theta_x - \Psi_z \Delta \Psi_x + \Psi_z \Delta \Psi_z, \\
-\Delta \Theta = -\sqrt{\mathcal{P} \mathcal{R}} \Psi_x + \Psi_z \Theta_x - \Psi_z \Theta_z.
\end{cases}
\]
From the boundary conditions, the functions $\Psi$ and $\Theta$ can be assumed to have the following representations:

$$\Psi = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin(amx) \sin(nz), \quad \Theta = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} B_{mn} \cos(amx) \sin(nz). \quad (5)$$

We now define the following function spaces for integers $k \geq 0$:

$$X^k := \{ \Psi = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin(amx) \sin(nz) | A_{mn} \in \mathbb{R}, \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} ((am)^{2k} + n^{2k}) A_{mn}^2 < \infty \},$$

$$Y^k := \{ \Theta = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} B_{mn} \cos(amx) \sin(nz) | B_{mn} \in \mathbb{R}, \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} ((am)^{2k} + n^{2k}) B_{mn}^2 < \infty \}.$$ 

In order to get the enclosure of the exact solutions for the problem (4), we need some appropriate finite dimensional subspaces. For $M_1, N_1, M_2 \geq 1$ and $N_2 \geq 0$, we set $N := (M_1, N_1, M_2, N_2)$ and define the finite dimensional approximate subspaces by

$$S_{N}^{(1)} = \left\{ \sum_{m=1}^{M_1} \sum_{n=1}^{N_1} \hat{A}_{mn} \sin(amx) \sin(nz) \mid \hat{A}_{mn} \in \mathbb{R} \right\},$$

$$S_{N}^{(2)} = \left\{ \sum_{m=0}^{M_2} \sum_{n=1}^{N_2} \hat{B}_{mn} \cos(amx) \sin(nz) \mid \hat{B}_{mn} \in \mathbb{R} \right\},$$

$$S_N = S_N^{(1)} \times S_N^{(2)}.$$ 

Let denote an approximate solution of (4) by $\hat{u}_N := (\hat{\Psi}_N, \hat{\Theta}_N) \in S_N$. We now set

$$\begin{cases} 
    f_1(\Psi, \Theta) := \sqrt{\mathcal{P}} \Theta_x - \Psi_z \Delta \Psi_x + \Psi_x \Delta \Psi_z, \\
    f_2(\Psi, \Theta) := -\sqrt{\mathcal{P}} \Psi_x + \Psi_z \Theta_x - \Psi_x \Theta_z,
\end{cases}$$

where $\Psi = \hat{\Psi}_N + w^{(1)}$, $\Theta = \hat{\Theta}_N + w^{(2)}$. Then (4) is rewritten as the problem with respect to $(w^{(1)}, w^{(2)}) \in X^4 \times Y^2$ satisfying

$$\begin{cases} 
    \mathcal{P} \Delta^2 w^{(1)} = f_1(\hat{\Psi}_N + w^{(1)}, \hat{\Theta}_N + w^{(2)}) - \mathcal{P} \Delta^2 \hat{\Psi}_N, \\
    -\Delta w^{(2)} = f_2(\hat{\Psi}_N + w^{(1)}, \hat{\Theta}_N + w^{(2)}) + \Delta \hat{\Theta}_N,
\end{cases} \quad (6)$$

which is so-called a residual equation. Setting $w = (w^{(1)}, w^{(2)})$ and

$$\begin{aligned}
    h_1(w) &= f_1(\hat{\Psi}_N + w^{(1)}, \hat{\Theta}_N + w^{(2)}) - \mathcal{P} \Delta^2 \hat{\Psi}_N, \\
    h_2(w) &= f_2(\hat{\Psi}_N + w^{(1)}, \hat{\Theta}_N + w^{(2)}) + \Delta \hat{\Theta}_N, \\
    h(w) &= (h_1(w), h_2(w)),
\end{aligned}$$
by virtue of the Sobolev embedding theorem and the definition of $f_1$ and $f_2$, $h$ is a bounded continuous map from $X^3 \times Y^1$ to $X^0 \times Y^0$. Moreover, it is easily shown that for all $(g_1, g_2) \in X^0 \times Y^0$, the linear problem:

$$\begin{aligned}
\Delta^2 \Psi &= g_1, \\
-\Delta \Theta &= g_2
\end{aligned}$$

has a unique solution $(\Psi, \Theta) \in X^4 \times Y^2$. We denote this mapping by $\Psi = (\Delta^2)^{-1}g_1$ and $\Theta = (-\Delta)^{-1}g_2$, then the operator:

$$K := (\mathcal{P}^{-1}(\Delta^2)^{-1}, (-\Delta)^{-1}): X^0 \times Y^0 \to X^3 \times Y^1$$

is a compact map because of the compactness of the imbedding $X^4 \hookrightarrow X^3$ and $Y^2 \hookrightarrow Y^1$ and the boundedness of $(\Delta^2)^{-1}: X^0 \to X^4$, $(\Delta)^{-1}: Y^0 \to Y^2$. Thus, (6) is rewritten by a fixed-point equation:

$$w = Fw$$

for the compact operator $F := K \circ h$ on $X^3 \times Y^1$. Therefore, by the Schauder fixed-point theorem, if we find a nonempty, closed, bounded and convex set $W \subset X^3 \times Y^1$, satisfying

$$FW \subset W$$

then there exists a solution of (8) in $W$. The set $W$ in (9) is referred as a candidate set of solutions[2, 3].

3 Extended System

Moreover, in order to obtain the enclosure of the bifurcation point, we set

$$Z := X^3 \times Y^1, \quad G := I - F$$

and an operator $S: Z \to Z$ by

$$Sw = S(\Psi, \Theta) := (\Psi(x + \pi/a, z), \Theta(x + \pi/a, z))$$

satisfying $SGw = GSw$. Using this "symmetric" operator $S$, we have the decomposition

$$Z = Z_s \oplus Z_a,$$

where $Z_s = \{w \in Z; Sw = w\}$ and $Z_a = \{w \in Z; Sw = -w\}$. Next, considering $\mathcal{R}$ as a variable, let $\mathcal{G}$ on $Z_s \times Z_a \times \mathbb{R}$ be a map defined by

$$\mathcal{G}(w, v, \mathcal{R}) := \begin{pmatrix}
G(w, \mathcal{R}) \\
D_w G[w, \mathcal{R}]v \\
\mathcal{L}(v) - 1.
\end{pmatrix}$$

(10)
Here $\mathcal{L}$ is an appropriate functional on $Z_a$. We will check the extended system $\mathcal{G}(w, v, \mathcal{R}) = 0$ has an isolate solution $(w_*, v_*, \mathcal{R}_*) \in Z_s \times Z_a \times \mathbb{R}$ and show a sufficient condition such that $\mathcal{R}_*$ is a symmetry-breaking bifurcation point [4] of $G(w, \mathcal{R}) = 0$ by computer-assisted proof.

参考文献


