数理解析研究所講究録
1510 巻 2006 年 169-181

Completeness of the Generalized Eigenfunctions for relativistic Schrödinger operators I

Dabi Wei
魏 大比

April 28, 2006

Abstract

Generalized eigenfunctions of the odd-dimensional ($n\geq 3$) relativestic Schrödinger operator $\sqrt{-\Delta}+V(x)$ with $|V(x)|\leq C\langle x\rangle^{-\sigma}$, $\sigma>1$, are considered. We compute the integral kernels of the boundary values $R^{\pm}(\lambda)=(\sqrt{-\Delta}-(\lambda\pm i0))^{-1}$, and prove that the generalized eigenfunctions $\varphi^{\pm}(x,k):=\varphi_{0}(x,k)-R^{\mp}(|k|)V\varphi_{0}(x,k)$ ($\varphi_{0}(x,k):=e^{ix\cdot k}$) are bounded for $(x,k)\in \mathbb{R}^{n}\times\{k|a\leq|k|\leq b\}$, where $[a,b]\subset(0,\infty)\backslash\sigma_{p}(H)$. This fact, together with the completeness of the wave operators, enables us to obtain the eigenfunction expansion for the absolutely continuous spectrum.

Introduction

This paper considers the odd-dimensional ($n\geq 3$) relativistic Schrödinger operator

$$H=H_{0}+V(x),\quad H_{0}=\sqrt{-\Delta},\quad x\in \mathbb{R}^{n}$$

with a short range potential $V(x)$.

Throughout the paper we assume that $V(x)$ is a real-valued measurable function on $\mathbb{R}^{n}$ satisfying

$$|V(x)|\leq C\langle x\rangle^{-\sigma},\quad \sigma>1.$$

When we deal with the boundness and the completeness of the generalized eigenfunctions, σ will be required to satisty the assumption $\sigma>(n+1)/2$ and n to be an odd integer with $n\geq 3$.

In general, the schrödinger operator is written as $-\Delta+V(x)$, $x\in \mathbb{R}^{n}$. In [8], the completeness of the generalized eigenfunctions for operator $-\Delta+V(x)$ was proved. However, it was considered by 3-dimensional case. And, in the case of N-body, the completeness was proved in [11, 12]. When the speed of the particles approach light, we have to consider the relativistic case. the schrödinger operator is written by $\sqrt{-\Delta+m}+V(x)$, $x\in \mathbb{R}^{n}$, where m is the mass of the particle. In recent years, there have been some works on the decay of eigenfunctions associated to the discrete spectra of these operators[4, 20]. On the asymptotic behaviour of the eigenfunctions of the relativistic N-body Schrödinger operator, some works have been done in [21].

But, like a photon, the zero mass particle exists. Then, the relativitic Schrödinger operator is written by $H = \sqrt{-\Delta} + V(x), \quad x \in \mathbb{R}^n$. H is essentially self adjoint on $C_0^\infty(\mathbb{R}^n)$ [27]. And in the paper [28], T. Umeda considered the 3-dimensional case and proved that the generalized eigenfuctions $\varphi^\pm(x,k)$ are bounded for $(x,k) \in \mathbb{R}^3 \times \{k|k \in \mathbb{R}^3, \ a \le |k| \le b\}$, $[a,b] \subset (0,\infty)\backslash\sigma_p(H)$. In the part II, T.Umeda announced that he will deal with the completeness of the generalized eigenfunctions. But, he was too busy to collect his result.

For the purpose of making a comparison, let us briefly recall some results done before. For $z \in \rho(H)$, the resolvents of H and H_0 will be written as

$$R_0(z) = (H_0 - z)^{-1}, \ \ R(z) = (H - z)^{-1}.$$

Clearly, for any $\lambda \in (0,\infty)\backslash\sigma_p(H)$ and $s > 1/2$, there exist the limits (see [2, Theorem 4A])

$$\begin{aligned} R_0^\pm(\lambda) &= \lim_{\mu \downarrow 0} R_0(\lambda \pm i\mu) \quad in \quad \mathbf{B}(L^{2,s}, H^{1,-s}), \\ R^\pm(\lambda) &= \lim_{\mu \downarrow 0} R(\lambda \pm i\mu) \quad in \quad \mathbf{B}(L^{2,s}, H^{1,-s}). \end{aligned}$$

Following S. Agmon [1], we define two families of generalized eigenfunctions of H by

$$\varphi^\pm(x,k) := \varphi_0(x,k) - R^\mp(|k|)\{V(\cdot)\varphi_0(\cdot,k)\}(x)$$

for k with $|k| \in (0,\infty)\backslash\sigma_p(H)$. In the paper [28, section 8], T. Umeda considered the 3-dimensional case and proved that the generalized eigenfunctions $\varphi^\pm$ satisfy

$$\varphi^\pm(x,k) = \varphi_0(x,k) - R_0^\mp(|k|)\{V(\cdot)\varphi^\pm(\cdot,k)\}(x)$$

for $(x,k) \in \mathbb{R}^3 \times \{k|k \in \mathbb{R}^3, \ a \le |k| \le b\}$, $[a,b] \subset (0,\infty)\backslash\sigma_p(H)$, which is called modified Leppmann-Schwinger equations. Moreover, he showed that the generalized eigenfunctions $\varphi^\pm(x,k)$ are bounded for $(x,k) \in \mathbb{R}^3 \times \{k|k \in \mathbb{R}^3, \ a \le |k| \le b\}$, $[a,b] \subset (0,\infty)\backslash\sigma_p(H)$, (see T. Umeda [28, section 9]). T. Umeda [29] announced that he will deal with the completeness of the generalized eigenfunctions.

Under the condition of the odd-dimension ($n \ge 3$), the present paper shows that the same equation is valid,

$$\varphi^\pm(x,k) = \varphi_0(x,k) - R_0^\mp(|k|)\{V(\cdot)\varphi^\pm(\cdot,k)\}(x)$$

for $(x,k) \in \mathbb{R}^n \times \{k|a \le |k| \le b\}$, $[a,b] \subset (0,\infty)\backslash\sigma_p(H)$, but when $n > 3$, the resolvent is defferent from the case $n = 3$. The computations show that there exists some polynomials $a_j(z)$, $b_j(z)$, $c_j(z)$ of z in $\mathbb{C}$ such that the integral kernel of the resolvent of $\sqrt{-\Delta}$ is given by

$$R_0(z)u(x) = \int_{\mathbb{R}^n} g_z(x-y)u(y)dy$$

for $z \in \mathbb{C}\setminus[0,\infty)$, where

$$\begin{aligned} g_z(x) = & -\frac{c_n}{2m}|x|^{-2m} + b_{m-1}(z)M_z(x)|x|^{-(m-1)} \\ & + \sum_{j=m}^{2m-1}\Big(a_j(z) + b_j(z)M_z(x) + c_j(z)N_z(x)\Big)|x|^{-j}, \end{aligned}$$

and

$$\begin{aligned} M_z(x) = & \frac{1}{|x|}\Big\{\mathrm{ci}(-|x|z)\sin(|x|z) - \mathrm{si}(-|x|z)\cos(|x|z)\Big\}, \\ N_z(x) = & \Big\{\mathrm{ci}(-|x|z)\cos(|x|z) + \mathrm{si}(-|x|z)\sin(|x|z)\Big\}. \end{aligned}$$

For the definitions of the cosine and sine integral functions $\mathrm{ci}(z)$ and $\mathrm{si}(z)$, see section 5. We compute the limit $g_\lambda^\pm(x) := \lim_{\mu\downarrow 0} g_{\lambda\pm i\mu}(x)$ as follows,

$$\begin{aligned} g_\lambda^\pm(x) = & \left\{a_{2m}(\lambda) + b_{2m}\left(e^{\pm i\lambda|x|} + m_\lambda(x)\right)\right\}|x|^{-2m} \\ & + \sum_{j=m}^{2m-1} a_j(\lambda)|x|^{-j} + \sum_{j=m}^{2m-1} b_j(\lambda)\left(e^{\pm i\lambda|x|} + m_\lambda(x)\right)|x|^{-j} \\ & + \sum_{j=m}^{2m-1} c_j(\lambda)\left(e^{\pm i(\lambda|x|+\pi/2)} + n_\lambda(x)\right)|x|^{-j}, \end{aligned}$$

where

$$\begin{aligned} m_\lambda(x) = & \mathrm{ci}(\lambda|x|)\sin(\lambda|x|) + \mathrm{si}(\lambda|x|)\cos(\lambda|x|), \\ n_\lambda(x) = & \mathrm{ci}(\lambda|x|)\cos(\lambda|x|) - \mathrm{si}(\lambda|x|)\sin(\lambda|x|). \end{aligned}$$

We then prove that the generalized eigenfunctions $\varphi^\pm(x,k)$ are bounded for $(x,k) \in \mathbb{R}^n \times \{k|\ \ a \le |k| \le b\}$, $[a,b] \subset (0,\infty)\setminus\sigma_p(H)$, with

$$R_0^\pm(\lambda)u(x) = \int_{\mathbb{R}^n} g_\lambda^\pm(x-y)u(y)dy.$$

$H = H_0 + V$ defines a selfadjoint operator in $L^2(\mathbb{R}^n)$, whose domain is $H^1(\mathbb{R}^n)$ (see section 2, or T. Umeda [27, Theorem 5.8]). Moreover, H is essentially selfadjoint on $C_0^\infty(\mathbb{R}^n)$ (see T. Umeda [27]). It follows from Reed-Simon [22, P113, Corollary 2] that

$$\sigma_e(H) = \sigma_e(H_0) = [0,\infty).$$

The fact that $\sigma_p(H) \cap (0,\infty)$ is a discrete set was first proved by B. Simon [23, Theorem 2.1]. He also proved that each eigenvalue in the set $\sigma_p(H)\cap(0,\infty)$ has finite multiplicity [23, Theorem 2.1]. From V. Enss's idea (see V. Enss [5]), we obtain that the wave operators $W_\pm$ defined by

$$W_\pm = \lim_{t\to\infty} e^{itH}e^{-itH_0}$$

are complete. Finally, by the idea of H. Kitada [12] and S.T. Kuroda [15], we obtain the completeness of the generalized eigenfunctions as follows.

Theorem Assume the dimension n ($n \geq 3$) is an odd integer, $\sigma > (n+1)/2$, $s > n/2$ and $[a,b] \subset (0,\infty)\backslash\sigma_p(H)$. For $u \in L^{2,s}(\mathbb{R}^n)$, let $\mathcal{F}_\pm$ be defined by

$$\mathcal{F}_\pm u(k) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} u(x)\overline{\varphi^\pm(x,k)}dx.$$

Then for an arbitrary $L^{2,s}(\mathbb{R}^n)$-function $f(x)$,

$$E_H([a,b])f(x) = (2\pi)^{-n/2} \int_{a\leq|k|\leq b} \mathcal{F}_\pm f(k)\varphi^\pm(x,k)dk$$

where E_H is the spectral measure on H.

The plan of the paper In section 1, we construct generalized eigenfunctions of $\sqrt{-\Delta} + V(x)$ on R^n. We compute the resolvent kernel of $\sqrt{-\Delta}$ on $\mathbb{R}^n$ in the integral form in section 2. Section 3 prove the generalized eigenfunctions are bounded in the case of odd-dimension $n \geq 3$. We studies the asymptotic completeness of wave operators in section 4. In the last section 5, we deal with the completeness of the generalized eigenfunctions. We explained about the theorems without proving in this paper for the limitation of the number of pages.

Notation We introduce the notation which will be used in the present paper.

For $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean norm of x and $\langle x\rangle = \sqrt{1+|x|^2}$. The Fourier transform of a function u is denoted by $\mathcal{F}u$ or $\hat{u}$, and is defined by

$$\mathcal{F}u(\xi) = \hat{u}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi}u(x)dx.$$

For s and l in $\mathcal{R}$, we define the weighted L^2-space and the weighted Sobolev space by

$$L^{2,s}(\mathbb{R}^n) = \{f|\langle x\rangle^s f \in L^2(\mathbb{R}^n)\}, \quad H^{l,s}(\mathbb{R}^n) = \{f|\langle x\rangle^s\langle D\rangle^l f \in L^2(\mathbb{R}^n)\}$$

respectively, where D stands for $-i\partial/\partial x$ and $\langle D\rangle = \sqrt{1+|D|^2} = \sqrt{1-\Delta}$. The inner products and the norm in $L^{2,s}(\mathbb{R}^n)$ and $H^{l,s}(\mathbb{R}^n)$ are given by

$$(f,g)_{L^{2,s}} = \int_{\mathbb{R}^n} \langle x\rangle^{2s} f(x)\overline{g(x)}dx, \quad (f,g)_{H^{l,s}} = \int_{\mathbb{R}^n} \langle x\rangle^{2s}\langle D\rangle^l f(x)\overline{\langle D\rangle^l g(x)}dx,$$
$$\|f\|_{L^{2,s}} = \{(f,f)_{L^{2,s}}\}^{1/2}, \quad \|f\|_{H^{l,s}} = \{(f,f)_{H^{l,s}}\}^{1/2},$$

respectively. For $s = 0$ we write

$$(f,g) = (f,g)_{L^{2,0}} = \int_{\mathbb{R}^n} f(x)\overline{g(x)}dx, \quad \|f\|_{L^2} = \|f\|_{L^{2,0}}.$$

For a pair of $f \in L^{2,-s}(\mathbb{R}^n)$ and $g \in L^{2,s}(\mathbb{R}^n)$, we also define $(f,g) = \int_{\mathbb{R}^n} f(x)\overline{g(x)}dx$.

By $C_0^\infty(\mathbb{R}^n)$ we mean the space of C^∞-functions of compact support. By $\mathcal{S}(\mathbb{R}^n)$ we mean the Schwartz space of rapidly decreasing functions, and by $\mathcal{S}'(\mathbb{R}^n)$ the space of tempered distributions.

The operator $\sqrt{-\Delta}e^{ix\cdot k}$ is formally defined by

$$\int_{\mathbb{R}^n} e^{ix\cdot\xi}|\xi|\delta(\xi-k)d\xi,$$

where $\delta(x)$ is the Dirac's delta function. As the symbol $|\xi|$ of $\sqrt{-\Delta}$ is singular at the origin $\xi=0$, making sense of the expression $\sqrt{-\Delta}e^{ix\cdot k}$ is one of the main tasks in the present paper.

For a pair of Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, $\mathbf{B}(\mathcal{H},\mathcal{K})$ denotes the Banach space of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$.

For a selfadjoint operator H in a Hilbert space, $\sigma(H)$ and $\rho(H)$ denote the spectrum of H and the resolvent set of H, respectively. The essential spectrum, the continuous spectrum and the absolutely continuous spectrum of H will be denoted by $\sigma_e(H)$, $\sigma_c(H)$, and $\sigma_{ac}(H)$ respectively. E_H denotes the spectral measure on T, and $E_H(\lambda)=E_H((-\infty,\lambda])$, $\quad E_H((a,b])=E_H(b)-E_H(a)$.

The continous subspace and the absolutely continuous subspace of H will be denoted by $\mathcal{H}_c$, $\mathcal{H}_{ac}$, respectively.

1 Generalized eigenfuction

We construct in this section generalized eigenfunctions of $\sqrt{-\Delta}+V(x)$ on $\mathbb{R}^n$, and show that the generalized eigenfunctions satisfy the equation

$$\varphi^\pm(x,k)=\varphi_0(x,k)-R_0^\mp(|k|)V\varphi^\pm(x,k),$$

where $R_0(z)$ is the resolvent of $H_0=\sqrt{-\Delta}$ defined by

$$R_0(z):=(H_0-z)^{-1}=\mathcal{F}^{-1}(|\xi|-z)^{-1}\mathcal{F},$$

and $\varphi_0(x,k)$ is definded by

$$\varphi_0(x,k)=e^{ix\cdot k}.$$

Similarly $R(z)$ is the resolvent of $H=\sqrt{-\Delta}+V(x)$ on $\mathbb{R}^n$ and we assume that $V(x)$ is a real-valued measurable function on $\mathbb{R}^n$ satistying $|V(x)|<C\langle x\rangle^{-\sigma}$ for some $\sigma>1$. To show the above equation for eigenfunctions, we uese two theorems demonstrated by Ben-Artzi and Nemirovki. (see [2, Section 2 and Theorem 4A])

Theorem 1.1 (Ben-Artzi and Nemirovki) Let $s>1/2$. Then
(1) For any $\lambda>0$, there exist the limits $R_0^\pm(\lambda)=\lim_{\mu\downarrow 0}R_0(\lambda\pm i\mu)$ in $\mathbf{B}(L^{2,s},H^{1,-s})$.
(2) The operator-valued functions $R_0^\pm(z)$ defined by

$$R_0^\pm(z)=\begin{cases}R_0(z) & if\quad z\in\mathbb{C}^\pm\\ R_0^\pm(\lambda) & if\quad z=\lambda>0\end{cases}$$

are $\mathbf{B}(L^{2,s}, H^{1,-s})$-valued continuous functions, where $\mathbb{C}^+$ and $\mathbb{C}^-$ are the upper and the lower half-planes respectively: $\mathbb{C}^\pm = \{z \in \mathbb{C} \mid \pm \operatorname{Im} z > 0\}$.

Theorem 1.2 (Ben-Artzi and Nemirovki) Let $s > 1/2$ and $\sigma > 1$. Then
(1) The continuous spectrum $\sigma_c(H) = [0, \infty)$ is absolutely continuous, except possibly for a discrete set of embedded eigenvalues $\sigma_p(H) \cap (0, \infty)$, which can accumulate only at 0 and ∞.
(2) For any $\lambda \in (0, \infty) \backslash \sigma_p(H)$, there exist the limits

$$R^\pm(\lambda) = \lim_{\mu \downarrow 0} R(\lambda \pm i\mu) \quad in \quad \mathbf{B}(L^{2,s}, H^{1,-s}).$$

(3) The operator-valued functions $R^\pm(z)$ defined by

$$R^\pm(z) = \begin{cases} R(z) & if \quad z \in \mathbb{C}^\pm \\ R^\pm(\lambda) & if \quad z = \lambda > 0 \backslash \sigma_p(H) \end{cases}$$

are $\mathbf{B}(L^{2,s}, H^{1,-s})$-valued continuous functions.

The main results of this section are

Theorem 1.3 Let $\sigma > (n+1)/2$, if $|k| \in (0, \infty) \backslash \sigma_p(H)$, then generalized eigenfunctions

$$\varphi^\pm(x, k) := \varphi_0(x, k) - R^\mp(|k|)\{V(\cdot)\varphi_0(\cdot, k)\}(x)$$

satisfy the equation

$$(\sqrt{-\Delta_x} + V(x))u = |k|u \quad in \quad \mathcal{S}'(\mathbb{R}^n_x)$$

where $\varphi_0(x, k)$ is definded by $\varphi_0(x, k) = e^{ix\cdot k}$.

Theorem 1.4 Let $\sigma > (n+1)/2$. If $|k| \in (0, \infty) \backslash \sigma_p(H)$, and $n/2 < s < \sigma - 1/2$, then we have

$$\varphi^\pm(x, k) = \varphi_0(x, k) - R_0^\mp(|k|)\{V(\cdot)\varphi^\pm(\cdot, k)\}(x) \quad in \quad L^{2,-s}(\mathbb{R}^n).$$

2 The integral kernel of the resolvents of H_0

This section is devoted to computing the resolvent kernel of $H_0 = \sqrt{-\Delta}$ on $\mathbb{R}^n$, where $n = 2m+1$, $m \geq 1$ and $m \in \mathbb{N}$. Then we compute the limit of $g_z(x)$ as $\mu \downarrow 0$, where $z = \lambda + i\mu$ and $\lambda > 0$, and study the properties of the integral operator $G_\lambda^\pm$. In this section we suppose that (cf. [6, p. 269, Formula (46) and (47)])

$$\begin{aligned}
&(1) \quad n = 2m+1, \quad m \geq 1 \text{ and } m \in \mathbb{N}, \\
&(2) \quad M_z(x) = \int_0^\infty e^{tz} \frac{1}{t^2 + |x|^2} dt = \frac{1}{|x|}\Big\{\mathrm{ci}(-|x|z)\sin(|x|z) - \mathrm{si}(-|x|z)\cos(|x|z)\Big\}, \\
&\qquad N_z(x) = \int_0^\infty e^{tz} \frac{t}{t^2 + |x|^2} dt = \mathrm{ci}(-|x|z)\cos(|x|z) + \mathrm{si}(-|x|z)\sin(|x|z), \\
&(3) \quad m_\lambda(x) = \mathrm{ci}(\lambda|x|)\sin(\lambda|x|) + \mathrm{si}(\lambda|x|)\cos(\lambda|x|), \\
&\qquad n_\lambda(x) = \mathrm{ci}(\lambda|x|)\cos(\lambda|x|) - \mathrm{si}(\lambda|x|)\sin(\lambda|x|).
\end{aligned}$$

Where ci(x) and si(x) are definded by

$$\mathrm{ci}(x)=\int_x^\infty \frac{\cos t}{t}dt\ , \qquad \mathrm{si}(x)=-\int_x^\infty \frac{\sin t}{t}dt\ , \quad x>0.$$

We see that si(x) has an analytic continuation si(z) (see [6, P145]),

$$\mathrm{si}(z)=-\frac{\pi}{2}+\sum_{m=0}^\infty \frac{(-1)^m}{(2m+1)!(2m+1)}z^{2m+1} \tag{2.1}$$

The cosine integral function ci(x) has an analytic continuation ci(z), which is a many-valued function with a logarithmic branch-point at $z=0$ (see [6, P145]). In this paper, we choose the principal branch

$$\mathrm{ci}(z)=-\gamma-\mathrm{Log} z-\sum_{m=1}^\infty \frac{(-1)^m}{(2m)!2m}z^{2m}, \quad z\in\mathbb{C}\backslash(-\infty,0], \tag{2.2}$$

where γ is the Euler's constant. The main theorems are

Theorem 2.1 Let $n\geq 3$, Re$z<0$, then

$$R_0(z)u=G_z u$$

for all $u\in C_0^\infty(\mathbb{R}^n)$, where

$$G_z u(x)=\int_{\mathbb{R}^n} g_z(x-y)u(y)dy,\ \ g_z(x)=\int_0^\infty e^{tz}\frac{c_n t}{(t^2+|x|^2)^{\frac{n+1}{2}}}dt,$$
$$c_n=\pi^{-\frac{n+1}{2}}\Gamma\left(\frac{n+1}{2}\right),\ \ \Gamma(x)=\int_0^\infty s^{x-1}e^{-s}ds. \tag{2.3}$$

Theorem 2.2 Let $n=2m+1$, $m\geq 1$ ($m\in\mathbb{N}$) and $s>1/2$, $u\in L^{2,s}(\mathbb{R}^n)$. Let $[a,b]\subset(0,\infty)$ and $\lambda\in[a,b]$.

(1) There exist some functions $a_j(\lambda)$, $b_j(\lambda)$, $c_j(\lambda)$ which are polynomials of λ for $j=m,m+1,\cdots,2m$,

$$\begin{aligned}
R_0^\pm(\lambda)u(x)&=G_\lambda^\pm u(x)=\int_{\mathbb{R}^n} g_\lambda^\pm(x-y)u(y)dy\\
g_\lambda^\pm(x)&:=\lim_{\mu\downarrow 0} g_{\lambda\pm i\mu}(x)=\left\{a_{2m}(\lambda)+b_{2m}\left(e^{\pm i\lambda|x|}+m_\lambda(x)\right)\right\}|x|^{-2m}\\
&\qquad+\sum_{j=m}^{2m-1}a_j(\lambda)|x|^{-j}+\sum_{j=m}^{2m-1}b_j(\lambda)\left(e^{\pm i\lambda|x|}+m_\lambda(x)\right)|x|^{-j}\\
&\qquad+\sum_{j=m}^{2m-1}c_j(\lambda)\left(e^{\pm i(\lambda|x|+\pi/2)}+n_\lambda(x)\right)|x|^{-j},
\end{aligned}$$

where $R_0^\pm(\lambda):=\lim_{\mu\downarrow 0}R_0(\lambda\pm i\mu)$.

(2) There exist some positive constants C_{abj} for $j = m, m+1, \cdots, 2m$ such that

$$|R_0^{\pm}(\lambda)u(x)| = |G_\lambda^{\pm}u(x)| \leq \sum_{j=m}^{2m} |D_j u(x)|,$$

$$D_j(\lambda)u(x) := C_{abj} \int_{\mathbb{R}^n} |x-y|^{-j} u(y)dy.$$

3 Boundness of the generalized eigenfunctions

In this section, we assume that n, $V(x)$ and k satisfy the next inequalitys:

(1) $n = 2m+1$ ($m \in \mathbb{N}$) and $m \geq 1$

(2) $|V(x)| \leq C\langle x\rangle^{-\sigma}, \quad \sigma > \dfrac{n+1}{2}$

(3) $k \in \left\{k \middle| a \leq |k| \leq b\right\}$ and $[a,b] \subset (0,\infty)\backslash\sigma_p(H)$.

Applying Theorem 1.4, we see that generalized eigenfuction $\varphi^{\pm}(x,k)$ defined by

$$\varphi^{\pm}(x,k) = \varphi_0(x,y) - R^{\mp}(|k|)\{V(\cdot)\varphi_0(\cdot,k)\}(x), \tag{3.1}$$

satisfies the equation

$$\varphi^{\pm}(x,k) = \varphi_0(x,k) - R_0^{\mp}(|k|)\{V(\cdot)\varphi^{\pm}(\cdot,k)\}(x), \tag{3.2}$$

where $\varphi_0(x,k) = e^{ix\cdot k}$.

In this section, we let $\{D_j V(\cdot)\varphi^{\pm}(\cdot,k)\}(x)$. be denoted by $D_j V(x)\varphi^{\pm}(x,k)$. Moreover, let $V(x)D_{j_r}V(x)D_{j_{r-1}}\cdots V(x)D_{j_1}V(x)\varphi^{\pm}(x,k)$, be denoted by

$$\left(\prod_{p=1}^{r} V(x)D_{j_p}\right)\left\{V(x)\varphi^{\pm}(x,k)\right\}.$$

This section proves boundness of the generalized eigenfunctions. And the main theorem is

Theorem 3.1 Let $n = 2m+1$, $m \geq 1$ ($m, n \in \mathbb{N}$), and $[a,b] \subset (0,\infty)\backslash\sigma_p(H)$. Then there exists a constant C_{ab} such that generalized eigenfunctions defined by $\varphi^{\pm}(x,k) := \varphi_0(x,y) - R^{\mp}(|k|)\{V(\cdot)\varphi_0(\cdot,k)\}(x)$ satisfy

$$|\varphi^{\pm}(x,k)| \leq C_{ab},$$

for all $(x,k) \in \mathbb{R}^n \times \{a \leq |k| \leq b\}$, where $\varphi_0(x,k) = e^{ix\cdot k}$.

To prove the main theorem, we gave the lemmas as follows.

Lemma 3.1 Let $m+1 \leq j \leq 2m$ ($j \in \mathbb{N}$) and $p > n/(n-j)$. If $u(x,k) \in L^2(\mathbb{R}_x^n) \cap L^p(\mathbb{R}_x^n)$, and $\|u(x,k)\|_{L^2} \leq C'_{ab}$, $\|u(x,k)\|_{L^p} \leq C''_{ab}$, ($C'_{ab}$ and C''_{ab} are positive constants) for all $(x,k) \in \mathbb{R}^n \times \{a \le |k| \le b\}$, then there exists a positive constant C_{ab}, such that

$$\left|D_j u(x,k)\right| \le C_{ab},$$

for all $(x,k) \in \mathbb{R}^n \times \{a \le |k| \le b\}$.

Lemma 3.2 Let r, $j_p \in \mathbb{N}$ and $s > 1/2$. If $m+1 \le j_p \le 2m$ for $1 \le p \le r$, then

$$\left(\prod_{p=1}^{r} V(x) D_{j_p}\right) \left\{V(x)\varphi^{\pm}(x,k)\right\} \in L^{2,s}(\mathbb{R}^n_x)$$

for all $r \in \mathbb{N}$. Moreover, there exits a positive constant C_{ab}, such that

$$\left\|\left(\prod_{p=1}^{r} V(x) D_{j_p}\right) \left\{V(x)\varphi^{\pm}(x,k)\right\}\right\|_{L^{2,s}} \le C_{ab}$$

for all $(x,k) \in \mathbb{R}^n \times \{a \le |k| \le b\}$.

Lemma 3.3 Let $0 < \alpha < n$, $1 < p < q < \infty$ and $f \in L^p(R^n)$. Let $I_\alpha f(x)$ be definded by $I_\alpha f(x) := \int_{\mathbb{R}^n} |x-y|^{-n+\alpha} f(y)dy$. If $1/q = 1/p - \alpha/n$, there exists a positive constant C_{pq}, such that

$$\|I_\alpha f\|_{L^q} \le C_{pq}\|f\|_{L^p}.$$

For the proof of the theorem, see [24, P119].

Lemma 3.4 Let $r \in \mathbb{N}$. If $m+1 \le j_p \le 2m$ $(1 \le q \le r)$, and $2\sum_{p=1}^{q} j_p > (2q-1)n$ for all $q \le r$, then

$$\left(\prod_{p=1}^{r} V(x) D_{j_p}\right) \left\{V(x)\varphi^{\pm}(x,k)\right\} \in L^{\frac{2n}{2\sum_{p=1}^{r} j_p - (2r-1)n}}(\mathbb{R}^n_x)$$

for all $r \le n-1$. Moreover, there exits a positive constant C_{ab}, such that

$$\left\|\left(\prod_{p=1}^{r} V(x) D_{j_p}\right) \left\{V(x)\varphi^{\pm}(x,k)\right\}\right\|_{L^{2n/\{2\sum_{p=1}^{r} j_p-(2r-1)n\}}} \le C_{ab}$$

for all $(x,k) \in \mathbb{R}^n \times \{a \le |k| \le b\}$.

Lemma 3.5 Let $r \in \mathbb{N}$ and $r \le n$. If $m \le j_p \le 2m$ for all $1 \le p \le r$, then there exist a positive constant C_{ab}, such that

$$\sum_{2\sum_{p=1}^{r} j_p < (2r-1)n} \left| D_{j_r} \left(\prod_{p=1}^{r-1} V(x) D_{j_p}\right) \left\{V(x)\varphi^{\pm}(x,k)\right\}\right| \le C_{ab},$$

for all $(x,k) \in \mathbb{R}^n \times \{a \le |k| \le b\}$.

4 Asymptotic completeness

We investigate the asymptotic completeness of wave operators in this section. We assum that the potential $V(x)$ is a real-valued measurable function on $\mathbb{R}^n$ satisfying

$$|V(x)| \leq C\langle x\rangle^{-\sigma} \qquad \sigma > 1 \tag{4.1}$$

Under this assumption, it is obvious that V is a bounded selfadjoint operator in $L^2(\mathbb{R}^n)$, and that $H = H_0 + V$ defines a selfadjoint operator in $L^2(\mathbb{R}^n)$, whose domain is $H^1(\mathbb{R}^n)$ (see T. Umeda [27, Theorem 5.8]). Moreover H is essentially selfadjoint on $C_0^\infty(\mathbb{R}^n)$ (see T. Umeda [27]). Since V is relatively compact with respect to H_0, it follows from Reed-Simon [22, P113, Corollary 2] that

$$\sigma_e(H) = \sigma_e(H_0) = [0, \infty).$$

In this section, we prove the next main theorem with V. Enss's idea (see V. Enss [5] and H. Isozaki [9]).

Theorem 4.1 Let $H_0 = \sqrt{-\Delta}$, $H = H_0 + V(x)$ and $V(x)$ satisfying (4.1). Then there exists the limits

$$W_\pm = \lim_{t\to\pm\infty} e^{itH} e^{-itH_0},$$

and the asymptotic completeness hold:

$$\mathcal{R}(W_\pm) = \mathcal{H}_{ac}(H).$$

5 Eigenfunction expansions

In this section, we assum that the dimension n is an odd integer, $n \geq 3$, and $\sigma > (n+1)/2$. We consider the completeness of the generalized eigenfunction in this section. The main idea is the same as the idea in H. Kitada [12] and S.T. Kuroda [15], besides, in this section, we use the method in T. Ikebe [8, section 11]. It is known that

$$\sigma_e(H) = \sigma_e(H_0) = [0, \infty).$$

We need to remark that $\sigma_p(H) \cap (0, \infty)$ is a discrete set. This fact was first proved by B. Simon [23, Theorem 2.1]. Moreover, B. Simon [23, Theorem 2.1] proved that each eigenvalue in the set $\sigma_p(H) \cap (0, \infty)$ has finite multiplicity.

The main theorem is

Theorem 5.1 Assume the dimension n ($n \geq 3$) is an odd integer, $\sigma > (n+1)/2$, $s > n/2$ and $[a, b] \subset (0, \infty)\backslash\sigma_p(H)$. For $u \in L^{2,s}(\mathbb{R}^n)$, let $\mathcal{F}_\pm$ be defined by

$$\mathcal{F}_\pm u(k) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} u(x)\overline{\varphi^\pm(x,k)}dx. \tag{5.1}$$

For an arbitrary $L^{2,s}(\mathbb{R}^n)$-function $f(x)$,

$$E_H([a,b])f(x) = (2\pi)^{-n/2} \int_{a\leq|k|\leq b} \mathcal{F}_\pm f(k)\varphi^\pm(x,k)dk,$$

where E_H is the spectral measure on H, and $\varphi^{\pm}(x,k)$ are defined in Theorem 1.3.

Acknowledgements The author wishes to express his sincere thanks to Professor H. Kitada for his encouraging and stimulating discussions with him, and thanks to Professor T. Umeda for his encouraging commucations. The author also wishes to express his sincere thanks to his family and Ms. T. Cai for their constant supports.

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Dabi Wei : Department of Information Physics and Computing, Graduate School of Information Science and Technology, University of Tokyo Hongou 7-3-1, bunkyo-ku, Tokyo 113-8656, Japan. *Email* : dabi@alab.t.u-tokyo.ac.jp