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<td>引用</td>
<td>数理解析研究所講究録</td>
</tr>
<tr>
<td>期日</td>
<td>2006-07</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/58514">http://hdl.handle.net/2433/58514</a></td>
</tr>
<tr>
<td>タイプ</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
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Rayleigh-Bénard Problems

Consider a plane horizontal layer (see Fig.1) of an incompressible viscous fluid heated from below. At the lower boundary: $z = 0$ the layer of fluid is maintained at temperature $T + \delta T$ and the temperature of the upper boundary ($z = h$) is $T$.

As well known, under the vanishing assumption in $y$-direction, the two-dimensional $(x-z)$ heat convection model can be described as the following Oberbeck-Boussinesq approximations [1]:

$$
\begin{align*}
    u_t + uu_x + wu_z &= -p_x/\rho_0 + \nu \Delta u, \\
    w_t + uw_x + ww_z &= -(p_z + g\rho)/\rho_0 + \nu \Delta w, \\
    u_x + w_z &= 0, \\
    \theta_t + u\theta_x + w\theta_z &= \kappa \Delta \theta.
\end{align*}
$$

(1)

Here, $u$, $w$: velocity in $x$ and $z$, respectively, $p$: pressure, $\theta$: temperature, $\rho$: fluid density, $\rho_0$: density at temperature $T + \delta T$, $\nu$: kinematic viscosity, $g$: gravitational acceleration, $\kappa$: coefficient of thermal diffusivity, $\alpha := \partial / \partial \xi (\xi = x, z, t), \Delta := \partial^2 / \partial x^2 + \partial^2 / \partial z^2$. And $\rho$ is assumed to be represented by $\rho - \rho_0 = -\rho_0 \alpha (\theta - T - \delta T)$, where $\alpha$ is the coefficient of thermal expansion.
The Oberbeck-Boussinesq equations (1) have the following stationary solution:

\[ u^* = 0, \quad w^* = 0, \quad \theta^* = T + \delta T - \frac{\delta T}{h} z, \quad p^* = p_0 - g \rho_0 (z + \frac{\alpha \delta T}{2h} z^2), \]

where \( p_0 \) is a constant. By setting \( \hat{u} := u, \) \( \hat{w} := w, \) \( \hat{\theta} := \theta^* - \theta, \) \( \hat{p} := p^* - p, \) we obtain the transformed equations:

\[
\begin{aligned}
\hat{u}_t + \hat{u} \hat{u}_x + \hat{w} \hat{u}_z &= \frac{\hat{p}_x}{\rho_0} + \nu \Delta \hat{u}, \\
\hat{w}_t + \hat{u} \hat{w}_x + \hat{w} \hat{w}_z &= \frac{\hat{p}_z}{\rho_0} - g \alpha \hat{\theta} + \nu \Delta \hat{w}, \\
\hat{u}_x + \hat{w}_z &= 0, \\
\hat{\theta}_t + \delta T \hat{w}/h + \hat{u} \hat{\theta}_x + \hat{w} \hat{\theta}_z &= \kappa \Delta \hat{\theta}.
\end{aligned}
\]

By further transforming to dimensionless variables:

\[
t \rightarrow \kappa t, \quad u \rightarrow \hat{u}/\kappa, \quad w \rightarrow \hat{w}/\kappa, \quad \theta \rightarrow \hat{\theta}/\delta T, \quad p \rightarrow \hat{p}/(\rho_0 \kappa^2)
\]

of (2), we have the dimensionless equations:

\[
\begin{aligned}
u \Delta^2 \Psi &= \sqrt{\mathcal{P} \mathcal{R}} \Theta_x - \Psi_z \Delta \Psi_x + \Psi_x \Delta \Psi_z, \\
-\Delta \Theta &= -\sqrt{\mathcal{P} \mathcal{R}} \Psi_x + \Psi_z \Theta_x - \Psi_x \Theta_z.
\end{aligned}
\]

Here \( \mathcal{R} := (\delta T \alpha g)/(\kappa \nu h) \) is the Rayleigh number and \( \mathcal{P} := \nu/\kappa \) is the Prandtl number.

\section{Fixed-point formulation of problem}

We describe the problem concerned as a fixed-point equation of a compact map on the appropriate function space. Since we only consider the steady-state solutions, \( u_t, \ w_t \) and \( \theta_t \) vanish in (3). And also assume that all fluid motion is confined to the rectangular region \( \Omega := \{0 < x < 2\pi/a, \ 0 < z < \pi\} \) for a given wave number \( a > 0 \).

Let us impose periodic boundary condition (period \( 2\pi/a \)) in the horizontal direction, stress-free boundary conditions \( (u_z = w = 0) \) for the velocity field and Dirichlet boundary conditions \( (\theta = 0) \) for the temperature field on the surfaces \( z = 0, \pi \), respectively. Furthermore, we assume the following evenness and oddness conditions:

\[ u(x, z) = -u(-x, z), \quad w(x, z) = w(-x, z), \quad \theta(x, z) = \theta(-x, z). \]

We use the stream function \( \Psi \) satisfying \( u = -\Psi_z, \ w = \Psi_x \) so that \( u_x + w_z = 0 \). By some simple calculations in (3) with setting \( \Theta := \sqrt{\mathcal{P} \mathcal{R}} \theta \), we obtain

\[
\begin{aligned}
\mathcal{P} \Delta^2 \Psi &= \sqrt{\mathcal{P} \mathcal{R}} \Theta_x - \Psi_z \Delta \Psi_x + \Psi_x \Delta \Psi_z, \\
-\Delta \Theta &= -\sqrt{\mathcal{P} \mathcal{R}} \Psi_x + \Psi_z \Theta_x - \Psi_x \Theta_z.
\end{aligned}
\]
From the boundary conditions, the functions $\Psi$ and $\Theta$ can be assumed to have the following representations:

$$
\Psi = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin(amx) \sin(nz), \quad \Theta = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} B_{mn} \cos(amx) \sin(nz). \quad (5)
$$

We now define the following function spaces for integers $k \geq 0$:

$$
X^{k} := \left\{ \Psi = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin(amx) \sin(nz) \mid A_{mn} \in \mathbb{R}, \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} ((am)^{2k} + n^{2k}) A_{mn}^{2} < \infty \right\},
$$

$$
Y^{k} := \left\{ \Theta = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} B_{mn} \cos(amx) \sin(nz) \mid B_{mn} \in \mathbb{R}, \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} ((am)^{2k} + n^{2k}) B_{mn}^{2} < \infty \right\}.
$$

In order to get the enclosure of the exact solutions for the problem (4), we need some appropriate finite dimensional subspaces. For $M_1, N_1, M_2 \geq 1$ and $N_2 \geq 0$, we set $N := (M_1, N_1, M_2, N_2)$ and define the finite dimensional approximate subspaces by

$$
S_{N}^{(1)} = \left\{ \sum_{m=1}^{M_1} \sum_{n=1}^{N_1} \hat{A}_{mn} \sin(amx) \sin(nz) \mid \hat{A}_{mn} \in \mathbb{R} \right\},
$$

$$
S_{N}^{(2)} = \left\{ \sum_{m=0}^{M_2} \sum_{n=1}^{N_2} \hat{B}_{mn} \cos(amx) \sin(nz) \mid \hat{B}_{mn} \in \mathbb{R} \right\},
$$

$$
S_{N} = S_{N}^{(1)} \times S_{N}^{(2)}.
$$

Let denote an approximate solution of (4) by $\hat{u}_{N} := (\hat{\Psi}_{N}, \hat{\Theta}_{N}) \in S_{N}$. We now set

$$
\begin{align*}
    f_1(\Psi, \Theta) & := \sqrt{PR} \Theta_x - \Psi_x \Delta \Psi_x + \Psi_x \Delta \Psi_z, \\
    f_2(\Psi, \Theta) & := -\sqrt{PR} \Psi_x + \Psi_z \Theta_x - \Psi_z \Theta_z,
\end{align*}
$$

where $\Psi = \hat{\Psi}_{N} + w^{(1)}$, $\Theta = \hat{\Theta}_{N} + w^{(2)}$. Then (4) is rewritten as the problem with respect to $(w^{(1)}, w^{(2)}) \in X^{4} \times Y^{2}$ satisfying

$$
\begin{align*}
    \mathcal{P} \Delta^2 w^{(1)} & = f_1(\hat{\Psi}_{N} + w^{(1)}, \hat{\Theta}_{N} + w^{(2)}) - \mathcal{P} \Delta^2 \hat{\Psi}_{N}, \\
    -\Delta w^{(2)} & = f_2(\hat{\Psi}_{N} + w^{(1)}, \hat{\Theta}_{N} + w^{(2)}) + \Delta \hat{\Theta}_{N},
\end{align*}
$$

which is so-called a residual equation. Setting $w = (w^{(1)}, w^{(2)})$ and

$$
\begin{align*}
    h_1(w) & = f_1(\hat{\Psi}_{N} + w^{(1)}, \hat{\Theta}_{N} + w^{(2)}) - \mathcal{P} \Delta^2 \hat{\Psi}_{N}, \\
    h_2(w) & = f_2(\hat{\Psi}_{N} + w^{(1)}, \hat{\Theta}_{N} + w^{(2)}) + \Delta \hat{\Theta}_{N}, \\
    h(w) & = (h_1(w), h_2(w)),
\end{align*}
$$

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by virtue of the Sobolev embedding theorem and the definition of $f_1$ and $f_2$, $h$ is a bounded continuous map from $X^3 \times Y^1$ to $X^0 \times Y^0$. Moreover, it is easily shown that for all $(g_1, g_2) \in X^0 \times Y^0$, the linear problem:

\[
\begin{cases}
\Delta^2 \overline{\Psi} = g_1, \\
-\Delta \overline{\Theta} = g_2
\end{cases}
\] (7)

has a unique solution $(\overline{\Psi}, \overline{\Theta}) \in X^4 \times Y^2$. We denote this mapping by $\overline{\Psi} = (\Delta^2)^{-1}g_1$ and $\overline{\Theta} = (-\Delta)^{-1}g_2$, then the operator:

$\mathcal{K} := (\mathcal{P}^{-1}(\Delta^2)^{-1}, (-\Delta)^{-1}) : X^0 \times Y^0 \to X^3 \times Y^1$

is a compact map because of the compactness of the imbedding $X^4 \hookrightarrow X^3$ and $Y^2 \hookrightarrow Y^1$ and the boundedness of $(\Delta^2)^{-1} : X^0 \to X^4$, $(-\Delta)^{-1} : Y^0 \to Y^2$. Thus, (6) is rewritten by a fixed-point equation:

\[ w = Fw \] (8)

for the compact operator $F := \mathcal{K} \circ h$ on $X^3 \times Y^1$. Therefore, by the Schauder fixed-point theorem, if we find a nonempty, closed, bounded and convex set $W \subset X^3 \times Y^1$, satisfying

\[ FW \subset W \] (9)

then there exists a solution of (8) in $W$. The set $W$ in (9) is referred as a candidate set of solutions [2, 3].

3 Extended System

Moreover, in order to obtain the enclosure of the bifurcation point, we set

$Z := X^3 \times Y^1$, \quad $G := I - F$

and an operator $S : Z \to Z$ by

$Sw = S(\Psi, \Theta) := (\Psi(x + \pi/a, z), \Theta(x + \pi/a, z))$

satisfying $SGw = GSw$. Using this "symmetric" operator $S$, we have the decomposition

$Z = Z_s \oplus Z_a$,

where $Z_s = \{w \in Z; Sw = w\}$ and $Z_a = \{w \in Z; Sw = -w\}$. Next, considering $\mathcal{R}$ as a variable, let $\mathcal{G}$ on $Z_s \times Z_a \times \mathbb{R}$ be a map defined by

$\mathcal{G}(w, v, \mathcal{R}) := \begin{pmatrix} G(w, \mathcal{R}) \\ D_vG[w, \mathcal{R}]v \\ L(v) - 1 \end{pmatrix}$ (10)
Here $\mathcal{L}$ is an appropriate functional on $Z_a$. We will check the extended system $\mathcal{G}(w, v, \mathcal{R}) = 0$ has an isolate solution $(w_*, v_*, \mathcal{R}_*) \in Z_s \times Z_a \times \mathbb{R}$ and show a sufficient condition such that $\mathcal{R}_*$ is a symmetry-breaking bifurcation point [4] of $G(w, \mathcal{R}) = 0$ by computer-assisted proof.

参考文献


