数理解析研究所講究録
1507巻 2006年 135-139

「マクロに異なる状態の重ね合わせ」の物理

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(Dated: March 31, 2006)

「マクロに異なる状態の重ね合わせ」は、量子論ができたばかりのころから様々な関心を呼んできた。しかし、単純な(自明な)状態以外には、その定義すら明らかではなく、その一般的性質や、物理現象における役割もほとんど未解明である。我々のグループは、数年前からこのような状態の物理を明らかにすべく、研究を続けている。本研究会では、一般の混合状態に対する「マクロに異なる状態の重ね合わせ」の reasonable な定義と、その実験的な検出法についての我々の最近の仕事を報告する。(Phys. Rev. Lett. **95** (2005) 090401)

PACS numbers: 03.65.Ud,03.67.Mn,05.70.Fh

Superposition of macroscopically distinct states has been attracting much attention since the birth of quantum theory [1–5]. We say a quantum state, represented by a density operator $\hat{\rho}$, is *entangled macroscopically* if $\hat{\rho}$ has such superposition. However, the term 'superposition of macroscopically distinct states' is quite ambiguous in general. For example, do the following states of a system composed of N ($\gg 1$) spins have such superposition?

(i) $|\psi_1\rangle \equiv \sqrt{1-1/N}\, |\downarrow\downarrow\cdots\downarrow\rangle + \sqrt{1/N}\, |\uparrow\uparrow\cdots\uparrow\rangle$,

(ii) $|\psi_2\rangle \equiv (|\downarrow\downarrow\downarrow\cdots\downarrow\rangle + |\uparrow\downarrow\downarrow\cdots\downarrow\rangle + |\uparrow\uparrow\downarrow\cdots\downarrow\rangle + \cdots + |\uparrow\uparrow\uparrow\cdots\uparrow\rangle)/\sqrt{N+1}$,

(iii) classical mixtures of macroscopically entangled states.

For *pure* states a reasonable criterion has been given in Refs. [5, 6], using which we can show that $|\psi_2\rangle$ is macroscopically entangled whereas $|\psi_1\rangle$ is not. Importantly, macroscopic entanglement *as defined by this criterion* is closely related to fundamental stabilities of quantum states [5]. It was also shown that in quantum computers macroscopically entangled states are *always* used to solve hard problems quickly [7, 8]. In experiments, however, it would be hard to generate and confirm pure states for macroscopic systems, hence the criterion for pure states may be difficult to apply. Thus the following questions arise: How can we *detect* macroscopic entanglement of an *unknown* state? How can we define macroscopic entanglement for *mixed* states?

The purpose of this paper is to answer these questions. We first show that macroscopic entanglement of unknown states can *not* be detected if one looks only at the expectation values of low-order polynomials [9] of additive variables (which are fundamental macroscopic variables; see below). Hence, it should be detected by some many-point correlations of local observables. Among such correlations, we point out that Mermin's correlation [3, 4] can detect macroscopic entanglement only for special states. We thus propose a new correlation $\hat{C}_{\hat{A}\hat{\eta}}$, which is a function of two operators $\hat{A}$ and $\hat{\eta}$ (see below), for general macroscopic systems composed of N ($\gg 1$) sites. It can be measured by measuring local observables of all sites and collecting the data thereby obtained. For a state represented by a density operator $\hat{\rho}$, we focus on the maximum value of the expectation value

$$\langle C\rangle = \mathrm{Tr}(\hat{\rho}\hat{C}_{\hat{A}\hat{\eta}})$$

over all possible choices of $\hat{A}$ and $\hat{\eta}$, and define an index q of $\hat{\rho}$ by

$$\max_{\hat{A},\hat{\eta}} (\langle C\rangle, N) = O(N^q). \tag{1}$$

Here and after, we say that $f(N) = O(g(N))$ if

$$\lim_{N\to\infty} f(N)/g(N) = \text{constant} \neq 0.$$

We will show that $1 \le q \le 2$, and that it is reasonable to call states with $q = 2$ macroscopically entangled states. Hence, one can detect macroscopic entanglement by measuring $\langle C\rangle$.

Basic idea — We consider quantum states which are homogeneous, or effectively homogeneous as in Refs. [7, 13]. We say a quantum state (or system) is *macroscopic* if for every quantity of interest the term that is leading order in N gives the dominant contribution. In general, macroscopic states are characterized by macroscopic variables, among which additive variables are fundamental because macroscopic states can be fully specified by (a proper set of) additive variables [6, 10]. Hence, two states are *macroscopically distinct* iff there is an additive variable A such that its difference is $O(N)$ between the two states. In quantum systems, additive variables are represented by additive observables;

$$\hat{A} = \sum_{l=1}^{N} \hat{a}(l),$$

where $\hat{a}(l)$ is a local operator at site l. Throughout this paper, we assume that all $\hat{a}(l)$'s are hermitian. For a

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spin system, for example, such observables include the magnetization $\hat{M}_\alpha = \sum_l \hat{\sigma}_\alpha(l)$ ($\alpha = x, y, z$) and the staggered magnetization $\hat{M}_\alpha^{\rm st} = \sum_l (-1)^l \hat{\sigma}_\alpha(l)$, in which $\hat{a}(l) = (-1)^l \hat{\sigma}_\alpha(l)$. Note that $\hat{a}(l')$ for $l' \neq l$ is not necessarily the spatial translation of $\hat{a}(l)$. To avoid mathematical complexities, we henceforth assume that $\|\hat{a}(l)\|$ is finite and independent of N, and thus $\|\hat{A}\| = O(N)$.

Let $\hat{A}$ be an additive observable, and $|A\nu\rangle$ its eigenstate; $\hat{A}|A\nu\rangle = A|A\nu\rangle$, where ν labels degenerate eigenstates. According to the above argument, a quantum state $\hat{\rho}$ has more superposition of macroscopically distinct states, i.e., is more entangled macroscopically, if $|\langle A\nu|\hat{\rho}|A'\nu'\rangle|$'s with $|A - A'| = O(N)$ are larger for a certain additive observable $\hat{A}$. Our task is thus to propose a way of detecting such $\langle A\nu|\hat{\rho}|A'\nu'\rangle$'s for general $\hat{\rho}$.

Expectation values of low-order polynomials of additive observables — One might expect that $\langle A\nu|\hat{\rho}|A'\nu'\rangle$ could be detected, if exists, through the expectation value of another additive observable $\hat{B}$. Unfortunately, this is impossible for $|A - A'| = O(N)$. For example, suppose that $\hat{\rho} = |\psi\rangle\langle\psi|$ and, neglecting degeneracies of $|A\nu\rangle$'s for simplicity, $|\psi\rangle = (|A_1\rangle + |A_2\rangle)/\sqrt{2}$, where $|A_1 - A_2| = O(N)$. Then, for any additive observable $\hat{B} = \sum_l \hat{b}(l)$, we have

$$\mathrm{Tr}(\hat{\rho}\hat{B}) = \mathrm{Tr}(\hat{\rho}_{\rm mix}\hat{B}),$$

where

$$\hat{\rho}_{\rm mix} = \frac{1}{2}|A_1\rangle\langle A_1| + \frac{1}{2}|A_2\rangle\langle A_2|,$$

because $\hat{B}$ is the sum of single-site operators and thus $\langle A_1|\hat{B}|A_2\rangle = 0$.

More generally, we recall that genuine quantum natures, such as the violation of Bell-type inequalities, come from non-commutativity of observables. For additive observables $\hat{A} = \sum_l \hat{a}(l)$ and $\hat{B} = \sum_l \hat{b}(l)$, however, we have

$$\left\| [\hat{A}/N, \hat{B}/N] \right\| = \left\| \sum_l [\hat{a}(l), \hat{b}(l)] \right\| / N^2 \leq O(1/N).$$

This implies that higher accuracy of experiments is required for larger N to detect genuine quantum natures of a macroscopic state $\hat{\rho}$ through expectation values of $\hat{A}$, $\hat{B}$ and $\hat{A}\hat{B}$ (and low-order polynomials [9] of them). In other words, *any macroscopic states can be well described by local classical theories if one looks only at such expectation values* [11]. This seems to be a foundation of macroscopic physics, such as thermodynamics and fluid dynamics, which are local classical theories.

As a simple example, let us consider the Clauser-Horne-Shimony-Holt (CHSH) correlation [12] of macroscopic variables. Suppose that the system is hypothetically decomposed into two subsystems, each having $N/2$ sites. Let $\hat{A}, \hat{A}'$ and $\hat{B}, \hat{B}'$ are additive observables of one subsystem and the other, respectively. If we normalize them in such a way that their norms are $N/2$, we may define their CHSH correlation by

$$\hat{C}_{\rm CHSH}^{\rm macro} \equiv (\hat{A}\hat{B} + \hat{A}'\hat{B} - \hat{A}\hat{B}' + \hat{A}'\hat{B}')/(N/2)^2.$$

The expectation value $\langle C_{\rm CHSH}^{\rm macro}\rangle_{\rm cl}$ of the corresponding classical correlation satisfies the CHSH inequality

$$|\langle C_{\rm CHSH}^{\rm macro}\rangle_{\rm cl}| \leq 2$$

for any local classical theories. Since $\hat{A}/N, \hat{A}'/N, \hat{B}/N$, and $\hat{B}'/N$ all commute with each other in the $N \to \infty$ limit, we find that

$$\max_{\hat{\rho}, \hat{A}, \hat{A}', \hat{B}, \hat{B}'} \mathrm{Tr}(\hat{\rho}\hat{C}_{\rm CHSH}^{\rm macro}) \to 2$$

as $N \to \infty$, *however anomalous the quantum state is.*

Limitation of Mermin's correlation — The above result suggests that one should look at many-point correlations of *local* observables in order to detect macroscopic entanglement. Mermin proposed one of such correlations $\hat{C}_{\rm M}$ and proved a generalized Bell inequality for it, which is violated by an exponentially large factor $2^{(N-1)/2}$ by a 'cat state,' i.e., superposition with equal weights of *two* states which are macroscopically distinct [3]. Since such a state is entangled macroscopically, one might expect that

$$\langle C_{\rm M}\rangle = \mathrm{Tr}(\hat{\rho}\hat{C}_{\rm M})$$

could be a good measure of macroscopic entanglement if operators in $\hat{C}_{\rm M}$ are properly taken for each state [4]. However, this is not the case in general. For example, the state $|\psi_1\rangle$ in the introduction also violates Mermin's inequality by an exponentially large factor $\simeq 2^{(N-\log_2 N+1)/2}$. However, this state is *not* entangled macroscopically because $q = 1$ (and $p = 1$, see below). Hence, $\langle C_{\rm M}\rangle$ can *not* detect macroscopic entanglement correctly, except for special states such as cat states. We must therefore seek a new correlation.

New correlation for detecting macroscopic entanglement and index q — Let $\mathcal{H}$ be the Hilbert space by which a given macroscopic system composed of N ($\gg 1$) sites is described. Take arbitrarily an additive observable $\hat{A}$ and a projection operator $\hat{\eta}$ on $\mathcal{H}$, satisfying $\hat{\eta}^2 = \hat{\eta}$. Using them, we define the following hermitian operator;

$$\hat{C}_{\hat{A}\hat{\eta}} \equiv [\hat{A}, [\hat{A}, \hat{\eta}]] = \hat{A}^2\hat{\eta} - 2\hat{A}\hat{\eta}\hat{A} + \hat{\eta}\hat{A}^2. \qquad (2)$$

To see its physical meaning, we decompose $\hat{\eta}$ as

$$\hat{\eta} \equiv \sum_{j=1}^{M} |\phi_j\rangle\langle\phi_j|,$$

where $|\phi_j\rangle$'s are orthonormalized vectors and $1 \leq M \leq \dim\mathcal{H}$. Using eigenstates of $\hat{A}$, we obtain the expectation value

$$\langle C\rangle = \mathrm{Tr}\left(\hat{\rho}\hat{C}_{\hat{A}\hat{\eta}}\right)$$

for a state $\hat{\rho}$ as

$$\langle C\rangle = \sum_{j=1}^{M} \sum_{A\nu A'\nu'} (A-A')^2 u_{A\nu}^{j*} \langle A\nu|\hat{\rho}|A'\nu'\rangle u_{A'\nu'}^{j}, \quad (3)$$

where $u_{A\nu}^{j} \equiv \langle A\nu|\phi_j\rangle$. For a given state $\hat{\rho}$, we focus on the N dependence of the maximum value $\max_{\hat{A},\hat{\eta}}\langle C\rangle$ for all possible choices of $\hat{A}$ and $\hat{\eta}$, and define an index q by Eq. (1). By definition, $q \geq 1$. As we will show shortly, the equality is satisfied, e.g., by every separable state (i.e., classical mixture of product states). On the other hand, we find that $q \leq 2$ because

$$|\langle C\rangle| \leq \|[\hat{A},[\hat{A},\hat{\eta}]]\| \leq 4\|\hat{A}\|^2\,\|\hat{\eta}\| = O(N^2),$$

where we have used $\|\hat{A}\| = O(N)$ and $\|\hat{\eta}\| = 1$. It is seen from Eq. (3) that $\hat{\rho}$ has a larger value of $\max_{\hat{A},\hat{\eta}}\langle C\rangle$ when $|\langle A\nu|\hat{\rho}|A'\nu'\rangle|$'s with $|A-A'| = O(N)$ are larger. Since such matrix elements represents quantum coherence between macroscopically distinct states, it is reasonable to call $\hat{\rho}$ with the maximum value $q=2$ a *macroscopically entangled state*. Note that the minimum value $q=1$ is taken also by the random state

$$\hat{\rho} = \hat{1}/\dim\mathcal{H},$$

for which $\langle C\rangle = 0$. Hence, the index q of macroscopic entanglement classifies separable states, for which quantum coherence exists only within each site, and the random state, for which any quantum coherence is absent, as a single group. This is reasonable because they do not have macroscopic entanglement at all.

To sum up, the index q of macroscopic entanglement, defined by Eq. (1), ranges over $1 \leq q \leq 2$. We say $\hat{\rho}$ is macroscopically entangled if $q=2$, whereas states with $q<2$ may be entangled but not macroscopically, among which states with $q=1$ are similar to separable states in view of macroscopic entanglement.

Properties of q for pure states — For pure states, a reasonable index p of macroscopic entanglement was given in Refs. [5, 6] as

$$\max_{\hat{A}}\langle\psi|(\Delta\hat{A})^2|\psi\rangle = O(N^p),$$

where $\Delta\hat{A} \equiv \hat{A} - \langle\psi|\hat{A}|\psi\rangle$ and $1 \leq p \leq 2$. We now investigate the relation between q and p for pure states.

If $\hat{\rho}$ is a pure state $|\psi\rangle\langle\psi|$, we can easily show that $\hat{\eta}|\psi\rangle \neq 0$ is necessary to maximize $\langle C\rangle$. Furthermore, any $\hat{\eta}$ such that $\hat{\eta}|\psi\rangle \neq 0$ can be expressed as

$$\hat{\eta} = |\phi\rangle\langle\phi| + \sum_{j=2}^{M} |\phi'_j\rangle\langle\phi'_j|,$$

where $|\phi\rangle \equiv \hat{\eta}|\psi\rangle/\|\hat{\eta}|\psi\rangle\|$, $\langle\psi|\phi'_j\rangle = \langle\phi|\phi'_j\rangle = 0$ and $\langle\phi'_j|\phi'_{j'}\rangle = \delta_{j,j'}$. Using this expression, we have

$$\begin{aligned}\langle C\rangle \;=\; & \Big(\langle\phi|\hat{A}^2|\psi\rangle\langle\psi|\phi\rangle + \text{c.c.}\Big) \\ & -2\Big|\langle\phi|\hat{A}|\psi\rangle\Big|^2 - 2\sum_{j=2}^{M}\Big|\langle\phi'_j|\hat{A}|\psi\rangle\Big|^2. \end{aligned} \quad (4)$$

Since this becomes maximum when $M=1$, we find

$$\max_{\hat{\eta}}\langle C\rangle = \max_{|\phi\rangle}\langle\phi|[\hat{A},[\hat{A},|\psi\rangle\langle\psi|]]|\phi\rangle. \quad (5)$$

Therefore,

$$\max_{\hat{A},\hat{\eta}}\langle C\rangle \geq \max_{\hat{A}}\langle\psi|[\hat{A},[\hat{A},|\psi\rangle\langle\psi|]]|\psi\rangle = 2\max_{\hat{A}}\langle\psi|(\Delta\hat{A})^2|\psi\rangle,$$

from which we immediately find that *if* $p=2$ *then* $q=2$, and *if* $q=1$ *then* $p=1$. We also note that Eq. (5) implies that $\max_{\hat{\eta}}\langle C\rangle$ is the maximum eigenvalue of the hermitian operator $[\hat{A},[\hat{A},|\psi\rangle\langle\psi|]]$. If we denote an eigenvector corresponding to the maximum eigenvalue by $|\phi_A\rangle$, we have

$$\begin{aligned}\max_{\hat{A},\hat{\eta}}\langle C\rangle &= \max_{\hat{A}}\langle\phi_A|[\hat{A},[\hat{A},|\psi\rangle\langle\psi|]]|\phi_A\rangle \\ &= \max_{\hat{A}}\mathrm{Tr}\left([|\phi_A\rangle\langle\phi_A|,A]^\dagger[|\psi\rangle\langle\psi|,A]\right) \\ &\leq 2\left[\max_{\hat{A}}\langle\phi_A|(\Delta_{\phi_A}\hat{A})^2|\phi_A\rangle\right]^{\frac{1}{2}}\left[\max_{\hat{A}'}\langle\psi|(\Delta_\psi\hat{A}')^2|\psi\rangle\right]^{\frac{1}{2}}, \end{aligned} \quad (6)$$

where we have used the Cauchy-Schwartz inequality,

$$|\mathrm{Tr}(\hat{J}^\dagger\hat{K})| \leq [\mathrm{Tr}(\hat{J}^\dagger\hat{J})]^{1/2}[\mathrm{Tr}(\hat{K}^\dagger\hat{K})]^{1/2},$$

and $\Delta_{\phi_A}\hat{A} \equiv \hat{A} - \langle\phi_A|\hat{A}|\phi_A\rangle$, $\Delta_\psi\hat{A}' \equiv \hat{A}' - \langle\psi|\hat{A}'|\psi\rangle$. We thus find that *if* $q=2$ *then* $p=2$. Moreover, since $|\phi_A\rangle$ is an eigenvector of

$$[\hat{A},[\hat{A},|\psi\rangle\langle\psi|]],$$

it is given by a linear combination

$$|\phi_A\rangle = x|\psi\rangle + \sum_l y_l\hat{a}(l)|\psi\rangle + \sum_{l,l'} z_{ll'}\hat{a}(l)\hat{a}(l')|\psi\rangle.$$

This implies that $|\phi_A\rangle$ is obtained from $|\psi\rangle$ by adding one- and two-particle excitations. Since addition of such microscopic excitations does not change the value of the index p of macroscopic entanglement [5, 6], $p=1$ for $|\phi\rangle$ if $p=1$ for $|\psi\rangle$. Thus, from inequality (6), we find that *if* $p=1$ *then* $q=1$. In particular, $q=1$ *for any product state* $|\psi\rangle = \bigotimes_{l=1}^{N}|\psi_l\rangle$ because $p=1$.

To sum up, we have found that $p=1 \Leftrightarrow q=1$ and that $p=2 \Leftrightarrow q=2$, for pure states.

Properties of q for mixed states — The above results demonstrate that q is a natural generalization of p, which was defined only for pure states [5, 6]. We now present basic properties of q for mixed states.

Any mixture $\hat{\rho} = \sum_\lambda \rho_\lambda|\psi_\lambda\rangle\langle\psi_\lambda|$ *of pure states* $|\psi_\lambda\rangle$*'s with* $q=1$ *has* $q=1$. In fact,

$$\max_{\hat{A},\hat{\eta}}\langle C\rangle \leq \sum_\lambda \rho_\lambda \max_{\hat{A},\hat{\eta}}\langle\psi_\lambda|\hat{C}_{\hat{A},\hat{\eta}}|\psi_\lambda\rangle = \sum_\lambda \rho_\lambda O(N) = O(N).$$

In particular, $q=1$ *for separable states* since $q=1$ for product states. On the other hand, *mixtures of pure states* $|\psi_\lambda\rangle$*'s with* $q=2$ *do not necessarily have* $q=2$.

A simple example for an N-spin system is the state with $\rho_\pm = 1/2$ and $|\psi_\pm\rangle = (|\downarrow\rangle^{\otimes N} \pm |\uparrow\rangle^{\otimes N})/\sqrt{2}$. Then,

$$\hat{\rho}_{\rm ex1} \equiv \frac{1}{2}|\psi_+\rangle\langle\psi_+| + \frac{1}{2}|\psi_-\rangle\langle\psi_-|$$

is equal to

$$\frac{1}{2}(|\downarrow\rangle\langle\downarrow|)^{\otimes N} + \frac{1}{2}(|\uparrow\rangle\langle\uparrow|)^{\otimes N},$$

which is a classical mixture of product states, and thus $q = 1$.

It is interesting to clarify the conditions for $q = 2$ for mixtures of states with $q = 2$. A *sufficient condition* is as follows. Suppose that for an additive operator $\hat{A}$ we have pure states $|\psi_1\rangle$, $|\psi_2\rangle$, $\cdots$ such that

$$\langle\psi_\lambda|\psi_{\lambda'}\rangle = \delta_{\lambda,\lambda'} \text{ for } \lambda, \lambda' = 1, 2, \cdots, \tag{7}$$

$$\langle\psi_\lambda|\hat{A}|\psi_{\lambda'}\rangle = 0 \text{ for } \lambda \neq \lambda', \tag{8}$$

$$\langle\psi_\lambda|(\Delta_\lambda \hat{A})^2|\psi_\lambda\rangle = O(N^2) \text{ for } \lambda \leq \Lambda, \tag{9}$$

$$\langle\psi_\lambda|(\Delta_\lambda \hat{A})^2|\psi_\lambda\rangle < O(N^2) \text{ for } \lambda > \Lambda, \tag{10}$$

where $\Delta_\lambda \hat{A} \equiv \hat{A} - \langle\psi_\lambda|\hat{A}|\psi_\lambda\rangle$ and Λ is a positive integer. Consider classical mixtures of these states,

$$\hat{\rho} = \sum_\lambda \rho_\lambda |\psi_\lambda\rangle\langle\psi_\lambda|,$$

where ρ_λ's are real numbers such that $0 \leq \rho_\lambda \leq 1$ and $\sum_\lambda \rho_\lambda = 1$. If

$$\lim_{N\to\infty} \sum_{\lambda\leq\Lambda} \rho_\lambda \neq 0, \tag{11}$$

then any such mixtures have $q = 2$, hence are entangled macroscopically. In fact, if we take $\hat{\eta} = \sum_\lambda |\psi_\lambda\rangle\langle\psi_\lambda|$, we find

$$\langle C\rangle = 2\sum_\lambda \rho_\lambda \langle\psi_\lambda|(\Delta_\lambda \hat{A})^2|\psi_\lambda\rangle = O(N^2),$$

hence $q = 2$.

For example, let

$$|\psi_\lambda\rangle = \frac{1}{\sqrt{2}}|\downarrow\rangle^{\otimes(\lambda-1)}|\uparrow\rangle|\downarrow\rangle^{\otimes(N-\lambda)} + \frac{1}{\sqrt{2}}|\uparrow\rangle^{\otimes(\lambda-1)}|\downarrow\rangle|\uparrow\rangle^{\otimes(N-\lambda)}$$

for $\lambda = 1, 2, \cdots, N$. Then, conditions (7)-(9) are all satisfied for $\hat{A} = \hat{M}_z = \sum_l \hat{\sigma}_z(l)$ and $\Lambda = N$. Therefore, any mixtures of these states, such as

$$\hat{\rho}_{\rm ex2} \equiv (1/N)\sum_{\lambda=1}^{N} |\psi_\lambda\rangle\langle\psi_\lambda|,$$

are entangled macroscopically, i.e., $q = 2$. This may be understood by noting that such mixtures are mixtures of the 'same sort' of superpositions of macroscopically distinct states in the sense that all $|\psi_\lambda\rangle$'s are superpositions of states with $M_z = \pm(N-2)$.

A more instructive example is the case where

$$|\psi_\lambda\rangle \equiv (|\lambda\rangle + |\bar{\lambda}\rangle)/\sqrt{2},$$

where $|\lambda\rangle$ ($|\bar{\lambda}\rangle$) is an arbitrary state in which λ spins are up (down) and $N-\lambda$ spins are down (up). If we limit the range of λ over, say, $1 \leq \lambda \leq N/3$, then conditions (7)-(9) are all satisfied for $\hat{A} = \hat{M}_z$ and $\Lambda = N/3$. Therefore, any mixtures of these states, such as

$$\hat{\rho}_{\rm ex3} \equiv (3/N)\sum_{\lambda=1}^{N/3} |\psi_\lambda\rangle\langle\psi_\lambda|,$$

are entangled macroscopically, i.e., $q = 2$. Intuitively, such mixtures are mixtures of the same sort of superpositions of macroscopically distinct states in the sense that all $|\psi_\lambda\rangle$'s are superpositions of states with positive and negative M_z.

Furthermore, $\hat{\rho}'_{\rm ex2} \equiv w\hat{\rho}_{\rm ex2} + (1-w)\hat{\rho}_{\rm ex1}$ and $\hat{\rho}'_{\rm ex3} \equiv w\hat{\rho}_{\rm ex3} + (1-w)\hat{\rho}_{\rm ex1}$ also have $q = 2$ if $w > 0$ and independent of N, because $|\downarrow\rangle^{\otimes N}$ and $|\uparrow\rangle^{\otimes N}$ satisfy the above conditions for $|\psi_\lambda\rangle$'s with $\lambda > \Lambda$.

Measurement of $\langle C\rangle$ *by local measurements* — When detecting entanglement of two particles by measuring the CHSH correlation,

$$\hat{C}_{\rm CHSH} = \hat{a}(\theta)\hat{b}(\phi) + \hat{a}(\theta')\hat{b}(\phi) - \hat{a}(\theta)\hat{b}(\phi') + \hat{a}(\theta')\hat{b}(\phi'),$$

one does *not* measure it using a *single* experimental setup, which performs a *global* (non-local) measurement. Instead, one measures $\hat{a}$'s and $\hat{b}$'s locally and simultaneously, which are observables of one particle and the other, respectively. Since $\hat{a}(\theta)$ and $\hat{a}(\theta')$ cannot be measured simultaneously because $[\hat{a}(\theta), \hat{a}(\theta')] \neq 0$, they should be measured independently using different experimental setups, and similarly for $\hat{b}(\phi)$ and $\hat{b}(\phi')$. That is, one performs local measurements with various setups. By collecting the data of such local measurements, one can obtain the expectation values of all terms in $\hat{C}_{\rm CHSH}$, and hence the value of $\langle C_{\rm CHSH}\rangle$.

In a similar manner, one can obtain $\langle C\rangle$ by measuring local observables with various setups and collecting the data thereby obtained. This might be obvious because in general any hermitian operator on $\mathcal{H} = \bigotimes_l \mathcal{H}_l$, where $\mathcal{H}_l$ is the local Hilbert space of site l, can be expressed as the sum of products of local hermitian operators. However, we show it in such a way that local observables to be measured can be seen easily. Let $|a_l\mu_l\rangle \in \mathcal{H}_l$ be an eigenvector of $\hat{a}(l)$;

$$\hat{a}(l)|a_l\mu_l\rangle = a_l|a_l\mu_l\rangle,$$

where μ_l labels degenerate eigenvectors. We can take

$$|A\nu\rangle = \bigotimes_l |a_l\mu_l\rangle,$$

where $A = \sum_l a_l$. Hence, denoting $\boldsymbol{a} = (a_1, a_2, \cdots, a_N)$

and $\boldsymbol{\mu} = (\mu_1, \mu_2, \cdots, \mu_N)$, we can express $\hat{C}_{\hat{A}\hat{\eta}}$ as

$$\hat{C}_{\hat{A}\hat{\eta}} = \sum_{j=1}^{M} \sum_{\boldsymbol{a}\boldsymbol{\mu}\boldsymbol{a}'\boldsymbol{\mu}'} \left(\sum_{l'} (a_{l'} - a'_{l'}) \right)^2 u^{j}_{\boldsymbol{a}'\boldsymbol{\mu}'} u^{j*}_{\boldsymbol{a}\boldsymbol{\mu}} \times \bigotimes_{l} \left(\hat{\varphi}'_{a'_l\mu'_l a_l\mu_l}(l) + i\hat{\varphi}''_{a'_l\mu'_l a_l\mu_l}(l) \right), \quad (12)$$

where

$$\hat{\varphi}'_{a'_l\mu'_l a_l\mu_l}(l) \equiv (|a'_l\mu'_l\rangle\langle a_l\mu_l| + \text{h.c.})/2$$

and

$$\hat{\varphi}''_{a'_l\mu'_l a_l\mu_l}(l) \equiv (|a'_l\mu'_l\rangle\langle a_l\mu_l| - \text{h.c.})/(2i)$$

are local hermitian operators on $\mathcal{H}_l$. By expanding Eq. (12), we obtain a polynomial of $\hat{\varphi}'(l)$'s and $\hat{\varphi}''(l)$'s, i.e., the sum of products of local observables. Therefore, $\langle C \rangle$ can be measured by measuring such local observables of each terms (using proper experimental setups for each) and collecting the data thereby obtained.

The operators $\hat{\varphi}'(l)$'s and $\hat{\varphi}''(l)$'s, which we denote $\hat{\boldsymbol{\varphi}}$, and the numbers $\boldsymbol{a}$, $\boldsymbol{\mu}$ in Eq. (12) correspond to $\hat{a}$, $\hat{b}$, $\theta, \theta', \phi, \phi'$ of $\hat{C}_{\text{CHSH}}$. To find the value of q, one should seek a particular set of $\hat{\boldsymbol{\varphi}}$, $\boldsymbol{a}$, $\boldsymbol{\mu}$ that maximizes $\langle C \rangle$ (or gives the same order of magnitude of $\langle C \rangle$ as the maximum value). If the state $\hat{\rho}$ is unknown, one should perform experiments for various choices of $\hat{\boldsymbol{\varphi}}$, $\boldsymbol{a}$, $\boldsymbol{\mu}$, and thereby find the maximum value of $\langle C \rangle$. This situation is the same as the case of detecting the violation of the CHSH inequality of two particles by an unknown state, where one should perform experiments for various choices of $\hat{a}$, $\hat{b}$, $\theta, \theta', \phi, \phi'$. In many practical experiments, however, one tries to generate some target state with a prescribed $\hat{\rho}$. In such a case, one can theoretically find $\hat{A}$ and $\hat{\eta}$ that should give the maximum value of $\langle C \rangle$ [14]. Then, one needs to measure $\langle C \rangle$ only for $\hat{\boldsymbol{\varphi}}$, $\boldsymbol{a}$, $\boldsymbol{\mu}$ corresponding to such $\hat{A}$ and $\hat{\eta}$.

Conversion of states with $q < 2$ to states with $q = 2$ — Entanglement is often defined in terms of possibility of converting a state in question to another state which is manifestly entangled [15]. In the present case, it is possible to convert $|\psi_1\rangle$ in the introduction, which has $q = 1$, to a cat state, which has $q = 2$, by a *single*-spin projective measurement. However, its success probability tends to vanish with increasing N. In our opinion, it is natural to exclude such rare events to *define* macroscopic entanglement, and to interpret the above possibility as an interesting possibility with a very small but non-vanishing (for finite N) success probability.

Possible experiments — It is very interesting to detect macroscopic entanglement experimentally. One way of producing states with $q = 2$ is to cool a symmetry-breaking system whose order parameter does not commute with the Hamiltonian, such as the Heisenberg antiferromagnet on a two-dimensional square lattice [6]. If the temperature can be made lower than the energy difference between the exact ground state (which is symmetric [5, 6, 16]) and the symmetry-breaking vacuum, then the equilibrium density operator becomes a macroscopically entangled state [14]. Another way may be to use quantum computers, in which one can manipulate quantum states rather freely [15], as a playground of many-body physics.

[1] E. Schrödinger, Naturwissenschaften **23** 807, 823, 844 (1935).
[2] A. J. Leggett, *Chance and Matter* (edited by J. Souletie et al., Elsevier, Amsterdam, 1987) 397.
[3] N. D. Mermin, Phys. Rev. Lett. **65**, 1838 (1990).
[4] S. M. Roy and V. Singh, Phys. Rev. Lett. **67**, 2761 (1991).
[5] A. Shimizu and T. Miyadera, Phys. Rev. Lett. **89**, 270403 (2002).
[6] T. Morimae, A. Sugita and A. Shimizu, Phys. Rev. A **71**, 032317 (2005).
[7] A. Ukena and A. Shimizu, Phys. Rev. A **69**, 022301 (2004).
[8] A. Ukena and A. Shimizu, e-Print: quant-ph/0505057.
[9] Low-order polynomials mean here m-th order polynomials with $m = O(1)$, although we expect that the same can be said for any m such that $m \ll N$.
[10] H. B. Callen, *Thermodynamics* (Wiley, New York, 1960).
[11] For example, macroscopic properties of superconductors are well described by the Ginzburg-Landau theory, which is surely a local classical theory, although its macroscopic variable is called the 'macroscopic wavefunction.'
[12] J. F. Clauser et al., Phys. Rev. Lett. **23**, 880 (1969).
[13] A. Sugita and A. Shimizu, J. Phys. Soc. Jpn. **74** (2005), in press.
[14] T. Morimae and A. Shimizu, unpublished.
[15] See. e.g., M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).
[16] See, e.g., A. Shimizu and T. Miyadera, Phys. Rev. E **64**, 056121 (2001), and references cited therein.