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On a Product Formula for Quantum Zeno Dynamics*

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1. Introduction and Result

We want to discuss what in quantum mechanics, after frequently repeated measurements for each short time t/n interval we shall have at time t as a result, although it seems to need to assume that each measurement is an instantaneous ideal measurement. The decay of the system can be slowed down or even prevented. This is called *quantum Zeno effect.* The problem was discussed in Beskow–Nilsson [BN], Misra–Sudarshan [MS]. Zeno is a Greek philosopher who said something like "A flying arrow does not fly".

The aim of this talk is to present the following product formula on quantum Zeno dynamics. This note is mainly based on the joint work with P.Exner [EI].

Theorem. Let H be a nonnegative selfadjoint operator in a separable Hilbert space $\mathcal{H}$, and P an orthogonal projection on $\mathcal{H}$. Assume that $H_P := (H^{1/2}P)^*(H^{1/2}P)$ is densely defined. Then it holds :

$$\lim_{n\to\infty}[Pe^{-itH/n}P]^n = e^{-itH_P}P, \tag{1.1}$$

in the topology of Frechét space $L^2_{loc}(\mathbf{R};\mathcal{H}) = L^2_{loc}(\mathbf{R})\otimes\mathcal{H}$. In other words, for every $\phi\in C_0^\infty(\mathbf{R})$,

$$\int \phi(t)\|[Pe^{-itH/n}P]^n f - e^{-itH_P}Pf\|^2\,dt \to 0, \quad n\to\infty,$$

or for every $T>0$,

$$\int_0^T \|[Pe^{-itH/n}P]^n f - e^{-itH_P}Pf\|^2\,dt \to 0, \quad n\to\infty.$$

So passig to subsequences, we have also:

Corollary. There exist a subset $M\subset\mathbf{R}$ of Lebesgue measure zero and a sequence $\{n'\}$ of strictly increasing positive integers along which for all $t\in\mathbf{R}\setminus M$,

$$[Pe^{-itH/n}P]^n f \to e^{-itH_P}Pf, \ \text{strongly}, \ f\in\mathcal{H}.$$

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Remark. If dim P is finite, then (1.1) holds strongly on $\mathcal{H}$.

2. Sketch of Proof.

We may think the time variable t on $[0,\infty)$ instead of $\mathbf{R}$. We use the method by Kato [K] of the proof of the Trotter product formula for the *form sum* $A\dot{+}B$ of two nonnegative selfadjoint operators A and B in a Hilbert space $\mathcal{H}$. He proved that s-$\lim_{n\to\infty}(e^{-tA/n}e^{-tB/n})^n = e^{-t(A\dot{+}B)}J$, where J stands for the orthogonal projection to the closure $\mathcal{K}$ of the subspace $D[A^{1/2}]\cap D[B^{1/2}]$, and $D[A^{1/2}]$ and $D[A^{1/2}]$ denote the form domains for the operators A and B, so that $A\dot{+}B$ would become a nonnegative selfadjoint operator in the Hilbert space $\mathcal{K}$.

With H and P in our Hilbert space $\mathcal{H}$ as in the statement of Theorem, put for $\mathrm{Re}\,\zeta \geq 0$ with $\zeta \neq 0$,

$$F(\zeta,\tau) := Pe^{-\zeta\tau H}P, \quad S(\zeta,\tau) := \tau^{-1}[I - F(\zeta,\tau)] = \tau^{-1}[I - Pe^{-\zeta\tau H}P].$$

The key ingredient of the proof is the following

Lemma.

$$(I + S(it,\tau))^{-1} \longrightarrow (I + itH_P)^{-1}P, \quad \tau \to 0, \tag{2.1}$$

strongly in $L^2_{loc}([0,\infty);\mathcal{H})$.

Proof of Theorem. By Lemma

$$P(I + S(it,\tau))^{-1}P \longrightarrow (I + itH_P)^{-1}P,\ \tau \to 0$$

strongly in $L^2_{loc}([0,\infty);\mathcal{H})$ and hence on $P\mathcal{H}$: as $\tau \to 0$,

$$\left((I + S(it,\tau)|_{P\mathcal{H}}\right)^{-1} \longrightarrow \left[(I + itH_P)|_{P\mathcal{H}}\right]^{-1},$$

strongly in $L^2_{loc}([0,\infty);\mathcal{H})$
It is also easy to see: on $(I-P)\mathcal{H} \ni f$

$$[Pe^{-itH/n}P]^n f = 0, \quad e^{-itH_P}Pf = 0.$$

Then by mimicking an argument used for Chernoff's Theorem we have: for $f \in \mathcal{H}$,

$$[Pe^{-itH/n}P]^n f \longrightarrow e^{-itH_P}Pf,\ n \to \infty,$$

strongly in $L^2_{loc}([0,\infty);\mathcal{H})$.

For the proof of Lemma, we employ the method of Kato [K] with Friedman's idea [F]. However, we would refer its details to [EI].

3. Qunatum Zeno Problem

Facchi, Pascazio, Scardicchio and Schulman considered in [FPSS] the following problem called "quantum Zeno problem" and asserted:

Let $H = -\Delta$ be the free Schrödinger operator, i.e. Laplacian in $L^2(\mathbf{R}^d)$. Let $\Omega \subset \mathbf{R}^d$ be a bounded open subset with smooth boundary. Let $P := \chi_\Omega$ be the indicator function of the set Ω, so that $PL^2(\mathbf{R}^d) = L^2(\Omega)$. Let $-\Delta_\Omega$ be the Dirichlet Laplacian in Ω, which has domain $D[-\Delta_\Omega] = W^2(\Omega) \cap W_0^1(\Omega)$.
Then it holds that

$$\lim_{n\to\infty} [Pe^{it\Delta/n}P]^n = e^{it\Delta_\Omega}P, \tag{3.1}$$

strongly in $L^2(\mathbb{R}^d)$.

But their proof contains a gap. What is sure up to now is what we have been able to show in our Theorem and its Corollary, namely, (3.1) holds only in a weaker sense of convergence, strongly in $L^2_{loc}(\mathbf{R}; L^2(\mathbf{R}^d))$ or in strong convergence on $\mathcal{H}$ for almost every $t \geq 0$ along a subsequence $\{n'\}$ of some increasing potitive integers.

To see our theorem can apply, we have only to check that the operator $(-\Delta)_P \equiv \left((-\Delta)^{1/2}P\right)^*\left((-\Delta)^{1/2}P\right)$ coincides with $-\Delta_\Omega$.

To prove it, let $u \in D[(-\Delta)_P]$ (then note $(-\Delta)^{1/2}P \in L^2(\mathbf{R}^d)$), so that u and $(-\Delta)_P u$ belong to $L^2(\mathbf{R}^d)$. We have

$$\langle(-\Delta)_P u, \varphi\rangle = \langle u, -\Delta\varphi\rangle = \langle -\Delta u, \varphi\rangle,$$

for any $\varphi \in C_0^\infty(\Omega)$ because φ has a compact support in Ω. Thus $(-\Delta)_P u = -\Delta u$ holds in Ω in the sense of distributions. Then $u|_\Omega \in W^2(\Omega)$. Then u admits boundary trace $u(\cdot)$ on the bry of Ω. Hence there exists an extension $\tilde{u}$ in $W^2(\mathbf{R}^d)$ s.t. $\tilde{u}|_{\overline{\Omega}} = u$. Next, since $\infty > \langle(-\Delta)_P u, u\rangle = \int |\nabla(\chi_\Omega u)|^2 dx$, we have

$$\nabla(\chi_\Omega u) = \nabla((\chi_\Omega)^2 u) = (\nabla\chi_\Omega)\chi_\Omega u(x) + \chi_\Omega\nabla(\chi_\Omega u).$$

For LHS to belong to $L^2(\mathbf{R}^d)$, the fun. $\nabla(\chi_\Omega u)$ must not contain the δ-type singular term, which requires $u(\cdot) = 0$ on the bry of Ω. This, combined with the fact that $u|_\Omega, \nabla u|_\Omega \in L^2(\Omega)$, implies $u|_\Omega \in W^2(\Omega) \cap W_0^1(\Omega)$.

Thus we have shown that $u|_\Omega \in D[-\Delta_\Omega]$ and $(-\Delta)_P u|_\Omega = -\Delta_\Omega(u|_\Omega)$ or $-\Delta_\Omega \supset (-\Delta)_P|_{L^2(\Omega)}$, but both operators are selfadjoint, so they coincide.

4. Criticism to the paper by Facchi-Pascazio et al.

Let $\{\lambda_j\}_{j=1}^\infty$ be the set of the eigenvalues of the Dirichlet Laplacian $-\Delta_\Omega$ and $\{\varphi_j\}_{j=1}^\infty$ the set of the corresponding orthonormal eigenfunctions:

$$-\Delta_\Omega\varphi_j = \lambda_j\varphi_j, \quad \lambda_j > 0.$$

Put $Z(\tau) := \chi_\Omega e^{-i\tau H}\chi_\Omega : L^2(\Omega) \mapsto L^2(\Omega)$ with $\tau = t/n$.

Since the linear combinations of $\{\varphi_j\}$ is dense, we have only to show, for j fixed,

$$
\begin{aligned}
&[Z(\tau)^n\varphi_j](x) \qquad (x = x_n)\\
&= \int_\Omega \cdots \int_\Omega e^{-i\tau H}(x_n, x_{n-1}) \cdots e^{-i\tau H}(x_1, x_0)\varphi_j(x_0)dx_{n-1}\cdots dx_1 dx_0\\
&= \frac{1}{(4\pi i\frac{t}{n})^{dn/2}} \int_\Omega \cdots \int_\Omega \exp\Big[i\frac{\sum_{j=1}^n (x_j - x_{j-1})^2}{4\frac{t}{n}}\Big]\varphi_j(x_0)dx_{n-1}\cdots dx_1 dx_0\\
&\longrightarrow e^{-itH_D}\varphi_j = e^{-it\lambda_j}\varphi_j, \quad \text{in } L^2(\Omega), \quad n \to \infty.
\end{aligned}
$$

We may take $j = 1$. If $x \in \Omega$ (x is an interior point of Ω), we have by stationary phase method

$$Z(\tau)\varphi_1 = (1 - i\lambda_1\tau)\varphi_1 + f_1,$$

where $\int_\Omega |f_1(x)|dx = O(\tau)$, $f_1(x) = O(\tau^2)$ (locally uniformly in Ω).
Then their argument goes as follows: Hence

$$
\begin{aligned}
&Z(\tau)^n\varphi_1 = (1 - i\lambda_1\tau)^n\varphi_1 + r_n(x),\\
&r_n(x) = \sum_{k=1}^n (1 - i\lambda_1\tau)^{n-k} f_k(x), \quad f_k := Z(\tau)^{k-1}f_1,\\
&(1 - i\lambda_1\tau)^n\varphi_1 \to e^{-it\lambda_1}\varphi_1, \text{ in } L^2(\Omega), \quad n \to \infty.
\end{aligned}
$$

Up to here, there seems to be no problem. But then they took for granted, without precisely checking, that "$r_n(x)$ also would become small." Here is a gap of their proof Their argument is insufficient to see it, because we shall have to take care in using the stationary phase method, not in the whole space $\mathbb{R}^d$ [though they considered in one dimension ($d = 1$)] but in a bounded domain Ω with *boundary*. For our argument around here, see also [TI].

5. Concluding Remarks with Open Problem

1^o. Once the theorem of symmetric product case is proved, the following non-symmeric product formula can also be proved of course:

$$\lim_{n\to\infty} [e^{-itH/n}P]^n = \lim_{n\to\infty} [Pe^{-itH/n}]^n = e^{-itH_P}P,$$

in the topology of $L^2_{loc}(\mathbf{R}; \mathcal{H})$.

2^o. We can extend Theorem to the case P may be t-dependent orthogonal projection $P(t)$ on $\mathcal{H}$ such that

$$
\begin{aligned}
&P(t)P(0) = P(t), \ P(0) = P,\\
&\text{s-}\lim_{t\to 0} \|H^{1/2}P(t)v\| = \|H^{1/2}Pv\|, \ v \in D[H^{1/2}P].
\end{aligned}
$$

It holds that

$$\lim_{n\to\infty} [P(1/n)e^{-itH/n}P(1/n)]^n = e^{-itH_P}P,$$

in the topology of $L^2_{loc}(\mathbf{R};\mathcal{H})$.

3^o. However, the strong convergence (1.1) on $\mathcal{H}$ holds, if we replace the operator H in $e^{-itH/n}\ H$ by its cut-off operator as well as $e^{-itH/n}$ by the resolvent $(I+itH/n)^{-1}$ (see [EINZ]).

4^o. The final result we actually expect to show is:

Conjecture. In case dim P is infinite, it also holds that

$$\text{s-}\lim_{n\to\infty}[Pe^{-itH/n}P]^n = e^{-itH_P}P,$$

uniformly on each compact t-interval in $[0,\infty)$.

Needless to say, this is solved if one is able to show (2.1) strongly on the Hilbert space $\mathcal{H}$.

We hope this conjecture will be proved up to when this progress report will be published (March 1, 2006).

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References

[BN] J. Beskow and J. Nilsson, *The concept of wave function and the irreducible representations of the Poincaré group, II. Unstable systems and the exponential decay law*, Arkiv Fys. **34** (1967), 561-569.

[EI] P. Exner and T. Ichinose, *A Product formula related to quantum Zeno dynamics*, Ann. H. Poincaré, **6** (2005), 195–215; also, *Product formula for quantum Zeno dynamics*, to appear soon in Proceedings of the XIV-th International Congress on Mathematical Physics (M $\cap\Phi$), [July 28–Aug.2, 2003, Lisbon, Portugal], World Scientific 2005.

[EINZ] P. Exner, T. Ichinose, H. Neidhardt and V. A. Zagrebnov, *Zeno Product formula revisited*, to appear in Integral Equations Operator Theory.

[FPSS] P. Facchi, S. Pascazio, A. Scardicchio, and L. S. Schulman, *Zeno dynamics yields ordinary constraints*, Phys. Rev. **A 65** (2002), 012108.

[F] C. Friedman, *Semigroup product formulas, compressions, and continual observations in quantum mechanics*, Indiana Math. J. **21**(1971/72), 1001–1011.

[K] T. Kato, *Trotter's product formula for an arbitrary pair of self-adjoint contraction semigroups*, In: Topics in Functional Analysis (I. Gohberg and Mark Kac, ed.), Academic Press, New York, 1978, pp.185–195.

[MS] B. Misra and E. C. G. Sudarshan, *The Zeno's paradox in quantum theory*, J. Math. Phys. **18** (1977), 756–763.

[TI] 田村英男・一瀬 孝, *Exponential product approximation to heat kernel of Dirichlet Laplacian and Zeno product*, to appear in 京都大学数理解析研究所講究録「スペクトル・散乱理論とその周辺, 2004.12.20–22」.